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On the Natural Numbers That Cannot Be Expressed as a Sum of Two Primes

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Abstract: In this paper, a structural property of prime numbers will be introduced and analyzed. Our analysis is based on the application of the sequence of even integers $2, 10, 16, 22, 26, 28\ldots$ which is defined as follows:
$F(0) = 2,\ for\, i > 0,\ F(i) = F(i-1) + l$, where l is the smallest even integer that is valid for:
$F(i-1) + 2k - 1 \in P, for\, k = 1, 2, \ldots, \frac{l}{2} - 1$
$F(i-1) + 2k - 1 \notin P, for\, k = \frac{l}{2}$
and P is the set of prime numbers.
The sequence F answers the question, which are the natural numbers that cannot be expressed as a sum of two primes. We prove that numbers expressed as $j = F(i) + 1$ cannot be written as the sum of two primes, for $i = 1, 2, \cdots$.

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1. Introduction

Goldbach's conjecture is one of the oldest mathematical problems in history. It was discovered by Goldbach in 1742 and remains unsolved up to this day. In 1742, Goldbach and Euler in conversation and in an exchange of letters discussed the representation of numbers as sums of at most three primes. The correspondence led to the formation of the Goldbach conjecture. It reads 'Every even number is the sum of two primes'. In fact, there are two conjectures, *the binary* and *the ternary Goldbach conjecture*. These conjectures can be stated as follows:

- Every even integer greater than two can be written as the sum of two primes (the binary conjecture).
- Every odd integer greater than five can be written as the sum of three primes (the ternary conjecture).

Over time, there have been many exciting results about conjecture. For a detailed review on this range of problems see the book [1]. Of course, there are many other resources. Interesting stories about primes and unresolved mathematical problems can be found in [2]. Various versions and names of the Goldbach conjecture are also known. The ternary conjecture is often called *the Goldbach weak conjecture*. In 2013, Harald Helfgott published a proof of Goldbach's weak conjecture [3]. As of 2018, the proof is widely accepted in the mathematics community, but it has not yet been published in a peer-reviewed journal.

In this work, the discussion will focus on the binary Goldbach conjecture. It is easy to show that the binary conjecture implies the ternary. We examine, what are the numbers which cannot be written as a sum of two primes. We shall show exactly that elements of a uniquely defined sequence $F(i) + 1$, for $i = 1, 2, \cdots$ (refer to Chapter 4 for definition) cannot be written as a sum of two primes. . The question of whether we can write all other integers $j \neq F(i) + 1$ greater than three as a sum of two primes remains open.

The paper is organized as follows. Chapter 1 contains a short historical overview about Goldbach's conjecture and a short description about obtained results. In Chapter 2 basic sets, notations and used technologies are introduced. Prime vectors, matrices and a control matrix function are discussed in Chapter 3. A sequence, the members of which cannot be written as the sum of two prime numbers, is investigated in Chapter 4. Chapter 5 provides a summary of the results achieved.

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2. Used Terms and Methods

2.1. Basic Sets and Definitions

The set of all natural numbers (non-negative integers) is denoted by $\mathbf{N} = \{0, 1, 2, \cdots\}$. A natural number greater than 1 is prime if its only divisors are 1 and itself. The set of all prime numbers is denoted by $\mathbf{P}$. If $n \in \mathbf{N}$ then $\mathbf{P}_n = \{p \in \mathbf{P}; p \leq n\}$ is the set of primes less or equal to n. For any set S the symbol $|S|$ will mean the number its elements. The array of numbers consisting of one row and n - elements is called an n-dimensional row vector and denoted by $a = (a_0, a_1, \cdots, a_{n-1})$ or $a = (a_j)$. The transpose of the vector $a = (a_0, a_1, \cdots, a_{n-1})$ is an n-dimensional column vector denoted by $a^T = (a_j)^T = (a_0, a_1, \cdots, a_{n-1})^T = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$. In this paper the value a_j - *component of the vector* $a = (a_0, a_1, \cdots, a_{n-1})$ is a non-negative integer, for $j = 0, 1, \cdots, n-1$.

2.2. Used Methods and Technologies

The Goldbach conjecture is one of the most intensively studied problem of number theory. There are many interesting results on this problem ([4–10]). We are not aiming to give a complete account of conjecture related literature here.

The analysis of the Goldbach conjecture is performed using *Toeplitz and circulant matrices*. We applied a similar analysis in [11]. That paper presents an application of *Toeplitz matrices* and a *max-algebraic claim* which is equivalent to Goldbach's conjecture.

In our work, the Goldbach conjecture is examined by methods of *combinatorics and classical linear algebra using unique circulant matrices*.

3. Matrices and Primes

In this section prime vectors, prime matrices and a control matrix function will be introduced and their main properties will be discussed. Everywhere in this paper n stands for a positive integer greater than 2.

3.1. Prime Vectors and Matrices

Definition 1. *The column vector $a^T = (a_0, a_1, \cdots, a_{n-1})^T = (a_i)^T$ is called an n - dimensional **extended prime vector** of size n if for all $i = 0, 1, \cdots, n-1$*

$$a_i = \begin{cases} 1, \text{ if } i \text{ is prime} \\ 0, \text{ otherwise} \end{cases}$$

The $n \times n$ matrix $A = (a_{ij})$ is called circulant if $a_{ij} = b_{i-j(mod\, n)}$, for some $b_0, b_1, ..., b_{n-1} \in \mathbf{N}$ (or $\mathbf{R}$ in general case). This matrix is fully specified by its first column $b^T = (b_0, \cdots, b_{n-1})^T$.

Definition 2. *Let $a^T = (a_0, a_1, \cdots, a_{n-1})^T$ be n - dimensional extended prime vector. The matrix $A_n = (a_{ij})$ is called an $n \times n$ **prime matrix**, if*

$$a_{ij} = a_{i-j(mod\, n)} \tag{1}$$

for all $i, j \in \{0, 1, \ldots, n-1\}$.

Clearly, the prime matrix is a special circulant matrix with entries from the set $\{0, 1\}$. Notice, the circulant matrices can also be defined by a row vector.

Example 1. *The matrix A_6 is a 6×6 prime matrix, and*

$$a^T = (0,0,1,1,0,1)^T = (a_0, a_1, a_2, a_3, a_4, a_5)^T$$

is an extended prime vector of size $n = 6$. For the sake of simplicity, we will also refer to $n \times n$ prime matrix as A, if this does not cause misunderstanding.

$$A_6 = (a_{ij}) = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_0 & a_5 & a_4 & a_3 & a_2 & a_1 \\ a_1 & a_0 & a_5 & a_4 & a_3 & a_2 \\ a_2 & a_1 & a_0 & a_5 & a_4 & a_3 \\ a_3 & a_2 & a_1 & a_0 & a_5 & a_4 \\ a_4 & a_3 & a_2 & a_1 & a_0 & a_5 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{pmatrix}$$

Let $A_n = (a_{ij})$ be an $n \times n$ prime matrix. The $i-th$ row of matrix A_n will be denoted $A_{n/i.}$ and the $j-th$ column $A_{n/.j}$ for all $i, j = 0, \ldots, n-1$. The vector $A_{n/.n-1}$ is called the **prime vector** of size n. The $0-th$ row vector $A_{n/0.}$ of matrix A_n is called the **reverse prime vector** of size n.

Definition 3. *The matrix T_n=($T_n[i,j]) = A_n^2$ is called a **control matrix** of order n.*

The entry

$$T_n[i,j] = A_{n/i.}A_{n/.j} = \sum_{l=0}^{n-1} a_{il}a_{lj}$$

*is a (dot) product of $i-th$ row and $j-th$ column of prime matrix A_n, for all $i, j = 0, \ldots, n-1$. The entry $T_n[0,0]$ is called the **main element** of T_n.*

The set of all control matrices of all orders $n > 2$ is denoted by $\mathscr{T}$. It is also important to note that the control matrix T_n as a product of circulant matrices $A_n \times A_n$ is also circulant, see [12].

Lemma 1. *Let $T_n \in \mathscr{T}$. The value*

$$T_n[0,0] = a_0a_0 + \sum_{i=1}^{n-1} a_ia_{n-i} \tag{2}$$

is the number of all prime pairs $p, q \in P_n$ such that $p + q = n$.

Proof. Suppose that $p, q \in P_n$ is an arbitrary pair of primes and $p + q = n$. Let $T_n[0,0] = A_{n/0.}A_{n/.0}$ and $A_{n/.0} = a^T = (a_0, a_1, \cdots, a_{n-1})^T$ is the extended prime vector, $A_{n/0.} = (a_0, a_{n-1}, a_{n-2}, \cdots, a_1)$ is the reverse prime vector . Clearly $q = n - p \in P_n$, therefore $a_pa_q = a_pa_{n-p} = 1$. If $i \notin P_n$ or $n - i \notin P_n$ then $a_i = 0$ or $a_{n-i} = 0$,thus $a_ia_{n-i} = 0$. □

We will examine when the equation $T_n[0,0] = 0$ holds.

Theorem 1. *An arbitrary natural number n can not be written as a sum of two primes if only if $T_n[0,0] = 0$ is fulfilled.*

Proof. If $T_n[0,0] = 0$ then there is no pair of primes $p, q \in P_n$ for which $p + q = n$. If n can not be written as a sum of two primes then for all $p, q \in P_n$, $p + q \neq n$, therefore $T_n[0,0] = 0$. □

Theorem 2. *If $T_n, T_{n+1} \in \mathscr{T}$ then $T_{n+1}[0,0] = 0$ if only if $T_n[0,n-1] = 0$.*

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Proof. $\Leftarrow$ Suppose that $T_n[0, n-1] = 0$. Therefore,

$$T_n[0, n-1] = A_{n/0}.A_{n/.n-1} = (a_0, a_{n-1}, a_{n-2}, \ldots, a_1)\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_0 \end{pmatrix} = 2a_0a_1 + \sum_{i=1}^{n-1} a_{n-i}a_{i+1} = 0 \quad (3)$$

It follows that $\sum_{i=1}^{n-1} a_{n-i}a_{i+1} = 0$. We know that $a_0 = a_1 = 0$, thus,

$$T_{n+1}[0, 0] = A_{n+1/0}.A_{n+1/.0} = (a_0, a_n, a_{n-1}, \ldots, a_1)\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} = a_0a_0 + a_na_1 + \sum_{i=1}^{n-1} a_{n-i}a_{i+1} = 0 \quad (4)$$

$\Rightarrow$ Equation 4 implies 3. $\square$

4. Not two Primes Sum Partition Sequence

4.1. Definition of an Auxiliary Sequence

Definition 4. *Let us define a sequence:*

$$F(0) = 2, \; for\; i > 0, \; F(i) = F(i-1) + l \quad (5)$$

where l is the smallest even integer with the properties:

$$F(i-1) + 2k - 1 \in P, \quad for \quad k = 1, 2, \ldots, \frac{l}{2} - 1$$
$$F(i-1) + 2k - 1 \notin P, \quad for \quad k = \frac{l}{2}$$

*$\{F(i)\}_{i=0}^{\infty}$ will be called the **auxiliary sequence**. The value $l = F(i) - F(i-1)$ is termed the **distance between adjacent elements** $F(i)$ and $F(i-1)$.*

It is not difficult to compute some first elements of sequence: $2, 10, 16, 22, 26, 28, \cdots$. $F(i)$ is the $i-th$ element of the sequence, for $i = 0, 1, 2, \cdots$.

Note that in the rest of the paper, notation $n \in F$ will mean, that there is an index $i \in \mathbf{N}$, such that $F(i) = n$ and $n \notin F$ will mean, that there is no such index $i \in \mathbf{N}$. We denote $F = \{F(i) : i \in \mathbf{N}\}$.

*4.2. Basic Properties of **the Auxiliary Sequence***

Lemma 2. *All elements of the sequence $\{F(i)\}_{i=0}^{\infty}$ are even numbers.*

Proof. The statement yields from Definition 5. The first item $F(0) = 2$ is even and, the distance $l = F(i) - F(i-1)$ is even for each i, therefore $F(i)$ is even for each i. $\square$

Lemma 3. *The sequence $\{F(i)\}_{i=0}^{\infty}$ is unbounded.*

Proof. Let $F(i)$ be an arbitrary element of sequence F. From Lemma 2 follows that $F(i) = 2k$ is an even number. Let $2k = 2^j n$, where $j \geq 1$ and n is the part of $2k$ prime factor, which contains primes greater than 2 or $n = 1$. Therefore, n is an odd number. Now we consider the number $3^j n$. This is not a prime, odd number greater than $2k$. Let $l-1$ be the smallest, non prime, odd number greater than $F(i) = 2k$. Then l is the smallest integer from Definition 5, therefore $F(i+1) = F(i) + l$, i.e., F is unbounded. The value $F(i+1)$ is the next member of sequence F, which is greater than $F(i)$. □

Lemma 4. *The distance $l = F(i+1) - F(i)$ is less than or equal to six, for $i > 1$.*

Proof. If $k > 2$ is even and $k+1$ and $k+3$ are primes then $k+1$ cannot be $0(mod\,3)$ and also cannot be $1(mod\,3)$ (because then $k+3$ would be $0(mod\,3)$). So $k+1$ is $2(mod\,3)$ and hence $k+5$ is $0(mod\,3)$. So if k is in the sequence and neither $k+2$, nor $k+4$ are then $k+6$ is in the sequence. □

Lemma 5. *If $l_0 = F(0)$, and $l_j = F(j) - F(j-1)$ for all $j = 1, 2, \cdots, i$ then $F(i) = \sum_{j=0}^{i} l_j$.*

Proof. We shall prove it by induction on k. If $k = 0$ then $F(0) = l_0$. For $k = i-1$ suppose, that $F(i-1) = \sum_{j=0}^{i-1} l_j$ and $l_j = F(j) - F(j-1)$ for $j = 1, 2, \cdots, i-1$. Now we consider the case $k = i$. Let denote $l_i = F(i) - F(i-1)$, then $F(i) = F(i-1) + l_i = \sum_{j=0}^{i-1} l_j + l_i = \sum_{j=0}^{i} l_j$ and $l_j = F(j) - F(j-1)$ for $j = 1, 2, \cdots, i$. □

4.3. *Not Two Primes Sum Partition Sequence*

Based on the Theorem 2 the sequence

$$\{F(i) + 1\}_{i=1}^{\infty} \tag{6}$$

describes those numbers that cannot be written as the sum of two prime numbers or shortly we can call the squence as the **not two primes sum partition sequence**.

We can define a set

$$M = \{j \in \mathbf{N}; (\forall i \in \mathbf{N}) j \neq F(i) + 1\} \tag{7}$$

by the sequence (6).

If it is proved that this set is the set of those numbers which can be written as the sum of two primes then this statement implies the binary Goldbach conjecture. In this case, the set M (it is also a sequence) can be called as **the two primes sum partition sequence**.

5. Conclusions

In summary, these results show that the sequence $\{F(i) + 1\}_{i=1}^{\infty}$ determines the set of natural numbers, which cannot be written as a sum of two primes.

The proof of the statement that $M = \{j \in \mathbf{N}; (\forall i \in \mathbf{N}) j \neq F(i) + 1\}$ is the set of numbers that can be written as the sum of two primes, would mean solving the binary Goldbach conjecture. However, this remains an open question. This work is at the same time an introduction, a description of the basic concepts, for proving the statement formulated above.

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References

1. John Forbes Nash, J.; Rassias, M.T., Eds. *Open Problems in Mathematic*; Springer, 2018. https://doi.org/DOI10.1007/978-3-319-32162-2.
2. du Sautoy, M. *The Music of the Primes: Why an Unsolved Problem in Mathematics Matters*; HarperCollins, 2003.

 doi:10.20944/preprints202408.1416.v1

3. Helfgott, H. The ternary Goldbach conjecture is true. *arXiv:1312.7748* **2013**.
4. Hardy, G.H.; Ramanujan, S. Asymptotic Formulae in Combinatory Analysis. *Proc. London Math. Soc* **1918**, *17*, 75–115.
5. Hardy, G.H.; Littlewood, J.E. Some problems of partitio numerorum, V: A further contribution to the study of Goldbach's problem. *Proc. London Math. Soc.* **1924**, *22*, 254–269.
6. Littlewood, J.E. On the zeros of the Riemann zeta-function. *Proc. Cam. Phil. Soc* **1924**, *22*, 295–318.
7. Vinogradov, I.M. Representation of an odd number as the sum of three primes. *Dokl. Akad. Nauk SSSR* **1937**, *16*, 179–195.
8. Vaughan, R.C. On the number of partitions into primes. *The Ramanujan Journal* **2008**, *15*, 109–121.
9. Lu, W.C. Exceptional set of Goldbach number. *Journal of Number Theory* **2010**, *130*, 2359–2392.
10. Liu, Z. Cubes of primes and almost prime. *Journal of Number Theory* **2013**, *132*, 1284–1294.
11. Szabó, P. Goldbach's conjecture in max-algebra. *Computational Management Science* **2017**, *14*, 81–89. https://doi.org/10.1007/s10287-016-0268-z.
12. Plavka, J. Eigenproblem for circulant matrices in max-algebra. *Optimization* **2001**, *50*, 477–483. https://doi.org/https://doi.org/10.1080/02331930108844576.