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Article

# On the Growth of Derivatives of Algebraic Polynomials in Regions with Piecewise Smooth Boundary

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## Abstract

In this paper, we study the behavior of the  $m - th$  derivatives of general algebraic polynomials in weighted Bergman spaces defined in domains of the complex plane bounded by piecewise smooth curves with nonzero exterior angles and zero interior angles. Our approach involves establishing upper bounds on the growth of these derivatives not only interior of the unbounded domain but also on the closure of given domain. Through this analysis, we reveal detailed patterns in the growth of the  $m - th$  derivatives of algebraic polynomials throughout the complex plane, depending on the properties of the weighted function and the domain.

**Keywords:** algebraic polynomial; quasiconformal mapping; smooth curve; Bernstein–Walsh–Markov inequality; polynomial estimates

## 1. Introduction and Preliminaries

Let  $\mathbb{C}$  denote the complex plane,  $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ ;  $G \subset \mathbb{C}$  be a bounded Jordan region (without loss of generality, we will assume that  $0 \in G$ ) and  $L := \partial G$ ;  $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = extL$ . For  $w \in \mathbb{C}$  and  $\delta > 0$ , we set:  $\Delta(w, \delta) := \{t \in \mathbb{C} : |t - w| > \delta\}$ ;  $\Delta := \Delta(0, 1)$ . Let  $w = \Phi(z)$  be the univalent conformal mapping of  $\Omega$  onto  $\Delta$  such that  $\Phi(\infty) = \infty$  and  $\lim_{z \rightarrow \infty} \frac{\Phi(z)}{z} > 0$ ;  $\Psi := \Phi^{-1}$ . For  $\rho \geq 1$ , let  $L_\rho$  denote the exterior level curve of  $L$ ,  $G_\rho$  denote the inner region of  $L_\rho$ , and  $\Omega_\rho$  denote the outer region of  $L_\rho$ .

Let us denote by

$$\wp_n := \left\{ P_n(z) = \sum_{k=0}^n a_k z^k : a_k \in \mathbb{C}, \deg P_n \leq n \right\}$$

the set of all complex-coefficient algebraic polynomials of degree at most  $n$ , where  $n \in \mathbb{N}$ .

The distance from a point  $z \in \mathbb{C}$  to a set  $A \subset \mathbb{C}$ , denoted by  $d(z, A)$ , is defined as

$$d(z, A) := \inf\{|z - \xi| : \xi \in A\}.$$

Let  $\{z_j\}_{j=1}^l \subset L$  be a fixed sequence of distinct points ordered in the positive direction along  $L$ , without loss of generality. Let  $R_0$  be a fixed constant such that  $1 < R_0 < \infty$ . Assume that the parameters  $\gamma_j$  satisfy  $\gamma_j > -2$  for all  $j = 1, \dots, l$ . Let  $h_0$  be a measurable function satisfying the inequality  $h_0(z) \geq c_0(G, h) > 0$ ,  $z \in G_{R_0}$ , for some constant  $c_0(G, h)$  depending only on  $G$  and  $h$ . Under these assumptions, we define the generalized Jacobi weight function by

$$h(z) := \begin{cases} h_0(z) \prod_{j=1}^l |z - z_j|^{\gamma_j}, & z \in \overline{G}_{R_0}, \\ 0, & z \in \mathbb{C} \setminus \overline{G}_{R_0}. \end{cases} \tag{1}$$

Let  $\sigma$  denote the two-dimensional Lebesgue measure. Given an arbitrary Jordan domain  $G$  and a real number  $p > 0$ , we present:

$$\begin{aligned} \|P_n\|_p &:= \|P_n\|_{A_p(h,G)} := \left( \iint_G h(z) |P_n(z)|^p d\sigma_z \right)^{1/p}, \quad 0 < p < \infty, \\ \|P_n\|_\infty &:= \|P_n\|_{A_\infty(1,G)} := \max_{z \in \overline{G}} |P_n(z)|, \quad p = \infty; \quad A_p(1,G) \equiv A_p(G), \end{aligned} \quad (2)$$

and, when  $L$  is rectifiable:

$$\begin{aligned} \|P_n\|_{\mathcal{L}_p(h,L)} &:= \left( \int_L h(z) |P_n(z)|^p |dz| \right)^{1/p}, \quad 0 < p < \infty, \\ \|P_n\|_{\mathcal{L}_\infty(1,L)} &:= \max_{z \in L} |P_n(z)|, \quad p = \infty; \quad \mathcal{L}_p(1,L) \equiv \mathcal{L}_p(L). \end{aligned} \quad (3)$$

For any  $P_n \in \wp_n$  the Bernstein-Walsh Lemma [49] states that the following inequality holds:

$$|P_n(z)| \leq |\Phi(z)|^n \|P_n\|_{C(\overline{G})}, \quad z \in \Omega. \quad (4)$$

Also, in [36], similar inequality of (4) is obtained for the space  $\mathcal{L}_p(L)$ , as following:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \leq R^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(L)}, \quad p > 0. \quad (5)$$

Assuming  $h(z) \neq 1$  is defined by parameters  $\gamma_j > -1$  for  $j = 1, 2, \dots, l$  as specified in (1), the weighted version of inequality (5) follows from [12, Lemma 2.4]:

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq R^{n+\frac{1+\gamma_{\max}}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \quad \gamma_{\max} = \max\{0, \gamma_j : j = 1, 2, \dots, l\}. \quad (6)$$

In order to recall an estimate analogous to (5) for the space  $A_p(h,G)$ , we begin by introducing the following definition.

**Definition 1.** [37, p.97] (see also [45]) The Jordan curve (or arc)  $L$  is called  $K$ -quasiconformal ( $K > 1$ ), if there is a  $K$ -quasiconformal mapping  $f$  of the region  $D \supset L$  such that  $f(L)$  is a circle (or line segment).

Let  $F(L)$  denote the set of all sense preserving plane homeomorphisms  $f$  of the region  $D \supset L$  such that  $f(L)$  is a circle (a line segment) and let

$$K_L := \inf\{K(f) : f \in F(L)\},$$

where  $K(f)$  is the maximal dilatation of  $f$ . Then  $L$  is a quasiconformal curve, if  $K_L < \infty$ , and  $L$  is a  $K$ -quasiconformal curve, if  $K_L \leq K$ .

A curve  $L$  is called a quasiconformal, if it is a  $K$ -quasiconformal for some  $K > 1$ .

An analogue of the estimates (4) and (6) for regions bounded by quasiconformal curve with the weight function  $h(z)$  defined as in (1) for all  $p > 0$  was found in [8] (see, also [6]) and has the following form:

$$\|P_n\|_{A_p(h,G_R)} \leq c_1 R^{*n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \quad (7)$$

where  $R^* := 1 + c_2(R-1)$ ,  $c_2 > 0$  and  $c_1 := c_1(G, p, c_2) > 0$  are constants independent of  $n$  and  $R$ . As shown in previous studies, not all quasiconformal curves are rectifiable (cf. [25], [37, p.104]).

In [9, Th.1.1], an analogous estimate was studied for the  $A_p(1, G)$ -norm,  $p > 0$ , for an arbitrary Jordan region and it was obtained that, for any  $P_n \in \wp_n$ ,  $R_1 = 1 + \frac{1}{n}$  and arbitrary  $R$ ,  $R > R_1$ , the following estimate

$$\|P_n\|_{A_p(G_R)} \leq cR^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{R_1})},$$

is true, where  $c = (\frac{2}{e^p-1})^{\frac{1}{p}} [1 + O(\frac{1}{n})]$  as  $n \rightarrow \infty$  and  $c$  is an asymptotically sharp constant.

In [46] the case was considered where the norm  $\|P_n\|_{C(\overline{G})}$  on the right-hand side (4) is replaced by  $\|P_n\|_{A_2(G)}$  which leads to new version of the Bernstein-Walsh lemma as follows: If the curve  $L = \partial G$  is quasiconformal and rectifiable, then there exists a constant  $c = c(L) > 0$  depending only on  $L$  such that

$$|P_n(z)| \leq c \frac{\sqrt{n}}{d(z, L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega, \quad (8)$$

holds for every  $P_n \in \wp_n$ .

On the other hand, using the mean value theorem, for an arbitrary Jordan region  $G$ , any polynomial  $P_n \in \wp_n$  and any  $p > 0$ , we can find that:

$$|P_n(z)| \leq \left( \frac{1}{\sqrt{\pi} d(z, L)} \right)^{\frac{2}{p}} \|P_n\|_{A_p(G)}, \quad z \in G. \quad (9)$$

In particular, estimates of the form (9) for orthonormal polynomials  $K_n(z) := K_n(z, h, G)$ ,  $\deg K_n = n$ ,  $n \in \mathbb{N}$ , over a region  $G$  and a specific weight function  $h$ , were investigated in [47, Th.2.1], [33, Th.5,  $h(z) \equiv 1$ ], [1, Th.6,  $h(z) \neq 1$ ], [2, Th.1,  $h(z) \neq 1$ ] and others. Similar estimates to (9) were studied, estimates in the space  $A_2(h, G)$  were obtained for any point  $z \in G$  in the following form:

$$|K_n(z)| \leq c_6 \frac{n^{-\beta}}{d^\alpha(z, L)}, \quad z \in G,$$

where  $c_6 = c_6(G) > 0$ ,  $\alpha = \alpha(G, h) > 0$  and  $\beta = \beta(G, h) > 0$ , are constants independent of  $n$ ,  $z$  and depending on the properties of the region  $G$  and the behaviour of the weight function  $h$ . Combining (8) and (9), we obtain an estimation on the growth of  $|P_n(z)|$  in  $\mathbb{C} \setminus L$ .

Thus, as can be seen from (8) and (9), to study the behavior of the derivatives of polynomials  $|P_n^{(m)}(z)|$ ,  $m \geq 0$ , in whole  $\mathbb{C}$  it is also necessary to know the behavior of these derivatives at the points of  $L$ . This means that we need to show that the whole plane can be represented as the union of a finite closed region and its complement—an infinite region—for example,  $\mathbb{C} = \overline{G_R} \cup \Omega_R$ ,  $R \geq 1$ ) and then study the behaviour of  $|P_n^{(m)}(z)|$ ,  $m \geq 0$ , in each of these regions separately. Based on this idea, we can break down  $\mathbb{C}$  into  $\mathbb{C} = \overline{G_R} \cup \Omega_R$ ,  $R = 1 + \frac{\varepsilon_0}{n}$  for some  $0 < \varepsilon_0 < 1$  and investigate the behaviour of  $|P_n^{(m)}(z)|$ ,  $m \geq 0$ , in each of the specified regions of  $\Omega_R$  and  $\overline{G_R}$ .

In the present study, we extend previous investigations on the problem of obtaining pointwise estimates for higher order derivatives of modulus of  $P_n(z)$  within certain unbounded regions characterized by interior zero and exterior non-zero angles. Our goal is to derive estimates in the following form:

$$|P_n^{(m)}(z)| \leq \eta_n \|P_n\|_p, \quad z \in \Omega, \quad m = 1, 2, \dots \quad (10)$$

where  $\eta_n := \eta_n(G, h, p, m, z)$ ,  $\eta_n(\cdot) \rightarrow \infty$  as  $n \rightarrow \infty$ , depends on the properties of  $G$  and  $h$ . Analogous results of the (10)-type for some norms and different unbounded regions were obtained by N.A. Lebedev, P.M. Tamrazov and V.K. Dzjadyk (see, for example, [31, pp. 418–428], [32, p. 383]). Also several studies have addressed the case  $m = 0$  with various  $p > 1$  values and region types. For instance, estimates have been obtained for regions with piecewise Dini-smooth boundaries and  $p > 1$  in [11], for piecewise smooth boundaries without cusps with  $p > 1$  in [10], for  $\kappa$ -quasidisks with  $p > 0$  in [13,15], for regions bounded by asymptotically conformal curves with  $p > 0$  in [14], for regions with piecewise smooth boundaries featuring interior angles and  $p > 0$  in [40], and for regions bounded by

piecewise asymptotically conformal curves and  $p > 0$  in [16], among others. For derivatives of order  $m \geq 1$ , estimates akin to those in (10) have been explored in various works. Notably, [19] investigates the case  $m = 1, p > 1$  for regions with piecewise smooth boundaries without cusps; [41] considers  $m \geq 1, p > 1$  for quasidisks under additional functional assumptions; [34] studies  $m = 1, p > 1$  for regions with piecewise smooth boundaries having both zero and nonzero interior angles; [43] addresses  $m \geq 1, p > 1$  for regions bounded by piecewise Dini-smooth curves with corners; [17] focuses on quasidisks with  $m \geq 1, p \geq 1$ ; and [28] studies regions with quasismooth boundaries and similar parameter ranges. These results often rely on a recurrence formula approach, where the estimate for each derivative order is derived from those of lower orders, a method that involves extensive calculations. However, [20] and [29]—which both consider  $m \geq 1, p \geq 1$  for regions bounded by asymptotically conformal curves—as well as our current work, employ a direct method to establish (10)-type estimates, without relying on recursive arguments.

Secondly, to finalize the analysis of the growth of  $|P_n^{(m)}(z)|$  in the entire complex plane, it is necessary to establish an inequality analogous to the Bernstein-Markov-Nikol'skii type within the regions under consideration, as demonstrated below:

$$\|P_n^{(m)}\|_\infty \leq \mu_n \|P_n\|_p, \quad m = 0, 1, 2, \dots, \quad (11)$$

where  $\mu_n := \mu_n(G, h, p, m) > 0, \mu_n \rightarrow \infty, n \rightarrow \infty$ , is a constant that depends on the properties of  $G$  and the weight function  $h$ , in general.

The investigation of inequalities of the type (11) began with the foundational works in [26], [27] and [48]. Subsequently, similar research has been explored in numerous other studies. Numerous studies have addressed these inequalities for  $m \geq 0$  within a range of function spaces, including those by [31, pp. 257-263], [38]-[39, pp.122-133], [44], [22], [30] and [35]. Further references can be found in the cited works. A more detailed study on this topic has been made by [3]-[10], [15] (for  $m = 0, p > 0$  and  $\kappa$ -quasidisks), [18] (for  $m \geq 1, p > 0$  and  $\kappa$ -quasidisks), [24] (for  $m = 0, p > 0$  and regions with piecewise smooth boundary with interior cusps), [40] (for  $m = 0, p > 0$  and regions with piecewise smooth boundaries containing interior angles), [42] (for  $m \geq 1, p > 1$  and regions with piecewise Dini-smooth boundaries having interior zero angles), [43] (for  $m \geq 1, p > 1$  and regions bounded by piecewise Dini-smooth curve with interior nonzero angles) and others.

Therefore, from the estimates (10) and (11), we find the growth of the  $m$ -th derivatives  $|P_n^{(m)}(z)|$ ,  $m = 0, 1, 2, \dots$ , on the whole complex plane as follows:

$$|P_n^{(m)}(z)| \leq c_7 \|P_n\|_p \begin{cases} \mu_n, & z \in \overline{G}_R, \\ \eta_n, & z \in \Omega_R, \end{cases} \quad (12)$$

where  $R = 1 + cn^{-1}$ ;  $c, 0 < c < 1, c_7 = c_7(G, p) > 0$  are constants that do not depend on  $n$  or  $P_n$ , but depend on the properties of  $G$  and  $h$ .

## 2. Definitions and Notations

Throughout this paper,  $c, c_0, c_1, c_2, \dots$  are positive constants and  $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$  are sufficiently small positive constants (generally, they are different in different relations). These constants generally depend on  $G$  and inessential parameters for the argument. Otherwise, the dependence will be explicitly stated. For any  $k \geq 0$  and  $m > k$ , the notation  $i = \overline{k, m}$  means  $i = k, k+1, \dots, m$ .

Let  $z = z(s), s \in [0, \text{mes}L]$  denote the natural representation of  $L$ .

**Definition 2.** We say that a Jordan curve or arc  $L$  is smooth, if  $L$  has a continuous tangent  $\theta(z) := \theta(z(s))$  at every point  $z(s)$ . The class of such smooth curves or arcs is denoted by  $C_\theta$ . Then we write  $G \in C_\theta$  to mean that  $\partial G \in C_\theta$ .

According to the "three-point" criterion [23, p.100], every piecewise smooth curve (without any cusps) or arc is quasiconformal. Moreover, according to [45], we have the following:

**Corollary 1.** *If  $L \in C_\theta$ , then  $L$  is  $(1 + \varepsilon)$ -quasiconformal for arbitrary small  $\varepsilon > 0$ .*

We say that a bounded Jordan curve (or arc)  $L$  is locally smooth at the point  $z \in L$ , if there exists a closed subarc  $\ell \subset L$  containing  $z$  such that every open subarc of  $\ell$  containing  $z$  is smooth.

In what follows, we introduce a certain category of regions enclosed by a piecewise smooth boundary curve, where the connecting arcs form interior angles of zero or non-zero exterior angles.

For a Jordan region  $G$  with boundary  $L = \partial G$ , let  $\{\zeta_j\}_{j=0}^l \subset L$  denote a finite, ordered set of distinct points placed along  $L$ , which, without loss of generality, are assumed to follow the positive orientation. We define each arc  $L_j$  as the segment of the boundary connecting the consecutive points  $\zeta_j$  and  $\zeta_{j+1}$ , where the indexing is taken modulo  $l + 1$ , i.e.,  $\zeta_{l+1} \equiv \zeta_0$ .

**Definition 3** ([10]). *We say that a Jordan region  $G \in PC_\theta(1; \lambda_1, \dots, \lambda_l)$ ,  $0 < \lambda_j \leq 2$ ,  $j = \overline{1, l}$ , if  $L = \partial G$  is the union of finitely many smooth arcs  $\{L_j\}_{j=0}^l$  connected at points  $\{\zeta_j\}_{j=0}^l \in L$  such that  $L$  has exterior (with respect to  $\overline{G}$ ) angles  $\lambda_j\pi$ ,  $0 < \lambda_j \leq 2$ , at the corner points  $\{\zeta_j\}_{j=1}^l \in L$  where two arcs  $L_{j-1}$  and  $L_j$ ,  $j = \overline{1, l}$ , intersect, and  $L$  is locally smooth at  $\zeta_0$ .*

Without loss of generality, we assume that these points on the curve  $L = \partial G$  are located in the positive direction such that  $G$  has exterior  $\lambda_j\pi$ ,  $0 < \lambda_j < 2$ ,  $j = \overline{1, l_1}$ , angles at the points  $\{\zeta_j\}_{j=1}^{l_1}$ ,  $l_1 \leq l$ , and interior zero angles (i.e.  $\lambda_j = 2$ —interior cusps) at the points  $\{\zeta_j\}_{j=l_1+1}^l$ .

It is clear from Definition 3, each region  $G \in PC_\theta(1; \lambda_1, \dots, \lambda_l)$ ,  $0 < \lambda_j \leq 2$ ,  $j = \overline{1, l}$ , may have exterior nonzero  $\lambda_j\pi$ ,  $0 < \lambda_j < 2$ , angles at the points  $\{z_j\}_{j=1}^{l_1} \in L$ , and interior zero angles ( $\lambda_j = 2$ ) at the points  $\{z_j\}_{j=l_1+1}^l \in L$ . If  $l_1 = l = 0$ , then the region  $G$  doesn't have such angles, in this case we will write:  $G \in PC_\theta(1) \equiv C_\theta$ ; if  $l_1 = l \geq 1$ , then  $G$  has only  $\lambda_i\pi$ ,  $0 < \lambda_i < 2$ ,  $i = \overline{1, l_1}$ , exterior nonzero angles, in this case we will write  $G \in PC_\theta(1; \lambda_1, \dots, \lambda_l)$ ,  $0 < \lambda_j < 2$ ,  $j = \overline{1, l}$ ; if  $l_1 = 0$  and  $l \geq 1$ , then  $G$  has only interior zero angles, in this case we will write  $G \in PC_\theta(1; 2, \dots, 2)$ .

Moreover, throughout the paper, we will assume that the  $\{z_j\}_{j=1}^l \in L$  points defined in (1) coincide with the  $\{\zeta_j\}_{j=1}^l \in L$  points specified in Definition 3.

To simplify the exposition and avoid cumbersome calculations, we will take  $l_1 = 1$ ,  $l = 2$  without loss of generality. After this assumption, we have the  $G \in PC_\theta(1; \lambda_1, 2)$ ,  $0 < \lambda_1 < 2$ , region which has exterior nonzero angle  $\lambda_1\pi$ ,  $0 < \lambda_1 < 2$ , at the point  $z_1 \in L$  and interior zero angle at the point  $z_2 \in L$ . Note that, the notation " $G \in PC_\theta(1; \lambda_1, \lambda_2)$ ,  $0 < \lambda_1, \lambda_2 < 2$ " means that the region  $G$  has two exterior nonzero  $\lambda_j\pi$ ,  $0 < \lambda_j < 2$ , angles at the point  $z_j \in L$ ,  $j = 1, 2$ .

For  $0 < \delta_j < \delta_0 := \frac{1}{4} \min\{|z_1 - z_2| : j = 1, 2\}$ , let us  $\delta := \min_{1 \leq j \leq 2} \delta_j$ ,  $\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$ ;  $\Omega(\delta) := \bigcup_{j=1}^2 \Omega(z_j, \delta)$ ,  $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$ .

Let  $w_j := \Phi(z_j)$ ,  $\varphi_j := \arg w_j$ . Without loss of generality, we will assume that  $\varphi_2 < 2\pi$ . For  $\eta_j = \min_{t \in \partial\Phi(\Omega(z_j, \delta_j))} |t - w_j| > 0$  and  $\eta := \min\{\eta_j, j = 1, 2\}$  let us set:

$\Delta_j(\eta_j) := \{t : |t - w_j| \leq \eta_j\} \subset \Phi(\Omega(z_j, \delta_j))$ ,  $\Delta(\eta) := \bigcup_{j=1}^2 \Delta_j(\eta)$ ,  $\widehat{\Delta}_j = \Delta \setminus \Delta(\eta_j)$ ;  $\widehat{\Delta}(\eta) := \Delta \setminus \Delta(\eta)$ ;  $\Delta'_1 := \Delta'_1(1)$ ,  $\Delta'_1(\rho) := \left\{t = R \cdot e^{i\theta} : R \geq \rho > 1, \frac{\varphi_0 + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2}\right\}$ ,  $\Delta'_2 := \Delta'_2(1)$ ,  $\Delta'_2(\rho) := \left\{t = R \cdot e^{i\theta} : R \geq \rho > 1, \frac{\varphi_1 + \varphi_2}{2} \leq \theta < \frac{\varphi_2 + \varphi_0}{2}\right\}$ , where  $\varphi_0 = 2\pi - \varphi_2$ ;  $\Omega_R^j := \Psi(\Delta'_j(R)$ ,  $L_R^j := L_R \cap \overline{\Omega}_R^j$ ,  $R \geq 1$ ,  $\Omega = \bigcup_{j=1}^2 \Omega^j$ .

### 3. Main Results

We begin by setting up the notational framework required for both the formulation of the main results and the subsequent analysis.

$$\begin{aligned}\tilde{\lambda} &= \begin{cases} \max\{1, \lambda\} + \varepsilon, & 0 < \lambda < 2, \\ 2, & \lambda = 2; \end{cases} \\ \tilde{\lambda}_j &= \max\{1, \lambda_j\} + \varepsilon, \quad 0 < \lambda_j < 2, \quad \lambda^* := \max\{\tilde{\lambda}_1, \tilde{\lambda}_2\}, \quad \forall \varepsilon > 0; \\ \gamma &:= \begin{cases} \gamma_1, & \text{if } 0 < \lambda < 2, \\ \gamma_2, & \text{if } \lambda = 2; \end{cases} \quad \tilde{\gamma}_i := \max\{0; \gamma_i\}, \quad i = 1, 2; \\ \hat{\gamma} &:= \begin{cases} \tilde{\gamma}_1, & \text{if } 0 < \lambda < 2, \\ \tilde{\gamma}_2, & \text{if } \lambda = 2; \end{cases} \\ \gamma^* &:= \max\{0, \gamma_1, \gamma_2\}; \quad \tilde{\gamma} := \gamma_j, \text{ for } z \in \Omega_R^j, \quad j = 1, 2; \quad \tilde{\gamma}^* := \max\{0; \tilde{\gamma}\}.\end{aligned}\tag{13}$$

In all theorems and corollaries throughout this section, the weight function  $h(z)$  is defined by (1) for the case  $l = 2$ . Also we assume that  $n \in \mathbb{N}$  and  $n \geq m$ , where  $m \in \mathbb{N}$  is specified individually in each result. Now let's start formulating the new results.

**Theorem 1.** Let  $p \geq 1$  and  $G \in PC_\theta(1; \lambda_1, 2)$  for some  $0 < \lambda_1 < 2$ . Suppose the weight function  $h$  is defined by (1) with  $\gamma > -2$ . Then, for every polynomial  $P_n \in \wp_n$ , all  $m = 0, 1, 2, \dots, n$ , and  $z \in \Omega_{R_1}$ , the following estimate holds:

$$\left| P_n^{(m)}(z) \right| \leq c_1 \frac{|\Phi^{n-m+1}(z)|}{d(z, L)} J_{n,p}^1(m) \|P_n\|_p, \tag{14}$$

where  $c_1 = c_1(G, p, \gamma, \lambda, m, \varepsilon) > 0$ , is a constant independent of  $n$  and  $z$ , and

$$J_{n,p}^1(m) := \begin{cases} n^{\left(\frac{\tilde{\gamma}+2}{p}+m-1\right)\tilde{\lambda}}, & 1 \leq p < \frac{\tilde{\lambda}(\tilde{\gamma}^*+2)+1}{\tilde{\lambda}+1}, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\tilde{\lambda}(\tilde{\gamma}^*+2)+1}{\tilde{\lambda}+1}, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}}, & p > \frac{\tilde{\lambda}(\tilde{\gamma}^*+2)+1}{\tilde{\lambda}+1}. \end{cases}$$

As a consequence, we can consider cases in which the region has either zero or non-zero angles at both points.

**Corollary 2.** Let  $p \geq 1$  and  $G \in PC_\theta(1; \lambda_1, \lambda_2)$  for some parameters  $0 < \lambda_1, \lambda_2 < 2$ . Then, for every polynomial  $P_n \in \wp_n$ , all  $m = 0, 1, 2, \dots, n$  and  $z \in \Omega_R^j$ , we have:

$$\left| P_n^{(m)}(z) \right| \leq c_2 \frac{|\Phi^{n-m+1}(z)|}{d(z, L)} J_{n,p}^2(m) \|P_n\|_p, \tag{15}$$

where  $c_2 = c_2(G, p, \gamma, \lambda, m, \varepsilon) > 0$ , is a constant independent of  $n$  and  $z$ ;

$$J_{n,p}^2(m) := \begin{cases} n^{\left(\frac{\tilde{\gamma}+2}{p}+m-1\right)\tilde{\lambda}_j}, & 1 \leq p < \frac{\tilde{\lambda}_j(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, \\ n^{m\tilde{\lambda}_j+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\tilde{\lambda}_j(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, \\ n^{m\tilde{\lambda}_j+1-\frac{1}{p}}, & p > \frac{\tilde{\lambda}_j(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}. \end{cases}$$

**Corollary 3.** Let  $p \geq 1$  and  $G \in PC_\theta(1; 2, 2)$  Then, for every polynomial  $P_n \in \wp_n$ , all  $m = 0, 1, 2, \dots, n$  and  $z \in \Omega_R^j$ , have:

$$\left| P_n^{(m)}(z) \right| \leq c_3 \frac{|\Phi^{n-m+1}(z)|}{d(z, L)} J_{n,p}^3(m) \|P_n\|_p, \tag{16}$$

where  $c_3 = c_3(G, p, \gamma, \lambda, m, \varepsilon) > 0$ , is a constant independent of  $n$  and  $z$ ;

$$J_{n,p}^3(m) := \begin{cases} n^{2\left(\frac{\tilde{\gamma}^*+2}{p}+m-1\right)}, & 1 \leq p < \frac{2(\tilde{\gamma}^*+2)+1}{\tilde{\lambda}+1}, \\ n^{2m+1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p = \frac{2(\tilde{\gamma}^*+2)+1}{\tilde{\lambda}+1}, \\ n^{2m+1-\frac{1}{p}}, & p > \frac{2(\tilde{\gamma}^*+2)+1}{\tilde{\lambda}+1}. \end{cases}$$

Now, we can state uniformly estimate of  $|P_n^{(m)}(z)|$ ,  $m \geq 0$ , for  $z \in \overline{G}$ .

**Theorem 2.** Let  $p > 0$  and  $G \in PC_\theta(1; \lambda, 2)$  for some parameter  $0 < \lambda < 2$ . Then, for every polynomial  $P_n \in \wp_n$ , all  $m = 0, 1, 2, \dots, n$  and arbitrarily small  $\varepsilon > 0$ , the following estimate holds:

$$\|P_n^{(m)}\|_\infty \leq c_4 \mu_{n,m} \|P_n\|_p, \quad (17)$$

where  $c_4 = c_4(G, \gamma, \lambda, p, m, \varepsilon) > 0$  is a constant, independent of  $z$  and  $n$ , the numbers  $\tilde{\gamma}, \tilde{\lambda}$  are defined as in (13) and

$$\mu_{n,m} := \begin{cases} n^{\left(\frac{2+\tilde{\gamma}}{p}+m\right)\tilde{\lambda}}, & \text{if } (2+\tilde{\gamma}) \cdot \tilde{\lambda} > 1, \\ n^{m\tilde{\lambda}+\frac{1}{p}}(\ln n)^{\frac{1}{p}}, & \text{if } (2+\tilde{\gamma}) \cdot \tilde{\lambda} = 1, \\ n^{m\tilde{\lambda}+\frac{1}{p}}, & \text{if } (2+\tilde{\gamma}) \cdot \tilde{\lambda} < 1. \end{cases}$$

We can consider separately the cases where the  $L$  curve has the same type of angles at both points: exterior nonzero or interior zero angles. In this case, from Theorem 2, we obtain the follows:

**Corollary 4.** Under the conditions of Theorem 2, the relation (17) is satisfied for  $G \in PC_\theta(1; \lambda_1, \lambda_2)$  for  $\mu_{n,m} = \tilde{\mu}_{n,m,1}$ , where

$$\tilde{\mu}_{n,m,1} = \begin{cases} n^{\left(\frac{2+\gamma^*}{p}+m\right)\lambda^*}, & \text{if } (2+\gamma^*) \cdot \lambda^* > 1, \\ n^{m\lambda^*+\frac{1}{p}}(\ln n)^{\frac{1}{p}}, & \text{if } (2+\gamma^*) \cdot \lambda^* = 1, \\ n^{m\lambda^*+\frac{1}{p}}, & \text{if } (2+\gamma^*) \cdot \lambda^* < 1, \end{cases} \quad (18)$$

and  $\gamma^*, \lambda^*$  are defined as in (13).

**Corollary 5.** Under the conditions of Theorem 2, the relation (17) is satisfied for  $G \in PC_\theta(1; 2, 2)$  for  $\mu_{n,m} = \tilde{\mu}_{n,m,2}$ , where

$$\tilde{\mu}_{n,m,2} = n^{2\left(\frac{2+\gamma^*}{p}+m\right)}, \quad (19)$$

and  $\gamma^*$  is defined as in (13).

According to (4) (applied to the polynomial  $Q_{n-m}(z) := P_n^{(m)}(z)$ ), the estimate (17) and their corollaries (18) and (19) are also valid for  $z \in \overline{G}_R$  with a different constant. Therefore, when the estimate (7) (for the  $z \in \overline{G}_R$ ) is combined with the estimates (2)-(3), the estimate of the growth of  $|P_n^{(m)}(z)|$  in the whole complex plane is obtained. Now, let us present the corresponding statements. Combined Theorem 1 with Theorem 2, we find the following:

**Theorem 3.** Let  $p \geq 1$  and  $G \in PC_\theta(1; \lambda, 2)$  for some parameter  $0 < \lambda < 2$ . Then, for every polynomial  $P_n \in \wp_n$ , all  $m = 0, 1, 2, \dots, n$  and arbitrarily small  $\varepsilon > 0$ , the following estimate holds:

$$\|P_n^{(m)}\|_\infty \leq c_5 \|P_n\|_p \begin{cases} \frac{|\Phi^{n-m+1}(z)|}{d(z, L)} J_{n,p}^1(m), & z \in \Omega_{1+\frac{1}{n}}, \\ \mu_{n,m}, & z \in \overline{G}_{1+\frac{1}{n}}, \end{cases}$$

where  $c_5 = c_5(G, \gamma, \lambda, p, m, \varepsilon) > 0$  is a constant, independent of  $z$  and  $n$ ;  $J_{n,p}^1(m)$  and  $\mu_{n,m}$  are defined as in (14) and (17), respectively.

By combining Corollaries 2 and 3 with Corollaries 4 and 5, respectively, we obtain the following:

**Corollary 6.** Let  $p \geq 1$  and  $G \in PC_\theta(1; \lambda_1, \lambda_2)$  for some parameters  $0 < \lambda_1, \lambda_2 < 2$ . Then, for every polynomial  $P_n \in \wp_n$ , all  $m = 1, 2, \dots$  and arbitrarily small  $\varepsilon > 0$ , the following estimate holds:

$$\left\| P_n^{(m)} \right\|_\infty \leq c_6 \|P_n\|_p \begin{cases} \frac{|\Phi^{n-m+1}(z)|}{d(z, L)} J_{n,p}^2(m), & z \in \Omega_{1+\frac{1}{n}}, \\ \tilde{\mu}_{n,m,1}, & z \in \overline{G}_{1+\frac{1}{n}}, \end{cases}$$

where  $c_6 = c_6(G, \gamma, \lambda, p, m, \varepsilon) > 0$  is a constant, independent of  $z$  and  $n$ ;  $J_{n,p}^2(m)$  and  $\tilde{\mu}_{n,m,1}$  are defined as in (15) and (18), respectively.

**Corollary 7.** Let  $p \geq 1$  and  $G \in PC_\theta(1; 2, 2)$ . Then, for every polynomial  $P_n \in \wp_n$ , all  $m = 0, 1, 2, \dots, n$  and  $z \in \Omega_R^j$ , have:

$$\left\| P_n^{(m)} \right\|_\infty \leq c_7 \|P_n\|_p \begin{cases} \frac{|\Phi^{n-m+1}(z)|}{d(z, L)} J_{n,p}^3(m), & z \in \Omega_{1+\frac{1}{n}}, \\ \tilde{\mu}_{n,m,2}, & z \in \overline{G}_{1+\frac{1}{n}}, \end{cases}$$

where  $c_7 = c_7(G, \gamma, \lambda, p, m, \varepsilon) > 0$  is a constant, independent of  $z$  and  $n$ ;  $J_{n,p}^3(m)$  and  $\tilde{\mu}_{n,m,2}$  are defined as in (16) and (19), respectively.

#### 4. Some Auxiliary Results

**Lemma 1 ([1]).** Let  $L = \partial G$  be a  $K$ -quasiconformal curve,  $z_1 \in L$ ,  $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \leq d(z_1, L_{R_0})\}$ ;  $w_j = \Phi(z_j)$ ,  $j = 1, 2, 3$ . Then

- The statements  $|z_1 - z_2| \preceq |z_1 - z_3|$  and  $|w_1 - w_2| \preceq |w_1 - w_3|$  are equivalent. So  $|z_1 - z_2| \asymp |z_1 - z_3|$  and  $|w_1 - w_2| \asymp |w_1 - w_3|$  are also equivalent.
- If  $|z_1 - z_2| \preceq |z_1 - z_3|$ , then

$$\left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{K-2} \preceq \left| \frac{z_1 - z_3}{z_1 - z_2} \right| \preceq \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{K^2},$$

where  $R_0 > 1$  is a constant, depending on  $G$ .

**Corollary 8.** Under the assumptions of Lemma 1, if  $z_3 \in L_{R_0}$ , then

$$|w_1 - w_2|^{K^2} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{K-2}.$$

**Corollary 9.** If  $L \in C_\theta$ , then

$$|w_1 - w_2|^{1+\varepsilon} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{1-\varepsilon},$$

for all  $\varepsilon > 0$ .

We will also use the following estimate for  $|\Psi'|$  (see, for example, [21, Th.2.8]):

$$|\Psi'(\tau)| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \quad (20)$$

The following lemma is a consequence of the results given in [50] and [33].

**Lemma 2.** Let  $G \in PC_\theta(1; \lambda, \dots, \lambda_l)$ ,  $0 < \lambda_j < 2$ ,  $j = 1, 2, \dots, l$ . Then

- i) for any  $w \in \Delta_j(\eta)$ ,  $|w - w_j|^{\lambda_j + \varepsilon} \preceq |\Psi(w) - \Psi(w_j)| \preceq |w - w_j|^{\lambda_j - \varepsilon}$ ,  $|w - w_j|^{\lambda_j - 1 + \varepsilon} \preceq |\Psi'(w)| \preceq |w - w_j|^{\lambda_j - 1 - \varepsilon}$ ;  
 ii) for any  $w \in \widehat{\Delta}(\eta)$ ,  $(|w| - 1)^{1 + \varepsilon} \preceq d(\Psi(w), L) \preceq (|w| - 1)^{1 - \varepsilon}$ ,  $(|w| - 1)^\varepsilon \preceq |\Psi'(w)| \preceq (|w| - 1)^{-\varepsilon}$ .

Let  $\{z_j\}_{j=1}^l$  be a fixed system of distinct points on curve  $L$  ordered in the positive direction and the weight function  $h(z)$  be defined as in (1).

**Lemma 3 ([11]).** Let  $L = \partial G$  is a  $K$ -quasiconformal curve;  $R = 1 + \frac{\varepsilon}{n}$ . Then, for any fixed  $\varepsilon \in (0, 1)$  there exist a level curve  $L_{1+\varepsilon(R-1)}$  such that the following holds for any polynomial  $P_n(z) \in \wp_n$ ,  $n \in \mathbb{N}$ :

$$\|P_n\|_{\mathcal{L}_p\left(\frac{h}{|\Phi'|}, L_{1+\varepsilon(R-1)}\right)} \preceq n^{\frac{1}{p}} \|P_n\|_p, \quad p > 0. \quad (21)$$

**Lemma 4 ([8]).** Let  $L = \partial G$  be a  $K$ -quasiconformal curve;  $h(z)$  be defined as in (1). Then, for arbitrary  $P_n(z) \in \wp_n$ , any  $R > 1$  and  $n = 1, 2, \dots$ , we have:

$$\|P_n\|_{A_p(h, G_R)} \preceq \widetilde{R}^{n+\frac{1}{p}} \|P_n\|_{A_p(h, G)}, \quad p > 0, \quad (22)$$

where  $\widetilde{R} = 1 + c(R - 1)$  and  $c$  is independent of  $n$  and  $R$ .

## 5. Proofs

**Proof of Theorem 1.** The proof of Theorems 1 will be carried separating the cases  $p \geq 2$  and  $1 < p < 2$ . Let  $G \in C_\theta(1; \lambda_1, 2)$  with  $0 < \lambda_1 < 2$ ,  $R = 1 + \frac{\varepsilon_0}{n}$ ,  $R_1 := 1 + \frac{R-1}{3}$  and  $R_2 := 1 + \frac{2(R-1)}{3}$ . For  $z \in \Omega$  and  $0 \leq m < n$ , let us define

$$H_{n,m}(z) := \frac{P_n^{(m)}(z)}{\Phi^{n-m+1}(z)}.$$

Noting that  $H_{n,m}(z)$  is analytic in the region  $\Omega$  and continuous on its closure  $\overline{\Omega}$ , also taking account that  $H_{n,m}(\infty) = 0$ , we can apply the Cauchy integral representation for unbounded regions to obtain:

$$H_{n,m}(z) = -\frac{1}{2\pi i} \int_{L_{R_1}} H_{n,m}(\zeta) \frac{d\zeta}{\zeta - z}, \quad z \in \Omega_{R_1}.$$

Taking the modulus of both parts and considering that  $|\Phi^{n-m+1}(\zeta)| = R_1^{n-m+1} > 1$ , for  $\zeta \in L_{R_1}$ , we obtain:

$$\left| \frac{P_n^{(m)}(z)}{\Phi^{n-m+1}(z)} \right| \leq \frac{1}{2\pi} \int_{L_{R_1}} \left| \frac{P_n^{(m)}(\zeta)}{\Phi^{n-m+1}(\zeta)} \right| \frac{|d\zeta|}{|\zeta - z|} \leq \frac{1}{2\pi d(z, L_{R_1})} \int_{L_{R_1}} |P_n^{(m)}(\zeta)| |d\zeta|.$$

Therefore,

$$\left| P_n^{(m)}(z) \right| \leq \frac{|\Phi^{n-m+1}(z)|}{2\pi d(z, L_{R_1})} \int_{L_{R_1}} |P_n^{(m)}(\zeta)| |d\zeta|. \quad (23)$$

Let us express the Cauchy  $m$ -th derivative formula for  $P_n(\zeta)$  within the integral

$$P_n^{(m)}(\zeta) = \frac{m!}{2\pi i} \int_{L_{R_2}} P_n(t) \frac{dt}{(t - \zeta)^{m+1}}, \quad \zeta \in G_{R_2},$$

and taking  $\zeta \in L_{R_1}$  we find:

$$\begin{aligned} |P_n^{(m)}(z)| &\leq \frac{|\Phi^{n-m+1}(z)|}{d(z, L_{R_1})} \int_{L_{R_1}} \left( \int_{L_{R_2}} |P_n(t)| \frac{|dt|}{|t-\zeta|^{m+1}} \right) |d\zeta| \\ &\leq \frac{|\Phi^{n-m+1}(z)|}{d(z, L_{R_1})} \sup_{t \in L_{R_2}} \left( \int_{L_{R_1}} \frac{|d\zeta|}{|t-\zeta|^{m+1}} \right) \int_{L_{R_2}} |P_n(t)| |dt|. \end{aligned} \quad (24)$$

Denote by

$$\int_{L_{R_1}} \frac{|d\zeta|}{|t-\zeta|^{m+1}} =: J_1(L_{R_1}); \quad \int_{L_{R_2}} |P_n(t)| |dt| =: J_2(L_{R_2}), \quad (25)$$

and estimate these integrals separately.

A) Estimation of  $J_1(L_{R_1})$ . Replacing the variable  $\tau = \Phi(\zeta), \zeta \in L_{R_1}, w = \Phi(t), t \in L_{R_2}$  and taking into account (20), we have:

$$J_1(L_{R_1}) = \int_{|\tau|=R_1} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w)|^{m+1}}. \quad (26)$$

To estimate the integral, for  $\rho \geq 1$  we introduce the following notations:

$$\begin{aligned} E_{\rho,j}^1 &:= \{\tau : \tau \in \Phi(L_\rho^j), |\tau - w_j| < c_j(\rho - 1)\}, \quad E_{\rho,j}^{11} := \{\tau \in E_{\rho,j}^1 : |\tau - w_j| \geq |\tau - w|\}, \\ E_{\rho,j}^{12} &:= E_{\rho,j}^1 \setminus E_{\rho,j}^{11}, \\ E_{\rho,j}^2 &:= \{\tau : \tau \in \Phi(L_\rho^j), c_j(\rho - 1) \leq |\tau - w_j| < \eta\}, \quad E_{\rho,j}^{21} := \{\tau \in E_{\rho,j}^2 : |\tau - w_j| \geq |\tau - w|\}, \\ E_{\rho,j}^{22} &:= E_{\rho,j}^2 \setminus E_{\rho,j}^{21}, \\ E_{\rho,j}^3 &:= \{\tau : \tau \in \Phi(L_\rho)\} \setminus (E_{\rho,j}^1 \cup E_{\rho,j}^2), \quad j = 1, 2 \end{aligned}$$

and perform for  $\rho = R_1$  the estimates  $J_1(L_{R_1})$  for the specified sets separately:

$$J_1(L_{R_1}) = \sum_{i,j=1}^2 J_1(E_{R_1,j}^i) + J_1(E_{R_1,j}^3). \quad (27)$$

Applying Lemma 1, Lemma 2 and (20), we obtain the following estimates, respectively:

1.1.

$$J_1(E_{R_1,1}^1) = \int_{E_{R_1,1}^1} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w)|^{m+1}} \asymp \int_{E_{R_1,1}^1} \frac{|\tau - w_1|^{\lambda_1-1-\varepsilon}}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}} |d\tau|.$$

1.1.1. For  $1 \leq \lambda_1 < 2$ :

$$\begin{aligned} J_1(E_{R_1,1}^1) &\asymp \int_{E_{R_1,1}^1} \frac{|\tau - w_1|^{\lambda_1-1-\varepsilon}}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}} |d\tau| \leq \left(\frac{1}{n}\right)^{\lambda_1-1-\varepsilon} \int_{E_{R_1,1}^1} \frac{|d\tau|}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}} \\ &\leq \left(\frac{1}{n}\right)^{\lambda_1-1-\varepsilon} n^{(m+1)(\lambda_1+\varepsilon)-1} = n^{m\lambda_1+\varepsilon}. \end{aligned}$$

1.1.2 For  $0 < \lambda_1 < 1$  :

$$\begin{aligned} J_1(E_{R_1,1}^1) &\asymp \int_{E_{R_1,1}^1} \frac{|\tau - w_1|^{\lambda_1-1-\varepsilon}}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}} |d\tau| = \int_{E_{R_1,1}^1} \frac{|d\tau|}{|\tau - w|^{1-\lambda_1+(m+1)(\lambda_1+\varepsilon)}} \\ &\preceq n^{1-\lambda_1+(m+1)(\lambda_1+\varepsilon)} \text{mes}(E_{R_1,1}^1) \preceq n^{1-\lambda_1+(m+1)(\lambda_1+\varepsilon)-1} = n^{m\lambda_1+\varepsilon}. \end{aligned}$$

1.2.

$$\begin{aligned} J_1(E_{R_1,1}^2) &= \int_{E_{R_1,1}^2} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w)|^{m+1}} \asymp \int_{E_{R_1,1}^2} \frac{|\tau - w_1|^{\lambda_1-1-\varepsilon}}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}} |d\tau| \\ &= \int_{E_{R_1,1}^{21}} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w)|^{m+1}} |d\tau| + \int_{E_{R_1,1}^{22}} \frac{|\tau - w_1|^{\lambda_1-1-\varepsilon}}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}} |d\tau|. \end{aligned}$$

1.2.1. For  $1 \leq \lambda_1 < 2$  :

$$\begin{aligned} J_1(E_{R_1,1}^{21}) &= \int_{E_{R_1,1}^{21}} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w)|^{m+1}} |d\tau| = \int_{\Psi(E_{R_1,1}^{21})} \frac{|d\zeta|}{|\zeta - t|^{m+1}} \\ &\preceq \int_{d(\zeta, L_{R_2})}^c \frac{|d\zeta|}{|\zeta - t|^{m+1}} \preceq \frac{1}{d^m(\zeta, L_{R_2})} \preceq n^{m\lambda_1+\varepsilon}; \end{aligned}$$

$$J_1(E_{R_1,1}^{22}) = \int_{E_{R_1,1}^{22}} \frac{|\tau - w_1|^{\lambda_1-1-\varepsilon}}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}} |d\tau| \preceq \int_{E_{R_1,1}^{22}} \frac{|d\tau|}{|\tau - w|^{(m+1)\lambda_1-\lambda_1+1+\varepsilon}} \preceq n^{m\lambda_1+\varepsilon}.$$

1.2.2. For  $0 < \lambda_1 < 1$  :

$$\begin{aligned} J_1(E_{R_1,1}^2) &= \int_{E_{R_1,1}^2} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w)|^{m+1}} \asymp \int_{E_{R_1,1}^2} \frac{|\tau - w_1|^{\lambda_1-1-\varepsilon}}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}} |d\tau| \\ &= \int_{E_{R_1,1}^{21}} \frac{|d\tau|}{|\tau - w|^{1-\lambda_1+(m+1)(\lambda_1+\varepsilon)}} + \int_{E_{R_1,1}^{22}} \frac{|d\tau|}{|\tau - w_1|^{1-\lambda_1+(m+1)(\lambda_1+\varepsilon)}} \\ &\preceq n^{m\lambda_1+\varepsilon} + n^{m\lambda_1+\varepsilon} \preceq n^{m\lambda_1+\varepsilon}. \end{aligned}$$

2.1. Applying (20) and Corollary 9, we get:

$$\begin{aligned} J_1(E_{R_1,2}^k) &\asymp \int_{E_{R_1,2}^k} \frac{d(\Psi(\tau), L)}{|\Psi(\tau) - \Psi(w)|^{m+1} (|\tau| - 1)} |d\tau| \preceq n \int_{E_{R_1,2}^k} \frac{|d\tau|}{|\tau - w|^{2m}} \preceq n^{2m}, \quad k = 1, 2; \\ J_1(E_{R_1,j}^3) &\asymp \int_{E_{R_1,j}^3} \frac{d(\Psi(\tau), L)}{|\Psi(\tau) - \Psi(w)|^{m+1} (|\tau| - 1)} |d\tau| \preceq n \int_{E_{R_1,j}^3} \frac{|d\tau|}{|\tau - w|^{m(1+\varepsilon)}} \preceq n^{2(1+\varepsilon)}, \quad j = 1, 2, \end{aligned}$$

and then, combining estimates obtained in cases 1.1-2.1, for all  $0 < \lambda_1 < 2$ , we obtain:

$$J_1(L_{R_1}) \preceq n^{m\tilde{\lambda}}. \quad (28)$$

B) Estimation of  $J_2(L_{R_2})$ .

Let us first assume that  $p > 1$ . Replacing the variable  $t = \Psi(w)$ , multiplying the integrands numerator and denominator by  $h^{\frac{1}{p}}(t)$  and using Hölder inequality, we have:

$$\begin{aligned} J_2(L_{R_2}) &= \int_{L_{R_2}} |P_n(t)| |dt| = \int_{|w|=R_2} \frac{h^{\frac{1}{p}}(\Psi(w)) |P_n(\Psi(w))| |\Psi'(w)|^{1-\frac{2}{p}} |\Psi'(w)|^{\frac{2}{p}}}{h^{\frac{1}{p}}(\Psi(w))} |dw| \\ &\leq \int_{|w|=R_2} \left( h(\Psi(w)) |P_n(\Psi(w))|^p |\Psi'(w)|^2 |dw| \right)^{\frac{1}{p}} \times \left( \int_{|w|=R_2} \frac{|\Psi'(w)|^{q(1-\frac{2}{p})}}{h^{\frac{q}{p}}(\Psi(w))} \right)^{\frac{1}{q}} \\ &=: J_{21}(L_{R_2}) \times J_{22}(L_{R_2}), \quad \frac{1}{p} + \frac{1}{q} = 1. \end{aligned} \quad (29)$$

By Lemma 3, we have:

$$J_{21}(L_{R_2}) \leq n^{\frac{1}{p}} \|P_n\|_p. \quad (30)$$

Now, it remains to estimate the following integral:

$$[J_{22}(L_{R_2})]^q := \int_{L_{R_2}} \frac{|\Psi'(w)|^{2-q}}{\prod_{j=1}^2 |\Psi(w) - \Psi(w_j)|^{\gamma_j(q-1)}} |dw|. \quad (31)$$

To estimate this integral, we set:

$$\begin{aligned} E_{R_2,j}^1 &:= \{w : w \in \Phi(L_{R_2}^j), |w - w_j| < c_j(R_2 - 1)\}; \\ E_{R_2,j}^2 &:= \{w : w \in \Phi(L_{R_2}^j), c_j(R_2 - 1) \leq |w - w_j| < \eta\}; \\ E_{R_2,j}^3 &:= \{w : w \in \Phi(L_{R_2}^j), \eta \leq |w - w_j| < \text{diam} \bar{G}\}, \end{aligned}$$

where  $0 < c_j < \eta$  is chosen so that  $\{w : |w - w_j| < c_j(R_2 - 1)\} \cap \Delta \neq \emptyset$  and  $\Phi(L_{R_2}) = \Phi(\bigcup_{j=1}^2 L_{R_2}^j) = \bigcup_{j=1}^2 \Phi(L_{R_2}^j) = \bigcup_{j=1}^2 \bigcup_{i=1}^3 E_{R_2,i}^j$ . Taking these notations into account, from (31), we get:

$$\begin{aligned} [J_{22}(L_{R_2})]^q &\leq \sum_{j=1}^2 \int_{\Phi(L_{R_2}^j)} \frac{|\Psi'(w)|^{2-q}}{\prod_{j=1}^2 |\Psi(w) - \Psi(w_j)|^{\gamma_j(q-1)}} |dw| \\ &\asymp \sum_{j=1}^2 \int_{\Phi(L_{R_2}^j)} \frac{|\Psi'(w)|^{2-q}}{|\Psi(w) - \Psi(w_j)|^{\gamma_j(q-1)}} |dw| =: \sum_{j=1}^2 B_{n,2}(w_j), \end{aligned} \quad (32)$$

since the points  $z_1$  and  $z_2$  are isolated. Therefore, we need to estimate  $B_{n,2}(w_j)$  for  $j = 1, 2$ :

$$B_{n,2}(w_j) = \sum_{i=1}^3 \int_{E_{R_2,i}^j} \frac{|\Psi'(w)|^{2-q}}{|\Psi(w) - \Psi(w_j)|^{\gamma_j(q-1)}} |dw| = \sum_{i=1}^3 B_{n,2}^i(w_j). \quad (33)$$

Case 1. Let  $1 < q \leq 2$  ( $p \geq 2$ ).

1.1. For  $\gamma_1, \gamma_2 \geq 0$ , we have:

$$\begin{aligned} B_{n,2}^1(w_1) &= \int_{E_{R_2,1}^1} \frac{|\Psi'(w)|^{2-q}}{|\Psi(w) - \Psi(w_1)|^{\gamma_1(q-1)}} |dw| \preceq \int_{E_{R_2,1}^1} \frac{|w - w_1|^{(\lambda_1-1-\varepsilon)(2-q)}}{|w - w_1|^{\gamma_1(q-1)(\lambda_1+\varepsilon)}} |dw| \\ &\preceq n^{\gamma_1(q-1)(\lambda_1+\varepsilon) - (\lambda_1-1-\varepsilon)(2-q)} \int_{E_{R_2,1}^1} |dw| \preceq n^{[\gamma_1(q-1)-(2-q)]\lambda_1+(2-q)+\varepsilon} \text{mes}(E_{R_2,1}^1) \\ &\preceq n^{[\gamma_1(q-1)-(2-q)]\lambda_1+(1-q)+\varepsilon}, \end{aligned} \quad (34)$$

$$\begin{aligned} B_{n,2}^2(w_1) &= \int_{E_{R_2,1}^2} \frac{|\Psi'(w)|^{2-q}}{|\Psi(w) - \Psi(w_1)|^{\gamma_1(q-1)}} |dw| \asymp \int_{E_{R_2,1}^2} \left( \frac{d(\Psi(w), L)}{|w| - 1} \right)^{2-q} \frac{|dw|}{|\Psi(w) - \Psi(w_1)|^{\gamma_1(q-1)}} \\ &\preceq n^{2-q} \int_{E_{R_2,1}^2} \frac{|dw|}{|w - w_1|^{[\gamma_1(q-1)-(2-q)](\lambda_1+\varepsilon)}} \preceq n^{2-q} \int_{E_{R_2,1}^2} \frac{|dw|}{|w - w_1|^{[\gamma_1(q-1)-(2-q)]\lambda_1+\varepsilon}} \\ &\preceq n^{2-q} \begin{cases} n^{[\gamma_1(q-1)-(2-q)]\lambda_1-1+\varepsilon}, & [\gamma_1(q-1)-(2-q)]\lambda_1+\varepsilon > 1, \\ \ln n, & [\gamma_1(q-1)-(2-q)]\lambda_1+\varepsilon = 1, \\ 1, & [\gamma_1(q-1)-(2-q)]\lambda_1+\varepsilon < 1; \end{cases} \end{aligned} \quad (35)$$

$$\begin{aligned} B_{n,2}^1(w_2) &= \int_{E_{R_2,2}^1} \frac{|\Psi'(w)|^{2-q}}{|\Psi(w) - \Psi(w_2)|^{\gamma_2(q-1)}} |dw| \asymp \int_{E_{R_2,2}^1} \left( \frac{d(\Psi(w), L)}{|w| - 1} \right)^{2-q} \frac{|dw|}{|\Psi(w) - \Psi(w_2)|^{\gamma_2(q-1)}} \\ &\preceq n^{2-q} \int_{E_{R_2,2}^1} \frac{|dw|}{|\Psi(w) - \Psi(w_2)|^{\gamma_2(q-1)-(2-q)}} \preceq n^{2-q} \int_{E_{R_2,2}^1} \frac{|dw|}{|w - w_2|^{2[\gamma_2(q-1)-(2-q)]}} \\ &\preceq n^{2-q+2[\gamma_2(q-1)-(2-q)]} \text{mes}(E_{R_2,2}^1) \preceq n^{2\gamma_2(q-1)-(2-q)-1}; \end{aligned} \quad (36)$$

$$\begin{aligned} B_{n,2}^2(w_2) &\preceq n^{2-q} \int_{E_{R_2,2}^2} \frac{|dw|}{|\Psi(w) - \Psi(w_2)|^{(\gamma_2+1)(q-1)-1}} \preceq n^{2-q} \int_{E_{R_2,2}^2} \frac{|dw|}{|w - w_2|^{2[\gamma_2(q-1)-(2-q)]}} \\ &\preceq n^{2-q} \begin{cases} n^{2[\gamma_2(q-1)-(2-q)]-1}, & 2[\gamma_2(q-1)-(2-q)] > 1, \\ \ln n, & 2[\gamma_2(q-1)-(2-q)] = 1, \\ 1, & 2[\gamma_2(q-1)-(2-q)] < 1, \end{cases} \\ &= \begin{cases} n^{2\gamma_2(q-1)-(2-q)-1}, & 2[\gamma_2(q-1)-(2-q)] > 1, \\ n^{2-q} \ln n, & 2[\gamma_2(q-1)-(2-q)] = 1, \\ n^{2-q}, & 2[\gamma_2(q-1)-(2-q)] < 1. \end{cases} \end{aligned} \quad (37)$$

1.2. Let us now  $\gamma_1, \gamma_2 < 0$ . Since  $(\lambda_1 - 1)(2 - q) + (-\gamma_1)(q - 1)\lambda_1 + 1 = \lambda_1(2 - q) + (-\gamma_1)(q - 1)\lambda_1 + 1 - (2 - q)$  and  $\lambda_1(2 - q) + (-\gamma_1)(q - 1)\lambda_1 > 0$  for all  $0 < \lambda_1 < 2$ ,  $1 < q \leq 2$  and  $\gamma_1 < 0$ , then we get:

$$\begin{aligned} B_{n,2}^1(w_1) &\preceq \int_{E_{R_2,1}^1} \frac{|w - w_1|^{(\lambda_1-1-\varepsilon)(2-q)}}{|w - w_1|^{\gamma_1(q-1)(\lambda_1+\varepsilon)}} |dw| \preceq \left( \frac{1}{n} \right)^{(\lambda_1-1)(2-q)+(-\gamma_1)(q-1)\lambda_1+\varepsilon} \text{mes}(E_{R_2,1}^1) \\ &\preceq \left( \frac{1}{n} \right)^{(\lambda_1-1)(2-q)+(-\gamma_1)(q-1)\lambda_1+1+\varepsilon} \preceq n^{2-q}; \end{aligned} \quad (38)$$

$$\begin{aligned}
B_{n,2}^2(w_1) &\preceq \int_{E_{R_2,1}^2} \frac{|w-w_1|^{(\lambda_1-1-\varepsilon)(2-q)}}{|w-w_1|^{\gamma_1(q-1)(\lambda_1+\varepsilon)}} |dw| = \int_{E_{R_2,1}^2} |w-w_1|^{(\lambda_1(2-q)+(-\gamma_1)(q-1)\lambda_1-(2-q)+\varepsilon)} |dw| \\
&\preceq \int_{E_{R_2,1}^2} |w-w_1|^{-(2-q)} |dw| \preceq n^{2-q} \int_{E_{R_2,1}^2} |dw| \preceq n^{2-q};
\end{aligned} \tag{39}$$

$$\begin{aligned}
B_{n,2}^1(w_2) + B_{n,2}^2(w_2) &= n^{2-q} \int_{E_{R_2,2}^1 \cup E_{R_2,2}^2} |\Psi(w) - \Psi(w_2)|^{-(\gamma_2+1)(q-1)+1} |dw| \\
&\preceq n^{2-q} \text{mes}(E_{R_2,2}^1 \cup E_{R_2,2}^2) \preceq n^{2-q} \left(\frac{1}{n} + 1\right) \preceq n^{2-q}.
\end{aligned} \tag{40}$$

For the  $w \in E_{R_2,1}^3$ , we have  $|\Psi(w) - \Psi(w_j)| \geq 1$ , then for  $j = 1, 2$ , we get:

$$B_{n,2}^3(w_j) \preceq \int_{E_{R_2,j}^3} \left( \frac{d(\Psi(w), L)}{|w| - 1} \right)^{2-q} |dw| \preceq \int_{E_{R_2,j}^3} (|w| - 1)^{-\varepsilon(2-q)} |dw| \preceq n^\varepsilon. \tag{41}$$

Combining estimates (29)-(41), in this case, we have:

$$J_2(L_{R_2}) \preceq n^{\frac{1}{p}} \|P_n\|_p$$

$$\begin{aligned}
&\times \left\{ \begin{aligned} &\left[ n^{[\gamma_1(q-1)-(2-q)]\lambda_1+(1-q)+\varepsilon} \right. \\ &+ \begin{cases} n^{[\gamma_1(q-1)-(2-q)]\lambda_1+(2-q)-1+\varepsilon}, & [\gamma_1(q-1)-(2-q)]\lambda_1+\varepsilon > 1, \\ n^{2-q} \ln n, & [\gamma_1(q-1)-(2-q)]\lambda_1+\varepsilon = 1, \\ n^{2-q}, & [\gamma_1(q-1)-(2-q)]\lambda_1+\varepsilon < 1, \end{cases} \\ &\left. + \left[ n^{2\gamma_2(q-1)-(2-q)-1} + \begin{cases} n^{2\gamma_2(q-1)-(2-q)-1}, & 2[\gamma_2(q-1)-(2-q)] > 1, \\ n^{2-q} \ln n, & 2[\gamma_2(q-1)-(2-q)] = 1, \\ n^{2-q}, & 2[\gamma_2(q-1)-(2-q)] < 1, \end{cases} \right] \right. \\ &\left. + n^\varepsilon \right] \end{aligned} \right\}^{\frac{1}{q}} \\
&\preceq \|P_n\|_p \left\{ \begin{aligned} &\left[ n^{\left(\frac{\gamma_1+2}{p}-1\right)\lambda_1+\varepsilon} + \begin{cases} n^{\left(\frac{\gamma_1+2}{p}-1\right)\lambda_1+\varepsilon}, & p < \frac{\lambda_1(\gamma_1+2)+1}{\lambda_1+1} + \varepsilon, & \gamma_1 > \frac{1-\varepsilon}{\lambda_1}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{\lambda_1(\gamma_1+2)+1}{\lambda_1+1} + \varepsilon, & \gamma_1 > \frac{1-\varepsilon}{\lambda_1}, \\ n^{1-\frac{1}{p}}, & p > \frac{\lambda_1(\gamma_1+2)+1}{\lambda_1+1} + \varepsilon, & \gamma_1 > \frac{1-\varepsilon}{\lambda_1}, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 < \gamma_1 \leq \frac{1-\varepsilon}{\lambda_1}, \end{cases} \right] \\ &+ \left[ n^{2\left(\frac{\gamma_2+2}{p}-1\right)} + \begin{cases} n^{2\left(\frac{\gamma_2+2}{p}-1\right)}, & 2 \leq p < \frac{2(\gamma_2+2)+1}{3}, & \gamma_2 > \frac{1}{2}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{2(\gamma_2+2)+1}{3}, & \gamma_2 > \frac{1}{2}, \\ n^{1-\frac{1}{p}}, & p > \frac{2(\gamma_2+2)+1}{3}, & \gamma_2 > \frac{1}{2}, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 < \gamma_2 \leq \frac{1}{2}, \end{cases} \right] \\ &+ n^\varepsilon \end{aligned} \right\}
\end{aligned}$$

$$\begin{aligned} &\preceq \|P_n\|_p \left[ \begin{cases} n^{\left(\frac{\gamma_1+2}{p}-1\right)\tilde{\lambda}_1}, & p < \frac{\tilde{\lambda}_1(\gamma_1+2)+1}{\tilde{\lambda}_1+1}, & \gamma_1 > \frac{1-\varepsilon}{\lambda_1}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{\tilde{\lambda}_1(\gamma_1+2)+1}{\tilde{\lambda}_1+1}, & \gamma_1 > \frac{1-\varepsilon}{\lambda_1}, \\ n^{1-\frac{1}{p}}, & p > \frac{\tilde{\lambda}_1(\gamma_1+2)+1}{\tilde{\lambda}_1+1}, & \gamma_1 > \frac{1-\varepsilon}{\lambda_1}, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 < \gamma_1 \leq \frac{1-\varepsilon}{\lambda_1}, \end{cases} \right. \\ &\quad \left. + \begin{cases} n^{2\left(\frac{\gamma_2+2}{p}-1\right)}, & p < \frac{2(\gamma_2+2)+1}{3}, & \gamma_2 > \frac{1}{2}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{2(\gamma_2+2)+1}{3}, & \gamma_2 > \frac{1}{2}, \\ n^{1-\frac{1}{p}}, & p > \frac{2(\gamma_2+2)+1}{3}, & \gamma_2 > \frac{1}{2}, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 < \gamma_2 \leq \frac{1}{2}, \end{cases} \right] \\ &\preceq \|P_n\|_p \left[ \begin{cases} n^{\left(\frac{\tilde{\gamma}+2}{p}-1\right)\tilde{\lambda}}, & p < \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, & \tilde{\gamma} > \frac{1}{\tilde{\lambda}}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, & \tilde{\gamma} > \frac{1}{\tilde{\lambda}}, \\ n^{1-\frac{1}{p}}, & p > \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, & \tilde{\gamma} > \frac{1}{\tilde{\lambda}}, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 < \tilde{\gamma} \leq \frac{1}{\tilde{\lambda}}, \end{cases} \right] \end{aligned}$$

where  $\tilde{\gamma} := \max\{0; \gamma\}$ . Therefore,

$$J_2(L_{R_2}) \preceq \|P_n\|_p \left[ \begin{cases} n^{\left(\frac{\tilde{\gamma}+2}{p}-1\right)\tilde{\lambda}}, & 2 \leq p < \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, & \tilde{\gamma} > \frac{1}{\tilde{\lambda}}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, & \tilde{\gamma} > \frac{1}{\tilde{\lambda}}, \\ n^{1-\frac{1}{p}}, & p > \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, & \tilde{\gamma} > \frac{1}{\tilde{\lambda}}, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 \leq \tilde{\gamma} \leq \frac{1}{\tilde{\lambda}}, \end{cases} \right] \quad (42)$$

Therefore, combining (24), (25), (28)-42), for case  $p \geq 2$ ,  $\gamma_1 > -2$ ,  $\gamma_2 > -2$ , and all  $z \in \Omega_{R_1}$ , we obtain:

$$\left| P_n^{(m)}(z) \right| \preceq \frac{|\Phi^{n-m+1}(z)|}{d(z, L_{R_1})} \|P_n\|_p \left[ \begin{cases} n^{\left(\frac{\tilde{\gamma}+2}{p}+m-1\right)\tilde{\lambda}}, & 2 \leq p < \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, & \tilde{\gamma} > \frac{1}{\tilde{\lambda}}, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, & \tilde{\gamma} > \frac{1}{\tilde{\lambda}}, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}}, & p > \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, & \tilde{\gamma} > \frac{1}{\tilde{\lambda}}, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}}, & p \geq 2, & -2 < \tilde{\gamma} \leq \frac{1}{\tilde{\lambda}}. \end{cases} \right] \quad (43)$$

Case 2. Now, let us  $q > 2$  ( $1 < p < 2$ ).

2.1. For  $\gamma_1, \gamma_2 \geq 0$ , according to Lemma's 1 and 2, consistently, we get:

$$B_{n,2}^1(w_1) \preceq \int_{E_{R_2,1}^1} \frac{|dw|}{|\Psi'(w)|^{q-2} |\Psi(w) - \Psi(w_1)|^{\gamma_1(q-1)}} \preceq \quad (44)$$

$$\begin{aligned} &\int_{E_{R_2,1}^1} \frac{|dw|}{|w - w_1|^{(\lambda_1-1+\varepsilon)(q-2)} |w - w_1|^{\gamma_1(q-1)(\lambda_1+\varepsilon)}} \\ &\preceq n^{\gamma_1(q-1)\lambda_1+(\lambda_1-1+\varepsilon)(q-2)} \text{mes}(E_{R_2,1}^1) \preceq n^{\gamma_1(q-1)\lambda_1+(\lambda_1-1+\varepsilon)(q-2)-1}, \end{aligned} \quad (45)$$

$$\begin{aligned}
B_{n,2}^2(w_1) &= \int_{E_{R_2,1}^2} \frac{|dw|}{|\Psi'(w)|^{q-2} |\Psi(w) - \Psi(w_1)|^{\gamma_1(q-1)}} \\
&\asymp \int_{E_{R_2,1}^2} \left( \frac{|w| - 1}{d(\Psi(w), L)} \right)^{q-2} \frac{|dw|}{|\Psi(w) - \Psi(w_1)|^{\gamma_1(q-1)}} \quad (46)
\end{aligned}$$

$$\begin{aligned}
&\preceq n^{2-q} \int_{E_{R_2,1}^2} \frac{|dw|}{|w - w_1|^{[\gamma_1(q-1)+(q-2)]\lambda_1+\varepsilon}} \\
&\preceq n^{2-q} \begin{cases} n^{[\gamma_1(q-1)+(q-2)]\lambda_1-1+\varepsilon}, & [\gamma_1(q-1) + (q-2)]\lambda_1 + \varepsilon > 1, \\ \ln n, & [\gamma_1(q-1) + (q-2)]\lambda_1 + \varepsilon = 1, \\ 1, & [\gamma_1(q-1) + (q-2)]\lambda_1 + \varepsilon < 1; \end{cases} \\
&\preceq \begin{cases} n^{[\gamma_1(q-1)+(q-2)]\lambda_1-1+(2-q)+\varepsilon}, & [\gamma_1(q-1) + (q-2)]\lambda_1 + \varepsilon > 1, \\ n^{2-q} \ln n, & [\gamma_1(q-1) + (q-2)]\lambda_1 + \varepsilon = 1, \\ n^{2-q}, & [\gamma_1(q-1) + (q-2)]\lambda_1 + \varepsilon < 1; \end{cases} \quad (47)
\end{aligned}$$

$$\begin{aligned}
B_{n,2}^1(w_2) &\preceq \int_{E_{R_2,2}^1} \frac{|dw|}{|\Psi'(w)|^{q-2} |\Psi(w) - \Psi(w_2)|^{\gamma_2(q-1)}} \quad (48) \\
&\asymp \int_{E_{R_2,2}^1} \left( \frac{|w| - 1}{d(\Psi(w), L)} \right)^{q-2} \frac{|dw|}{|\Psi(w) - \Psi(w_2)|^{\gamma_2(q-1)}} \\
&\preceq \left( \frac{1}{n} \right)^{q-2} \cdot n^{2[\gamma_2(q-1)+(q-2)]} \int_{E_{R_2,2}^1} |dw| \\
&\preceq n^{2[\gamma_2(q-1)+(q-2)]-(q-2)} \text{mes}(E_{R_2,2}^1) \preceq n^{2[\gamma_2(q-1)+(q-2)]+1-q}; \quad (49)
\end{aligned}$$

$$\begin{aligned}
B_{n,2}^2(w_2) &\preceq \int_{E_{R_2,2}^2} \frac{|dw|}{|\Psi'(w)|^{q-2} |\Psi(w) - \Psi(w_2)|^{\gamma_2(q-1)}} \quad (50) \\
&\asymp \int_{E_{R_2,2}^2} \left( \frac{|w| - 1}{d(\Psi(w), L)} \right)^{q-2} \frac{|dw|}{|\Psi(w) - \Psi(w_2)|^{\gamma_2(q-1)}}
\end{aligned}$$

$$\preceq n^{2-q} \int_{E_{R_2,2}^2} \frac{|dw|}{|w - w_2|^{2[\gamma_2(q-1)+(q-2)]}} \quad (51)$$

$$\begin{aligned}
&\preceq n^{2-q} \begin{cases} n^{2[\gamma_2(q-1)+(q-2)]-1}, & 2[\gamma_2(q-1) + (q-2)] > 1, \\ \ln n, & 2[\gamma_2(q-1) + (q-2)] = 1, \\ 1, & 2[\gamma_2(q-1) + (q-2)] < 1; \end{cases} \\
&\preceq \begin{cases} n^{2[\gamma_2(q-1)+(q-2)]+1-q}, & 2[\gamma_2(q-1) + (q-2)] > 1, \\ n^{2-q} \ln n, & 2[\gamma_2(q-1) + (q-2)] = 1, \\ n^{2-q}, & 2[\gamma_2(q-1) + (q-2)] < 1. \end{cases} \quad (52)
\end{aligned}$$

2.2. For  $-2 < \gamma_1, \gamma_2 < 0$ , we get:

$$\begin{aligned} B_{n,2}^1(w_1) &\preceq \int_{E_{R_2,1}^1} \frac{|\Psi(w) - \Psi(w_1)|^{(-\gamma_1)(q-1)}}{|\Psi'(w)|^{q-2}} |dw| \\ &\preceq n^{2-q} \int_{E_{R_2,1}^1} \frac{|dw|}{|w - w_1|^{(q-2)\lambda_1 + \varepsilon}} \preceq n^{2-q+(q-2)\lambda_1 + \varepsilon} \text{mes}(E_{R_2,1}^1) \preceq n^{1-q+(q-2)\lambda_1 + \varepsilon}; \end{aligned} \quad (53)$$

$$\begin{aligned} B_{n,2}^2(w_1) &\preceq \int_{E_{R_2,1}^2} \frac{|\Psi(w) - \Psi(w_1)|^{(-\gamma_1)(q-1)}}{|\Psi'(w)|^{q-2}} |dw| \\ &\preceq n^{2-q} \int_{E_{R_2,1}^2} \frac{|dw|}{|w - w_1|^{(q-2)\lambda_1 + \varepsilon}} \preceq n^{2-q} \begin{cases} n^{(q-2)\lambda_1 - 1 + \varepsilon}, & (q-2)\lambda_1 + \varepsilon > 1, \\ \ln n, & (q-2)\lambda_1 + \varepsilon = 1, \\ 1, & (q-2)\lambda_1 + \varepsilon < 1, \end{cases} \\ &= \begin{cases} n^{(q-2)\lambda_1 - (q-1) + \varepsilon}, & (q-2)\lambda_1 + \varepsilon > 1, \\ n^{2-q} \ln n, & (q-2)\lambda_1 + \varepsilon = 1, \\ n^{2-q}, & (q-2)\lambda_1 + \varepsilon < 1; \end{cases} \end{aligned} \quad (54)$$

$$\begin{aligned} B_{n,2}^1(w_2) &\preceq \int_{E_{R_2,2}^1} \frac{|\Psi(w) - \Psi(w_2)|^{(-\gamma_2)(q-1)}}{|\Psi'(w)|^{q-2}} |dw| \preceq n^{2-q} \int_{E_{R_2,2}^1} \frac{|dw|}{|w - w_2|^{2(q-2)}} \\ &\preceq n^{2-q+2(q-2)} \text{mes}(E_{R_2,2}^1) \preceq n^{(q-2)-1}; \end{aligned} \quad (55)$$

$$\begin{aligned} B_{n,2}^2(w_2) &\preceq n^{2-q} \int_{E_{R_2,2}^2} \frac{|dw|}{|w - w_2|^{2(q-2)}} \preceq n^{2-q} \begin{cases} n^{2(q-2)-1}, & 2(q-2) > 1, \\ \ln n, & 2(q-2) = 1, \\ 1, & 2(q-2) < 1, \end{cases} \\ &\preceq \begin{cases} n^{(q-2)-1}, & 2(q-2) > 1, \\ n^{2-q} \ln n, & 2(q-2) = 1, \\ n^{2-q}, & 2(q-2) < 1. \end{cases} \end{aligned} \quad (56)$$

For  $w \in E_{R_2,1}^3$ ,  $|\Psi(w) - \Psi(w_j)| \geq 1$ , and then, for  $j = 1, 2$ , we have:

$$B_{n,2}^3(w_j) \preceq \int_{E_{R_2,j}^3} \frac{|dw|}{|\Psi'(w)|^{q-2}} \preceq \int_{E_{R_2,j}^3} \frac{|dw|}{(|w| - 1)^\varepsilon} \preceq n^\varepsilon. \quad (57)$$

Combining (45)-(57), we get:

$$J_2(L_{R_2}) \preceq n^{\frac{1}{p}} \|P_n\|_p$$

$$\begin{aligned}
& \times \left\{ \left[ n^{\gamma_1(q-1)\lambda_1+(\lambda_1-1+\varepsilon)(q-2)-1} \right. \right. \\
& \quad + \begin{cases} n^{[\gamma_1(q-1)+(q-2)]\lambda_1-1+(2-q)+\varepsilon}, & [\gamma_1(q-1)+(q-2)]\lambda_1+\varepsilon > 1, \\ n^{2-q} \ln n, & [\gamma_1(q-1)+(q-2)]\lambda_1+\varepsilon = 1, \\ n^{2-q}, & [\gamma_1(q-1)+(q-2)]\lambda_1+\varepsilon < 1, \end{cases} \\
& \quad + \left[ n^{2[\gamma_2(q-1)+(q-2)]+1-q} + \begin{cases} n^{2[\gamma_2(q-1)+(q-2)]+1-q}, & 2[\gamma_2(q-1)+(q-2)] > 1, \\ n^{2-q} \ln n, & 2[\gamma_2(q-1)+(q-2)] = 1, \\ n^{2-q}, & 2[\gamma_2(q-1)+(q-2)] < 1, \end{cases} \right] \\
& \quad \left. + n^\varepsilon \right\}^{\frac{1}{q}} \\
& \preceq \|P_n\|_p \left\{ \left[ n^{\left(\frac{\gamma_1+2}{p}-1\right)\lambda_1+\varepsilon} + \begin{cases} n^{\left(\frac{\gamma_1+2}{p}-1\right)\lambda_1+\varepsilon}, & 1 < p < \frac{\lambda_1(\gamma_1+2)+1}{\lambda_1+1} + \varepsilon, \quad \gamma_1 \leq \frac{1-\varepsilon}{\lambda_1}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{\lambda_1(\gamma_1+2)+1}{\lambda_1+1} + \varepsilon, \quad \gamma_1 \leq \frac{1-\varepsilon}{\lambda_1}, \\ n^{1-\frac{1}{p}}, & 2 > p > \frac{\lambda_1(\gamma_1+2)+1}{\lambda_1+1} + \varepsilon, \quad \gamma_1 \leq \frac{1-\varepsilon}{\lambda_1}, \\ n^{\left(\frac{\gamma_1+2}{p}-1\right)\lambda_1+\varepsilon}, & 1 < p < 2, \quad \gamma_1 > \frac{1-\varepsilon}{\lambda_1}, \end{cases} \right] \right. \\
& \quad \left. + \left[ n^{2\left(\frac{\gamma_2+2}{p}-1\right)} + \begin{cases} n^{2\left(\frac{\gamma_2+2}{p}-1\right)}, & 1 < p < \frac{2(\gamma_2+2)+1}{3}, \quad 0 < \gamma_2 \leq \frac{1}{2}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{2(\gamma_2+2)+1}{3}, \quad 0 < \gamma_2 \leq \frac{1}{2}, \\ n^{1-\frac{1}{p}}, & 2 > p > \frac{2(\gamma_2+2)+1}{3}; \quad 0 < \gamma_2 \leq \frac{1}{2}, \\ n^{2\left(\frac{\gamma_2+2}{p}-1\right)}, & 1 < p < 2 \quad \gamma_2 > \frac{1}{2}, \end{cases} \right] \right. \\
& \quad \left. + n^\varepsilon \right\} \\
& \preceq \|P_n\|_p \left\{ \begin{aligned} & n^{\left(\frac{\tilde{\gamma}+2}{p}-1\right)\tilde{\lambda}}, \quad 1 < p < \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, \quad \tilde{\gamma} \leq \frac{1}{\tilde{\lambda}}, \\ & (n \ln n)^{1-\frac{1}{p}}, \quad p = \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, \quad \tilde{\gamma} \leq \frac{1}{\tilde{\lambda}}, \\ & n^{1-\frac{1}{p}}, \quad 2 > p > \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, \quad \tilde{\gamma} \leq \frac{1}{\tilde{\lambda}}, \\ & n^{\left(\frac{\tilde{\gamma}+2}{p}-1\right)\tilde{\lambda}}, \quad 1 < p < 2 \quad \tilde{\gamma} > \frac{1}{\tilde{\lambda}}, \end{aligned} \right\}
\end{aligned}$$

for  $\gamma_1, \gamma_2 \geq 0$ , and

$$J_2(L_{R_2}) \preceq n^{\frac{1}{p}} \|P_n\|_p$$

$$\begin{aligned}
& \times \left\{ \left[ n^{1-q+(q-2)\lambda_1+\varepsilon} + \begin{cases} n^{(q-2)\lambda_1-(q-1)+\varepsilon}, & (q-2)\lambda_1+\varepsilon > 1, \\ n^{2-q} \ln n, & (q-2)\lambda_1+\varepsilon = 1, \\ n^{2-q}, & (q-2)\lambda_1+\varepsilon < 1; \end{cases} \right] \right. \\
& \quad \left. + \left[ n^{(q-2)-1} + \begin{cases} n^{(q-2)-1}, & 2(q-2) > 1, \\ n^{2-q} \ln n, & 2(q-2) = 1, \\ n^{2-q}, & 2(q-2) < 1; \end{cases} \right] + n^\varepsilon \right\}^{\frac{1}{q}} \\
& \preceq \|P_n\|_p \left\{ \left[ n^{\left(\frac{2}{p}-1\right)\lambda_1+\varepsilon} + \begin{cases} n^{\left(\frac{2}{p}-1\right)\lambda_1+\varepsilon}, & p < \frac{2\lambda_1+1}{\lambda_1+1} + \varepsilon, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{2\lambda_1+1}{\lambda_1+1} + \varepsilon, \\ n^{1-\frac{1}{p}}, & p > \frac{2\lambda_1+1}{\lambda_1+1} + \varepsilon; \end{cases} \right] \right. \\
& \quad \left. + \left[ n^{2\left(\frac{2}{p}-1\right)} + \begin{cases} n^{2\left(\frac{2}{p}-1\right)}, & 1 < p < \frac{5}{3}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{5}{3}, \\ n^{1-\frac{1}{p}}, & 2 > p > \frac{5}{3}; \end{cases} \right] + n^\varepsilon \right\}
\end{aligned}$$

$$\leq \|P_n\|_p \begin{cases} n^{\left(\frac{2}{p}-1\right)\tilde{\lambda}}, & 1 < p < \frac{2\tilde{\lambda}+1}{\tilde{\lambda}+1}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{2\tilde{\lambda}+1}{\tilde{\lambda}+1}, \\ n^{1-\frac{1}{p}}, & 2 > p > \frac{2\tilde{\lambda}+1}{\tilde{\lambda}+1}, \end{cases}$$

for  $-2 < \gamma_1, \gamma_2 < 0$ , where  $\tilde{\gamma} := \gamma_j$ , for  $z \in \Omega_R^j$ ,  $j = 1, 2$ .

Substituting the last obtained estimates into (24) and taking into account (28), for the case  $1 < p < 2, \gamma_1 > -2, \gamma_2 > -2$ , we obtain:

$$|P_n^{(m)}(z)| \leq \frac{|\Phi^{n-m+1}(z)|}{d(z, L_{R_1})} \|P_n\|_p \begin{cases} n^{\left(\frac{\tilde{\gamma}+2}{p}+m-1\right)\tilde{\lambda}}, & 1 < p < \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, \quad \tilde{\gamma} \leq \frac{1}{\tilde{\lambda}}, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, \quad \tilde{\gamma} \leq \frac{1}{\tilde{\lambda}}, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}}, & 2 > p > \frac{\tilde{\lambda}(\tilde{\gamma}+2)+1}{\tilde{\lambda}+1}, \quad \tilde{\gamma} \leq \frac{1}{\tilde{\lambda}}, \\ n^{\left(\frac{\tilde{\gamma}+2}{p}+m-1\right)\tilde{\lambda}}, & 1 < p < 2 \quad \tilde{\gamma} > \frac{1}{\tilde{\lambda}}, \end{cases}$$

for  $\gamma_1, \gamma_2 \geq 0$ , and

$$|P_n^{(m)}(z)| \leq \frac{|\Phi^{n-m+1}(z)|}{d(z, L_{R_1})} \|P_n\|_p \begin{cases} n^{\left(\frac{2}{p}+m-1\right)\tilde{\lambda}}, & 1 < p < \frac{2\tilde{\lambda}+1}{\tilde{\lambda}+1}, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{2\tilde{\lambda}+1}{\tilde{\lambda}+1}, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}}, & 2 > p > \frac{2\tilde{\lambda}+1}{\tilde{\lambda}+1}, \end{cases}$$

for  $-2 < \gamma_1, \gamma_2 < 0$ . It remains for us to combine the last two inequalities corresponding to the cases  $\gamma_j \geq 0$  and  $-2 < \gamma_j < 0$ , for each  $j = 1, 2$ :

$$|P_n^{(m)}(z)| \leq \frac{|\Phi^{n-m+1}(z)|}{d(z, L_{R_1})} \|P_n\|_p \begin{cases} n^{\left(\frac{\tilde{\gamma}^*+2}{p}+m-1\right)\tilde{\lambda}}, & 1 < p < \frac{\tilde{\lambda}(\tilde{\gamma}^*+2)+1}{\tilde{\lambda}+1}, \quad -2 < \tilde{\gamma} \leq \frac{1}{\tilde{\lambda}}, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\tilde{\lambda}(\tilde{\gamma}^*+2)+1}{\tilde{\lambda}+1}, \quad -2 < \tilde{\gamma} \leq \frac{1}{\tilde{\lambda}}, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}}, & 2 > p > \frac{\tilde{\lambda}(\tilde{\gamma}^*+2)+1}{\tilde{\lambda}+1}, \quad -2 < \tilde{\gamma} \leq \frac{1}{\tilde{\lambda}}, \\ n^{\left(\frac{\tilde{\gamma}+2}{p}+m-1\right)\tilde{\lambda}}, & 1 < p < 2, \quad \tilde{\gamma} > \frac{1}{\tilde{\lambda}}. \end{cases}$$

Now, let us consider the case  $p = 1$ . Multiplying the numerator and denominator of the integrand of the inner integral by  $h$  in (24), we get:

$$\begin{aligned} |P_n^{(m)}(z)| &\leq \frac{|\Phi^{n-m+1}(z)|}{d(z, L_{R_1})} \cdot \sup_{t \in L_{R_2}} \left( \frac{1}{h(t)} \int_{L_{R_1}} \frac{|d\zeta|}{|t - \zeta|^{m+1}} \right) \left( \int_{L_{R_2}} h(t) |P_n(t)| |dt| \right) \\ &\leq \frac{|\Phi^{n-m+1}(z)|}{d(z, L_{R_1})} \|P_n\|_{\mathcal{L}_1(h, L_{R_2})} \cdot \sup_{t \in L_{R_2}} \left( \frac{1}{\prod_{j=1}^2 |t - z_j|^{\gamma_j}} \int_{L_{R_1}} \frac{|d\zeta|}{|t - \zeta|^{m+1}} \right). \end{aligned} \quad (58)$$

After replacing the variable  $t = \Psi(w)$  and using the Lemmas 1, 2 and according to (20), (26) and (28), we obtain:

$$\frac{1}{|t - z_1|^{\gamma_1}} = \frac{1}{|\Psi(w) - \Psi(w_1)|^{\gamma_1}} \leq \frac{1}{|w - w_1|^{\gamma_1 \tilde{\lambda}_1}} \leq \begin{cases} n^{\gamma_1 \tilde{\lambda}_1}, & \text{for } \gamma_1 \geq 0, \\ 1, & \text{for } \gamma_1 < 0; \end{cases} \quad (59)$$

$$\frac{1}{|t - z_2|^{\gamma_2}} = \frac{1}{|\Psi(w) - \Psi(w_2)|^{\gamma_2}} \leq \frac{1}{|w - w_2|^{2\gamma_2}} \leq \begin{cases} n^{2\gamma_2}, & \text{for } \gamma_2 \geq 0, \\ 1, & \text{for } \gamma_2 < 0; \end{cases} \quad (60)$$

$$\int_{L_{R_1}} \frac{|d\zeta|}{|t - \zeta|^{m+1}} = \int_{|\tau|=R_1} \frac{|\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w)|^{m+1}} = J_1(L_{R_1}) \leq n^{m\tilde{\lambda}}. \quad (61)$$

According to Lemmas 1, 3 and (20), we find:

$$\|P_n\|_{\mathcal{L}_1(h, L_{R_2})} = \int_{L_{R_2}} h(t) |P_n(t)| |dt| \leq \sup_{t \in L_{R_2}} |\Phi'(t)| \int_{L_{R_2}} \frac{h(t)}{|\Phi'(t)|} |P_n(t)| |dt| \leq n^{\tilde{\lambda}} \|P_n\|_1. \quad (62)$$

Therefore, combining estimates (58)-(62), we find:

$$|P_n^{(m)}(z)| \leq \frac{|\Phi^{n-m+1}(z)|}{d(z, L_{R_1})} n^{(\tilde{\gamma}^* + m + 1)\tilde{\lambda}} \|P_n\|_1.$$

Therefore, for any  $p = 1$ , the proof of Theorem 1 is completed.

In order to finalize the proof, it remains to verify that  $d(z, L_{R_1}) \geq d(z, L)$  for all  $z \in \Omega_R$ . To this end, we now introduce the notations that will be used in the remainder of the proof. For  $0 < \delta_j < \delta_0 := \frac{1}{4} \min\{|z_i - z_j| : i, j = 1, 2, \dots, l, i \neq j\}$ , let  $\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$ ;  $\delta := \min_{1 \leq j \leq l} \delta_j$ ; For  $L = \partial G$  we set:  $U_\infty(L, \delta) := \bigcup_{\zeta \in L} U(\zeta, \delta)$ —infinite open cover of the curve

$L$ ;  $U(L, \delta) := \bigcup_{j=1}^N U_j(L, \delta) \subset U_\infty(L, \delta)$ —finite open cover of the curve  $L$ ;  $\Omega(\delta) := \Omega(L, \delta) := \Omega \cap U_N(L, \delta)$ . Now, for  $z \notin \Omega(L_{R_1}, d(L_{R_1}, L_R))$ , we have:  $d(z, L_{R_1}) \geq \delta \geq d(z, L)$ . Next, let  $z \in \Omega(L_{R_1}, d(L_{R_1}, L_R))$ . Denote by  $\xi_1 \in L_{R_1}$  such that  $d(z, L_{R_1}) = |z - \xi_1|$ , and point  $\xi_2 \in L$ , such that  $d(z, L) = |z - \xi_2|$ . For  $w = \Phi(z)$ ,  $t_1 = \Phi(\xi_1)$ ,  $t_2 = \Phi(\xi_2)$ , we have:  $|w - t_1| \geq ||w - t_2| - |t_2 - t_1|| \geq ||w - t_2| - \frac{1}{2}|w - t_2|| \geq \frac{1}{2}|w - t_2|$ . Then, according to Lemma 1, we obtain:  $d(z, L_{R_1}) \geq d(z, L)$ .  $\square$

**Proof of Theorem 2.** First of all, we present the theorem that we will use, along with its corollaries. Then we prove the estimate for  $|P_n^{(m)}(z)|$ ,  $z \in \overline{G}$  for each  $m \geq 0$ .

**Theorem A [24]** Let  $p > 0$ ;  $G \in PC_\theta(1; \lambda, 2)$ , for some  $0 < \lambda < 2$ ;  $h(z)$  be defined as in (1). Then, for any  $P_n \in \wp_n$ ,  $n \in \mathbb{N}$ ,  $\gamma_j > -2$ ,  $j = 1, 2$ , and arbitrary small  $\varepsilon > 0$ , we have:

$$\|P_n\|_\infty \leq c_{10} \mu_{n,0} \|P_n\|_p, \quad (63)$$

where  $c_{10} = c_{10}(G, \gamma_1, \gamma_2, \lambda, p, \varepsilon) > 0$  is the constant, independent of  $z$  and  $n$ , and the numbers  $\mu_{n,0}$ ,  $\hat{\gamma}$ ,  $\tilde{\lambda}$  are defined as in (17) for  $m = 0$  and in (13), respectively.

**Corollary A.** Under the conditions of Theorem A, for  $G \in PC_\theta(1; \lambda_1, \lambda_2)$ , the relation (63) is satisfied for

$$\tilde{\mu}_{n,m,1} = \begin{cases} n^{\frac{2+\gamma^*}{p} \lambda^*}, & \text{if } (2 + \gamma^*) \cdot \lambda^* > 1, \\ (n \ln n)^{\frac{1}{p}}, & \text{if } (2 + \gamma^*) \cdot \lambda^* = 1, \\ n^{\frac{1}{p}}, & \text{if } (2 + \gamma^*) \cdot \lambda^* < 1, \end{cases}$$

where  $\gamma^*$  and  $\lambda^*$  are defined as in (13).

**Corollary B.** Under the conditions of Theorem A, for  $G \in PC_\theta(1; 2, 2)$ , the relation (1) is satisfied for

$$\tilde{\mu}_{n,1} = n^{\frac{2(2+\gamma^*)}{p}},$$

where  $\gamma^*$  is defined as in (13).

Now we proceed to the proof of the estimate for  $|P_n^{(m)}(z)|$ ,  $z \in \overline{G}$ , for each  $m \geq 0$  and for the region  $G \in PC_\theta(1; \lambda, 2)$ . Let  $z \in L$  is an arbitrary fixed point;  $B(z, d(z, L_{R_1})) := \{\xi : |\xi - z| < d(z, L_{R_1})\}$ . By the Cauchy integral formula for derivatives, we have:

$$P_n^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial B(z, d(z, L_{R_1}))} \frac{P_n(t)}{(t-z)^{m+1}} dt, \quad m = 0, 1, 2, \dots$$

Then, by applying well-known Bernstein-Walsh inequality (4), we obtain:

$$\begin{aligned} |P_n^{(m)}(z)| &\leq \frac{m!}{2\pi} \max_{z \in \partial B(z, d(z, L_{R_1}))} |P_n(t)| \cdot \int_{\partial B(z, d(z, L_{R_1}))} \frac{|dt|}{|t-z|^{m+1}} \\ &\leq \frac{m!}{2\pi} \max_{t \in \overline{G}_{R_1}} |P_n(t)| \cdot \frac{2\pi d(z, L_R)}{d^{m+1}(z, L_R)} \preceq \frac{\|P_n\|_{C(\overline{G})}}{d^m(z, L_R)}. \end{aligned}$$

If  $p = \infty$ , we get:

$$|P_n^{(m)}(z)| \preceq \frac{1}{d^m(z, L_R)} \|P_n\|_\infty, \quad \forall z \in L.$$

If  $0 < p < \infty$ , by applying Theorem A and using the Lemmas 1, 2, for all  $\forall z \in L$ , we get:

$$|P_n^{(m)}(z)| \preceq \mu_{n,0} \|P_n\|_p \cdot n^{m\tilde{\lambda}} \preceq \mu_{n,m} \|P_n\|_p. \quad (64)$$

Since  $z \in L$  is arbitrary, we complete the proof of Theorems 2.

The proofs of Corollaries 4 and 5 are obtained in a completely similar way, using Corollaries A and B in (64) in place of Theorem A.  $\square$

## 6. Discussion

We have studied the growth properties of  $m$ -th derivatives of algebraic polynomials in weighted Bergman spaces in bounded and unbounded regions. The regions we have studied are regions whose boundary is a piecewise smooth curve containing zero angles. We have clearly demonstrated the effects of the geometric properties of the region and the behaviour of the weight function on the growth of the polynomial by considering the zero-angle and non-zero-angle cases together and separately. Initially, we examine the issue within unbounded regions of the complex plane, and subsequently, we derive conclusions regarding the closure of said region. Thereafter, we integrate these findings to procure assessments of the growth behavior of derivatives of algebraic polynomials in the entirety of the complex plane. Consequently, our aim is to make a contribution to the study of polynomial estimation in the weighted Bergman spaces and to establish novel insights regarding the growth characteristics of polynomials.

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