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Article

Analytic Fourier–Feynman Transforms and Convolution Products Associated with Bounded Linear Operators on Abstract Wiener Space

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Abstract

In this paper, an analytic Fourier–Feynman transform (FFT) and a convolution product (CP) associated with bounded linear operator (BLOP)s on abstract Wiener space (AWS) $\mathbb{B}$ are defined. The existences of the FFT and the CP of certain bounded functionals on $\mathbb{B}$ are also provided. Additionally, three kinds of relationships between the FFT and the CP are investigated. It turned out in this paper that the relations between them as well as the concepts of the transform and the convolution involve previous researches performed with Gaussian processes on classical Wiener space $C_0[0, T]$. That is, the Gaussian processes used in previous researches are Banach space BLOPs on $C_0[0, T]$.

Keywords: abstract Wiener space; analytic Fourier–Feynman transform; convolution product; bounded linear operator

1. Introduction

Integral transforms, such as Fourier, Hankel, and Bessel transforms of functions, play crucial roles not only in functional analysis but also in applied mathematics, serving as tools for proving the algebraic structures of function spaces. A broad group of experts in mathematics have made contributions regarding the generalization of the notion of transform of functionals defined on infinite dimensional linear spaces. These approaches have many applications in different fields of mathematics. For example, the notion of the Fourier–Wiener transform for functionals on one-parameter Wiener space [1,2] was extended on AWS [3,4], in order to investigate the existence, uniqueness and regularity of solutions for Cauchy problems [5] associated with an infinite dimensional heat equation.

On the other hand, two kinds of identities,

$$\mathcal{T}(f * g) = \mathcal{T}(f)\mathcal{T}(g) \text{ and } \mathcal{T}(f) * \mathcal{T}(g) = \mathcal{T}(fg) \quad (1)$$

between a (well-defined) transform $\mathcal{T}$ and a corresponding convolution $*$ are important structures in the study area of the transforms $\mathcal{T}$.

The theory of the analytic FFT introduced by Brue [6] now is playing a central role in the analytic Feynman integration theory. Relationships, such as (1), between the analytic FFT and the corresponding CP on the one-parameter Wiener space have been developed in various research articles, see [7–11]. These structures involving the analytic FFT of functionals on AWSs can be found in [12–14]. There are tremendous works on the relationships between various transforms and convolutions of functionals on infinite dimensional spaces, see [15] and references cited therein.

The definitions of FFTs and CPs are based on the analytic Feynman integral. For more details, see [15,16]. In particular, the definitions of the FFT and the CP associated with Gaussian processes which are not stationary in time can be found in [16–21]. In recent paper [22], Choi defined an analytic Feynman integral and a FFT combined with BLOPs on AWS $\mathbb{B}$, and investigated various results involving iterated FFTs of functionals on $\mathbb{B}$. In this paper, an L_1 analytic FFT and a CP associated

with BLOPs on $\mathbb{B}$ are defined. The existences of the L_1 analytic FFT and the CP of certain bounded functionals on $\mathbb{B}$ are also provided. Additionally, three kinds of relationships between the FFT and the CP are investigated. It turned out in the last section of this paper that the relationships between them as well as the concepts of the transform and the convolution involve previous researches performed with Gaussian processes on the one-parameter Wiener space $C_0[0, T]$. That is, the Gaussian processes used in previous researches [16–21] are Banach space BLOPs on $C_0[0, T]$.

2. Background

In order to provide our results for the analytic FFT and the CP associated with BLOPs on AWSs, we first follow the expositions of [22–27].

Let $\mathbb{H}$ be a real separable Hilbert space with norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$, and let $\mathbb{B}$ be a real separable Banach space with norm $\|\cdot\|$. It is assumed that $\mathbb{H}$ is continuously, linearly, and densely embedded in $\mathbb{B}$ by a natural injection. Let ν be a centered Gaussian probability measure on $(\mathbb{B}, \mathcal{B}(\mathbb{B}))$, where $\mathcal{B}(\mathbb{B})$ denotes the Borel σ -field of $\mathbb{B}$. The triple $(\mathbb{H}, \mathbb{B}, \nu)$ is called an AWS if

$$\int_{\mathbb{B}} \exp(i(h, x)) d\nu(x) = \exp\left(-\frac{1}{2}|h|^2\right) \quad (2)$$

for any $h \in \mathbb{B}^*$, where $(\cdot, \cdot)$ denotes the $\mathbb{B}^*$ – $\mathbb{B}$ pairing, and where $\mathbb{B}^*$ is the topological dual of $\mathbb{B}$. Let $\mathbb{H}^*$ denote the topological dual of $\mathbb{H}$. Then the Banach space $\mathbb{B}^*$ is identified as a dense subspace of $\mathbb{H}^* \cong \mathbb{H}$ in the sense that, for all $y \in \mathbb{B}^*$ and $x \in \mathbb{H}$,

$$\langle y, x \rangle = (y, x). \quad (3)$$

Thus we have the triple

$$\mathbb{B}^* \subset \mathbb{H}^* \cong \mathbb{H} \subset \mathbb{B}. \quad (4)$$

Let $(\mathbb{H}, \mathbb{B}, \nu)$ be an AWS, and let $\{e_n\}$ be a complete orthonormal set in the Cameron–Martin space $\mathbb{H}$ such that e_j 's are in $\mathbb{B}^*$. For each $h \in \mathbb{H}$ and $x \in \mathbb{B}$, a stochastic linear functional $(h, x)^\sim$ is defined by

$$(h, x)^\sim = \begin{cases} \lim_{n \rightarrow \infty} \sum_{j=1}^n \langle h, e_j \rangle (e_j, x), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

By the definition of the stochastic linear functional $(\cdot, \cdot)^\sim$ and (3), it is clear that $(\theta, x)^\sim = (\theta, x)$ for all $\theta \in \mathbb{B}^*$ and $x \in \mathbb{B}$. It is well-known [24–27] that for every non-zero h in $\mathbb{H}$, $(h, x)^\sim$ is a non-degenerate Gaussian random variable on $\mathbb{B}$ with mean 0 and variance $|h|^2$. The stochastic linear functional $(h, x)^\sim$ given by (5) is essentially independent of the choice of the complete orthonormal set used in its definition.

By Kallianpur and Bromley's results [25], the limit in (5) exists for ν -a.e. $x \in \mathbb{B}$, and for each $h \in \mathbb{H}(\cong \mathbb{H}^*)$, the Gaussian random variable $(h, \cdot)^\sim$ is in $L_2(\mathbb{B}, \mathcal{B}(\mathbb{B}), \nu)$. For a more detailed study, we also refer the reader to reference [24]. In fact, the sequence $\{\sum_{j=1}^n \langle h, e_j \rangle (e_j, x)\}_{n=1}^\infty$ appeared in (5) is Cauchy in $L_2(\mathbb{B}, \mathcal{B}(\mathbb{B}), \nu)$.

Given a Banach space $\mathbb{X}$, let $\mathcal{L}(\mathbb{X})$ denote the space of all BLOPs from $\mathbb{X}$ to itself. Then, in view of the Riesz representation theorem, one can see that the $\mathbb{B}^*$ – $\mathbb{B}$ pairing $(\cdot, \cdot)$ is a bilinear form on $\mathbb{B}^* \times \mathbb{B}$. By the concept of the Banach space adjoint operator, given an operator $A \in \mathcal{L}(\mathbb{B})$, there exists an operator A^* in $\mathcal{L}(\mathbb{B}^*)$ such that for all $\theta \in \mathbb{B}^*$ and $x \in \mathbb{B}$,

$$(A^*\theta)x = \theta(Ax). \quad (6)$$

By the structure of the dual pairing and the triple (4), equation (6) can be rewritten by

$$(A^*\theta, x) = (\theta, Ax). \quad (7)$$

3. Fourier–Feynman Transform and Convolution Product Associated with Bounded Linear Operators

In order to define the analytic FFT and the CP associated with BLOPs of functionals on the AWS $(\mathbb{H}, \mathbb{B}, \nu)$, we need the concept of the “scale-invariant measurability”. The concept of the scale-invariant measurability, rather than Borel measurability or Wiener measurability, is precisely correct for the analytic Feynman integration theory on $\mathbb{B}$, see [27].

Let $\mathcal{W}(\mathbb{B})$ be the class of ν -Carathéodory measurable subsets of $\mathbb{B}$. A subset S of $\mathbb{B}$ is said to be scale-invariant measurable (s.i.m.) provided ρS is $\mathcal{W}(\mathbb{B})$ -measurable for every $\rho > 0$, and an s.i.m. subset N of $\mathbb{B}$ is said to be scale-invariant null provided $\nu(\rho N) = 0$ for every $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F on $\mathbb{B}$ is said to be s.i.m. provided F is defined on an s.i.m. set and $F(\rho \cdot)$ is $\mathcal{W}(\mathbb{B})$ -measurable for every $\rho > 0$. If two functionals F and G on $\mathbb{B}$ are equal s-a.e., i.e., for each $\rho > 0$, $\nu(\{x \in \mathbb{B} : F(\rho x) \neq G(\rho x)\}) = 0$, then we write $F \approx G$. For an s.i.m. functional F on $\mathbb{B}$, we denote by $[F]_s$ the equivalence class of functionals which are equal to F s-a.e..

Let $\mathbb{C}$, $\mathbb{C}_+$ and $\tilde{\mathbb{C}}_+$ denote the set of complex numbers, complex numbers with positive real part and non-zero complex numbers with nonnegative real part, respectively. For each $\lambda \in \tilde{\mathbb{C}}_+$, $\lambda^{1/2}$ denotes the principal square root of λ ; i.e., $\lambda^{1/2}$ is always chosen to have positive real part, so that $\lambda^{-1/2} = (\lambda^{-1})^{1/2}$ is in $\mathbb{C}_+$.

Given a ν -integrable functional F on $(\mathbb{B}, \mathcal{W}(\mathbb{B}), \nu)$ and an operator A in $\mathcal{L}(\mathbb{B})$, we define the Wiener integral (associated with the operator A) of F by the formula

$$\mathcal{I}_A[F] \equiv \mathcal{I}_{A,x}[F(Ax)] \equiv \int_{\mathbb{B}} F(Ax) d\nu(x).$$

Given an operator $A \in \mathcal{L}(\mathbb{B})$, let $F : \mathbb{B} \rightarrow \mathbb{C}$ be an s.i.m. functional such that

$$J_F(A; \lambda) = \mathcal{I}_{A,x}[F(\lambda^{-1/2} Ax)] = \int_{\mathbb{B}} F(\lambda^{-1/2} Ax) d\nu(x)$$

exists as a finite number for all $\lambda \in (0, +\infty)$. If there exists a function $J_F^*(A; \cdot)$ analytic on $\mathbb{C}_+$ such that $J_F^*(A; \lambda) = J_F(A; \lambda)$ for all $\lambda \in (0, +\infty)$, then $J_F^*(A; \lambda)$ is defined to be the analytic Wiener integral (associated with the operator A) of F over $\mathbb{B}$ with parameter λ . For $\lambda \in \mathbb{C}_+$ we write

$$\mathcal{I}_A^{\text{an}, w_\lambda}[F] \equiv \mathcal{I}_{A,x}^{\text{an}, w_\lambda}[F(Ax)] \equiv \int_{\mathbb{B}}^{\text{an}, w_\lambda} F(Ax) d\nu(x) = J_F^*(A; \lambda).$$

Let $q \neq 0$ be a real number, and let F be an s.i.m. functional whose analytic Wiener integral $\mathcal{I}_A^{\text{an}, w_\lambda}[F]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic Feynman integral (associated with the operator A) of F with parameter q , and we write

$$\mathcal{I}_A^{\text{an}, f_q}[F] \equiv \mathcal{I}_{A,x}^{\text{an}, f_q}[F(Ax)] \equiv \int_{\mathbb{B}}^{\text{an}, f_q} F(Ax) d\nu(x) = \lim_{\lambda \rightarrow -iq} \mathcal{I}_{A,x}^{\text{an}, w_\lambda}[F(Ax)]$$

where λ approaches $-iq$ through values in $\mathbb{C}_+$.

The definitions of the analytic FFT and the CP [7–21] on classical and abstract Wiener spaces are based on the structure of the analytic Feynman integral and the concept of the scale-invariant measurability. We define an L_1 analytic FFT and a CP associated with BLOPs on AWSs $\mathbb{B}$.

Definition 1. For $A \in \mathcal{L}(\mathbb{B})$ and $y \in \mathbb{B}$, let $F : \mathbb{B} \rightarrow \mathbb{C}$ be an s.i.m. functional such that the following analytic Wiener integral $T_{\lambda,A}(F)(y) = \mathcal{I}_{A,x}^{\text{an}, w_\lambda}[F(y + Ax)]$ exists for all $\lambda \in \mathbb{C}_+$ and for s-a.e. $y \in \mathbb{B}$. Let q be a non-zero real number. We define the L_1 analytic FFT (associated with the BLOP A), $T_{q,A}^{(1)}(F)$ of F , is defined by the formula

$$T_{q,A}^{(1)}(F)(y) = \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} T_{\lambda,A}(F)(y)$$

for s-a.e. $y \in \mathbb{B}$, whenever this limit exists. That is to say,

$$T_{q,A}^{(1)}(F)(y) = \mathcal{I}_{A,x}^{\text{an},f_q}[F(y + Ax)]$$

for s-a.e. $y \in \mathbb{B}$.

We note that if the L_1 analytic FFT $T_{q,A}^{(1)}(F)$ exists for a BLOP $A \in \mathcal{L}(\mathbb{B})$ and if $F \approx G$, then $T_{q,A}^{(1)}(G)$ exists and $T_{q,A}^{(1)}(G) \approx T_{q,A}^{(1)}(F)$.

Remark 1. If A is the identity operator on $\mathbb{B}$, then this definition agrees with the previous definition of the analytic (ordinary) FFT [12–14].

Next we give the definition of the CP. The following definition of the CP is based on the definition of the CP studied in [28].

Definition 2. Let F and G be s.i.m. functionals on $\mathbb{B}$. For $\lambda \in \widetilde{\mathbb{C}}_+$ and $C_1, C_2, A_1, A_2 \in \mathcal{L}(\mathbb{B})$, we define their CP with respect to $\{C_1, C_2, A_1, A_2\}$ (if it exists) by

$$\begin{aligned} (F * G)_\lambda^{(C_1, C_2; A_1, A_2)}(y) &= \begin{cases} \int_{\mathbb{B}}^{\text{an}, w_\lambda} F(C_1 y + A_1 x) G(C_2 y - A_2 x) d\nu(x), & \lambda \in \mathbb{C}_+ \\ \int_{\mathbb{B}}^{\text{an}, f_q} F(C_1 y + A_1 x) G(C_2 y - A_2 x) d\nu(x), & \lambda = -iq, q \in \mathbb{R}, q \neq 0. \end{cases} \end{aligned} \quad (8)$$

When $\lambda = -iq$, we denote $(F * G)_\lambda^{(C_1, C_2; A_1, A_2)}$ by $(F * G)_q^{(C_1, C_2; A_1, A_2)}$.

4. Fourier–Feynman Transform of Functionals in Banach algebra $\mathcal{F}(\mathbb{B}^*)$

In this section, we introduce the class $\mathcal{F}(\mathbb{B}^*)$, a Banach algebra (of equivalence classes) of functionals on AWS $\mathbb{B}$, and provide the existence theorem of the L_1 analytic FFT of functionals in the Banach algebra $\mathcal{F}(\mathbb{B}^*)$.

Let $\mathcal{M}(\mathbb{B}^*)$ denote the space of complex-valued countably additive (and hence finite) Borel measures on $\mathbb{B}^*$. Then it is well-known that under total variation norm $\|\cdot\|$ and with convolution as multiplication, the space $\mathcal{M}(\mathbb{B}^*)$ is a Banach algebra with identity. The class $\mathcal{F}(\mathbb{B}^*)$ is defined as the space of all s-equivalence classes of Fourier transforms of measures in $\mathcal{M}(\mathbb{B}^*)$, that is,

$$\mathcal{F}(\mathbb{B}^*) = \left\{ [F_\sigma]_s : F_\sigma(x) = \int_{\mathbb{B}^*} \exp(i(g, x)) d\sigma^*(g), x \in \mathbb{B}, \sigma^* \in \mathcal{M}(\mathbb{B}^*) \right\}.$$

We in this paper will identify a functional with its s-equivalence class and think of $\mathcal{F}(\mathbb{B}^*)$ as a collection of functionals on $\mathbb{B}^*$ rather than as a collection of s-equivalence classes. That is, $F_\sigma \in \mathcal{F}(\mathbb{B}^*)$ if and only if there exists a measure $\sigma^* \in \mathcal{M}(\mathbb{B}^*)$ such that

$$F_\sigma(x) = \int_{\mathbb{B}^*} \exp(i(g, x)) d\sigma^*(g) \quad (9)$$

for s-a.e. $x \in \mathbb{B}$. For more details, see [22].

In [22], Choi established the existence of the L_1 analytic FFT of functionals in $\mathcal{F}(\mathbb{B}^*)$ as follows.

Theorem 1. Let $F_\sigma \in \mathcal{F}(\mathbb{B}^*)$ be given by (9). Then, for all $A \in \mathcal{L}(\mathbb{B})$, the L_1 analytic FFT associated with the operator A , $T_{q,A}^{(1)}(F_\sigma)$ exists for each non-zero real q , and is given by the formula

$$T_{q,A}^{(1)}(F_\sigma)(y) = \int_{\mathbb{B}^*} \exp(i(g, y)) d(\sigma^*)_{t,q}^A(g) \quad (10)$$

for s-a.e. $y \in \mathbb{B}$, where $(\sigma^*)_{t,q}^A$ is the complex measure on $\mathbb{B}^*$ given by

$$(\sigma^*)_{t,q}^A(U) = \int_U \exp\left(-\frac{i}{2q}(g, AA^*g)\right) d\sigma^*(g) \quad (11)$$

for $U \in \mathcal{B}(\mathbb{B}^*)$. Thus $T_{q,A}^{(1)}(F_\sigma)$ is an element of $\mathcal{F}(\mathbb{B}^*)$.

The following corollary is a simple consequence from Theorems 1.

Corollary 1. Let $F_\sigma \in \mathcal{F}(\mathbb{B}^*)$ be given by (9). Then for all non-zero real q , and any $A \in \mathcal{L}(\mathbb{B})$,

$$T_{-q,A}^{(1)}(T_{q,A}^{(1)}(F_\sigma)) \approx F_\sigma. \quad (12)$$

From this, one can see that the L_1 analytic FFT, $T_{q,A}^{(1)}$ has the inverse transform $\{T_{q,A}^{(1)}\}^{-1} = T_{-q,A}^{(1)}$.

5. Convolution Product of Functionals in Banach algebra $\mathcal{F}(\mathbb{B}^*)$

In this section, we establish the existence of the CP of functionals in the class $\mathcal{F}(\mathbb{B}^*)$.

Theorem 2. Let F_{σ_1} and F_{σ_2} be functionals in $\mathcal{F}(\mathbb{B}^*)$ and let C_1, C_2, A_1 and A_2 be operators in $\mathcal{L}(\mathbb{B})$. Then for all real $q \in \mathbb{R} \setminus \{0\}$, the CP $(F_{\sigma_1} * F_{\sigma_2})^{(C_1, C_2; A_1, A_2)}$ of F_{σ_1} and F_{σ_2} exists and is given by the formula

$$\begin{aligned} & (F_{\sigma_1} * F_{\sigma_2})_q^{(C_1, C_2; A_1, A_2)}(y) \\ &= \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} \exp\left(i(C_1^*g_1 + C_2^*g_2, y) - \frac{i}{2q}|A_1^*g_1 - A_2^*g_2|^2\right) d\sigma_1^*(g_1) d\sigma_2^*(g_2) \end{aligned} \quad (13)$$

for s-a.e. $y \in \mathbb{B}$.

Proof. Using (8), (7), the Fubini theorem, and (2), it first follows that for all $\lambda \in (0, +\infty)$ and s-a.e. $y \in \mathbb{B}$,

$$\begin{aligned} & (F_{\sigma_1} * F_{\sigma_2})_\lambda^{(C_1, C_2; A_1, A_2)}(y) \\ &= \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} \exp\left(i(C_1^*g_1 + C_2^*g_2, y)\right) \left[\int_{\mathbb{B}} \exp\left(\frac{i}{\sqrt{\lambda}}(A_1^*g_1 - A_2^*g_2, x)\right) d\nu(x) \right] d\sigma_1^*(g_1) d\sigma_2^*(g_2) \\ &= \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} \exp\left(i(C_1^*g_1 + C_2^*g_2, y) - \frac{1}{2\lambda}|A_1^*g_1 - A_2^*g_2|^2\right) d\sigma_1^*(g_1) d\sigma_2^*(g_2). \end{aligned}$$

Now, for each $\lambda \in \mathbb{C}$, let

$$\begin{aligned} & J_{(F_{\sigma_1} * F_{\sigma_2})^{(C_1, C_2; \cdot, \cdot)}}^*(A_1, A_2; \lambda) \\ &= \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} \exp\left(i(C_1^*g_1 + C_2^*g_2, y) - \frac{1}{2\lambda}|A_1^*g_1 - A_2^*g_2|^2\right) d\sigma_1^*(g_1) d\sigma_2^*(g_2). \end{aligned}$$

Then

$$J_{(F_{\sigma_1} * F_{\sigma_2})^{(C_1, C_2; \cdot, \cdot)}}^*(A_1, A_2; \lambda) = (F_{\sigma_1} * F_{\sigma_2})_\lambda^{(C_1, C_2; A_1, A_2)}$$

for all $\lambda \in (0, +\infty)$, clearly. We note that

$$\begin{aligned}
 & |J_{(F_{\sigma_1}^* F_{\sigma_2}^*)}^{*(C_1, C_2; \cdot, \cdot)}(A_1, A_2; \lambda)| \\
 & \leq \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} \left| \exp \left(-\frac{\operatorname{Re}(\cdot)}{2|\lambda|^2} |A_1^* g_1 - A_2^* g_2|^2 \right) \right| d|\sigma_1^*|(g_1) d|\sigma_2^*|(g_2) \\
 & \leq \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} d|\sigma_1^*|(g_1) d|\sigma_2^*|(g_2) \\
 & = \left(\int_{\mathbb{B}^*} d|\sigma_1^*|(g_1) \right) \left(\int_{\mathbb{B}^*} d|\sigma_2^*|(g_2) \right) \\
 & = \|\sigma_1^*\| \|\sigma_2^*\| < +\infty
 \end{aligned} \tag{14}$$

for all $\lambda \in \mathbb{C}_+$, since $\operatorname{Re}(\lambda) > 0$. Thus, applying the dominated convergence theorem, we see that $J_{(F_{\sigma_1}^* F_{\sigma_2}^*)}^{*(C_1, C_2; \cdot, \cdot)}(A_1, A_2; \lambda)$ is a continuous function of λ on $\tilde{\mathbb{C}}_+$. Also, because

$$\Xi(\lambda) \equiv \exp \left(i(C_1^* g_1 + C_2^* g_2, y) - \frac{1}{2\lambda} |A_1^* g_1 - A_2^* g_2|^2 \right) d\sigma_1^*(g_1) d\sigma_2^*(g_2).$$

is analytic on $\mathbb{C}_+$, applying the Fubini theorem and the Cauchy integration theorem, it follows that

$$\begin{aligned}
 & \int_{\Delta} J_{(F_{\sigma_1}^* F_{\sigma_2}^*)}^{*(C_1, C_2; \cdot, \cdot)}(A_1, A_2; \lambda) d\lambda \\
 & = \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} \exp \left(i(C_1^* g_1 + C_2^* g_2, y) \right) \left(\int_{\Delta} \Xi(\lambda) d\lambda \right) d\sigma_1^*(g_1) d\sigma_2^*(g_2) = 0
 \end{aligned}$$

for all rectifiable simple closed curve Δ lying in $\mathbb{C}_+$. Thus, by the Morera theorem, $J_{(F_{\sigma_1}^* F_{\sigma_2}^*)}^{*(C_1, C_2; \cdot, \cdot)}(A_1, A_2; \lambda)$ is an analytic function of λ throughout $\mathbb{C}_+$ and is bounded as a function of λ in $\tilde{\mathbb{C}}_+$. Therefore the analytic Wiener integral

$$(F_{\sigma_1}^* F_{\sigma_2}^*)_{\lambda}^{(C_1, C_2; A_1, A_2)}(y) = J_{(F_{\sigma_1}^* F_{\sigma_2}^*)}^{*(C_1, C_2; \cdot, \cdot)}(A_1, A_2; \lambda)$$

exists for all $\lambda \in \mathbb{C}_+$. Finally, in view of Definition 2 and by the bounded convergence theorem (the use of which is justified by (14)), the CP $(F_{\sigma_1}^* F_{\sigma_2}^*)_{q}^{(C_1, C_2; A_1, A_2)}$ of F_{σ_1} and F_{σ_2} exists and is given by the right-hand side of (13) for all $q \in \mathbb{R} \setminus \{0\}$ and s-a.e. $y \in \mathbb{B}$. $\square$

Theorem 3. Let F_{σ_1} , F_{σ_2} , C_1 , C_2 , A_1 , and A_2 be as in Theorem 2. Then for each non-zero real number q , the CP $(F_{\sigma_1}^* F_{\sigma_2}^*)_{q}^{(C_1, C_2; A_1, A_2)}$ is in the Banach algebra $\mathcal{F}(\mathbb{B}^*)$.

Proof. Define a set function $(\sigma_1^* * \sigma_2^*)_{q}^{(A_1, A_2; \sigma_1^*, \sigma_2^*)} : \mathcal{B}(\mathbb{B}^* \times \mathbb{B}^*) \rightarrow \mathbb{C}$ by the formula

$$(\sigma_1^* * \sigma_2^*)_{q}^{(A_1, A_2; \sigma_1^*, \sigma_2^*)}(V) = \iint_V \exp \left(-\frac{i}{2q} |A_1^* g_1 - A_2^* g_2|^2 \right) d\sigma_1^*(g_1) d\sigma_2^*(g_2) \tag{15}$$

for V in $\mathcal{B}(\mathbb{B}^* \times \mathbb{B}^*)$, the Borel σ -field on $\mathbb{B}^* \times \mathbb{B}^*$. Then $(\sigma_1^* * \sigma_2^*)_{q}^{(A_1, A_2; \sigma_1^*, \sigma_2^*)}$ is a complex measure with finite total variation on $\mathcal{B}(\mathbb{B}^* \times \mathbb{B}^*)$. Next, let $\phi^{(C_1, C_2)} : \mathbb{B}^* \times \mathbb{B}^* \rightarrow \mathbb{B}^*$ be the continuous function given by

$$\phi^{(C_1, C_2)}(g_1, g_2) = C_1^* g_1 + C_2^* g_2.$$

Then $\phi^{(C_1, C_2)}$ is $\mathcal{B}(\mathbb{B}^* \times \mathbb{B}^*)$ - $\mathbb{B}^*$ -measurable. Finally, let the set function $(\sigma_1^* * \sigma_2^*)_{c, q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)} : \mathcal{B}(\mathbb{B}^*) \rightarrow \mathbb{C}$ be given by

$$(\sigma_1^* * \sigma_2^*)_{c, q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)} = (\sigma_1^* * \sigma_2^*)_{q}^{(A_1, A_2; \sigma_1^*, \sigma_2^*)} \circ (\phi^{(C_1, C_2)})^{-1}. \tag{16}$$

Then $(\sigma_1^* * \sigma_2^*)_{c,q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)}$ satisfies the countable additivity, obviously. Based on these structure above, it follows that

$$\begin{aligned} & \|(\sigma_1^* * \sigma_2^*)_{c,q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)}\| \\ & \leq \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} \left| \exp \left(-\frac{i}{2q} |A_1^* g_1 - A_2^* g_2|_2^2 \right) \right| d|\sigma_1^*|(g_1) d|\sigma_2^*|(g_2) \\ & = \left(\int_{\mathbb{B}^*} |\sigma_1^*|(g_1) \right) \left(\int_{\mathbb{B}^*} |\sigma_2^*|(g_2) \right) = \|\sigma_1^*\| \|\sigma_2^*\| < +\infty. \end{aligned}$$

Thus the set function $(\sigma_1^* * \sigma_2^*)_{c,q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)}$ is a complex measure on $\mathcal{B}(\mathbb{B}^*)$. Using equation (13) together with the complex measure $(\sigma_1^* * \sigma_2^*)_{c,q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)}$ given by (16), it follows that

$$\begin{aligned} & (F_{\sigma_1} * F_{\sigma_2})_q^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)}(y) \\ & = \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} \exp(i(C_1^* g_1 + C_2^* g_2, y)) d(\sigma_1^* * \sigma_2^*)_q^{(A_1, A_2; \sigma_1^*, \sigma_2^*)}(g_1, g_2) \\ & = \int_{\mathbb{B}} \exp(i(g, y)) d(\sigma_1^* * \sigma_2^*)_q^{(A_1, A_2; \sigma_1^*, \sigma_2^*)} \circ (\phi^{(C_1, C_2)})^{-1}(g) \\ & = \int_{\mathbb{B}^*} \exp(i(g, y)) d(\sigma_1^* * \sigma_2^*)_{c,q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)}(g) \\ & = F_{(\sigma_1^* * \sigma_2^*)_{c,q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)}}(y) \end{aligned} \quad (17)$$

for s-a.e. $y \in \mathbb{B}$. Hence the CP $(F_{\sigma_1} * F_{\sigma_2})_q^{(C_1, C_2; A_1, A_2)}$ belongs to the Banach algebra $\mathcal{F}(\mathbb{B}^*)$. $\square$

6. Relationships Between the Fourier–Feynman Transform and the Convolution Product

In this section, we provide three kinds of relationships between the FFT and the CP associated with BLOPs on AWS $\mathbb{B}$. In order to obtain our relationships between the FFT and the CP, we quote the following three expositions for BLOPs on $\mathbb{B}$ from [22].

(O1) Let A be an operator in $\mathcal{L}(\mathbb{B})$ such that $A(\mathbb{H}) \subseteq \mathbb{H}$. Then A is an element of $\mathcal{L}(\mathbb{H})$. Let $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H}) = \{A \in \mathcal{L}(\mathbb{B}) : A(\mathbb{H}) \subset \mathbb{H}\}$. Then the class $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$ is a linear space. For any operator A in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$, AA^* is positive definite on $\mathbb{H}$. Thus, by the square root lemma [29], there exists a positive operator $|A|$ on $\mathbb{H}$ such that $|A| = \sqrt{AA^*}$. We also note that every bounded positive definite operator on a Hilbert space is self-adjoint.

(O2) Given BLOPs A_1 and A_2 in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$, the operator $A_1 A_1^* + A_2 A_2^*$ is positive definite on $\mathbb{H}$. Thus, by the square root lemma, there is an operator $\sqrt{A_1 A_1^* + A_2 A_2^*}$, uniquely, in $\mathcal{L}(\mathbb{H})$. It is clear that the operator $\sqrt{A_1 A_1^* + A_2 A_2^*}$ is in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$.

In order to identify these operators, we consider the relation $\overset{\text{op}}{\sim}$ on $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$ given by

$$A_1 \overset{\text{op}}{\sim} A_2 \iff A_1 A_1^* = A_2 A_2^* \text{ on } \mathbb{H}.$$

Then $\overset{\text{op}}{\sim}$ is an equivalence relation. Let $[A]$ denote the equivalence class of an operator A in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$. In view of the observation (O1), it follows that there exists a positive definite operator $\mathfrak{S}(A)$ such that $A \overset{\text{op}}{\sim} \mathfrak{S}(A)$.

Given two operators A_1 and A_2 in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$, we will use the symbol ' $\mathfrak{S}(A_1, A_2)$ ' to indicate the representative element of the equivalence class

$$[\mathfrak{S}(A_1, A_2)] = \left\{ \mathfrak{S} \in \mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H}) : \mathfrak{S} \overset{\text{op}}{\sim} \sqrt{A_1 A_1^* + A_2 A_2^*} \text{ on } \mathbb{H} \right\}.$$

Then, in view of (O2), one can see that for any $\mathfrak{S}$ in $[\mathfrak{S}(A_1, A_2)]$ and all $g \in \mathbb{B}^*$,

$$|\mathfrak{S}^*g|^2 = (\mathfrak{S}^*g, \mathfrak{S}^*g) = (g, \mathfrak{S}\mathfrak{S}^*g) = (g, (A_1A_1^* + A_2A_2^*)g).$$

Throughout the remainder of this paper, we will regard $[\mathfrak{S}(A_1, A_2)] \equiv \mathfrak{S}(A_1, A_2)$, as an operator in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$, for the convenience of notation. Then we have that

$$\mathfrak{S}(A_1, A_2)\mathfrak{S}(A_1, A_2)^* = A_1A_1^* + A_2A_2^*.$$

(O3) Given a finite sequence $\mathcal{O} = (A_1, \dots, A_n)$ of operators in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$, let $\mathfrak{S}(\mathcal{O}) \equiv \mathfrak{S}(A_1, A_2, \dots, A_n)$ be the positive operators $\mathfrak{S}$ which satisfy the relation

$$\mathfrak{S}\mathfrak{S}^* = A_1A_1^* + \dots + A_nA_n^* \text{ on } \mathbb{H}.$$

By an induction argument, it follows that

$$\mathfrak{S}(\mathfrak{S}(A_1, A_2, \dots, A_{k-1}), A_k) = \mathfrak{S}(A_1, A_2, \dots, A_k)$$

for all $k \in \{2, \dots, n\}$. Also, for any permutation π of $\{1, \dots, n\}$, it also follows that

$$\mathfrak{S}(A_1, A_2, \dots, A_n) = \mathfrak{S}(A_{\pi(1)}, A_{\pi(2)}, \dots, A_{\pi(n)}).$$

For a notational convenience, we adopt the following notation: for A_{11}, A_{12}, A_{21} , and A_{22} in $\mathcal{L}(\mathbb{B})$, we let

$$\det \begin{pmatrix} A_{11} & A_{12} \\ A_{21}^* & A_{22}^* \end{pmatrix} = A_{11}A_{22}^* - A_{12}A_{21}^*$$

as the determinant formula by the cofactor expansion along the first row.

Under these expositions, we will provide three kinds of relationships between the FFT and the CP for functionals in the Banach algebra $\mathcal{F}(\mathbb{B}^*)$.

Our first theorem in this section shows that a FFT of a CP of functionals in the class $\mathcal{F}(\mathbb{B}^*)$ is a product of FFTs of each functional, under an appropriate hypothesis on BLOPs in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$.

Lemma 1 ([22]). Let $F_\sigma \in \mathcal{F}(\mathbb{B}^*)$ be given by (9), and let C be an operator in $\mathcal{L}(\mathbb{B})$. Then the functional F_σ^C given by $F_\sigma^C(x) = F_\sigma(Cx)$ belongs to the Banach algebra $\mathcal{F}(\mathbb{B}^*)$.

Theorem 4. Let F_{σ_1} and F_{σ_2} be functionals in $\mathcal{F}(\mathbb{B}^*)$, let C_1, C_2, A, A_1 , and A_2 be operators in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$. Suppose that

$$\det \begin{pmatrix} C_1A & A_1 \\ A_2^* & (C_2A)^* \end{pmatrix} = O$$

where O denotes the trivial operator on $\mathbb{B}$. Then for all real $q \in \mathbb{R} \setminus \{0\}$, it follows that

$$\begin{aligned} & T_{q,A}^{(1)} \left((F_{\sigma_1} * F_{\sigma_2})_q^{(C_1, C_2; A_1, A_2)} \right) (y) \\ &= T_{q, \mathfrak{S}(C_1A, A_1)}^{(1)} (F_{\sigma_1}) (C_1y) T_{q, \mathfrak{S}(C_2A, A_2)}^{(1)} (F_{\sigma_2}) (C_2y) \end{aligned} \quad (18)$$

for s-a.e. $y \in \mathbb{B}$, where $\mathfrak{S}(C_1A, A_1)$ and $\mathfrak{S}(C_2A, A_2)$ are operators in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$ which satisfy the relations

$$\mathfrak{S}(C_1A, A_1)\mathfrak{S}(C_1A, A_1)^* = C_1A(C_1A)^* + A_1A_1^* \quad (19)$$

and

$$\mathfrak{S}(C_2A, A_2)\mathfrak{S}(C_2A, A_2)^* = C_2A(C_2A)^* + A_2A_2^* \quad (20)$$

respectively.

Proof. Since the CP $(F_{\sigma_1} * F_{\sigma_2})_q^{(C_1, C_2; A_1, A_2)}$ of F_{σ_1} and F_{σ_2} is an element of $\mathcal{F}(\mathbb{B}^*)$ by Theorem 3, the L_1 analytic FFT $T_{q,A}^{(1)}((F_{\sigma_1} * F_{\sigma_2})_q^{(C_1, C_2; A_1, A_2)})$ of $(F_{\sigma_1} * F_{\sigma_2})_q^{(C_1, C_2; A_1, A_2)}$ in the left-hand side of (18) exists for all $q \in \mathbb{R} \setminus \{0\}$ in view of Theorems 1. Also, by Lemma 1 and Theorem 1, both FFTs, $T_{q,\mathfrak{S}(C_1 A, A_1)}^{(1)}(F_{\sigma_1})(C_1 \cdot)$ and $T_{q,\mathfrak{S}(C_2 A, A_2)}^{(1)}(F_{\sigma_2})(C_2 \cdot)$ in the right-hand side of (18) exist for all $q \in \mathbb{R} \setminus \{0\}$. Thus it suffices to show that the equality in (18) holds true.

Using (10) with F_σ replaced with $(F_{\sigma_1} * F_{\sigma_2})_q^{(C_1, C_2; A_1, A_2)}$, (11) with σ^* replaced with $(\sigma_1^* * \sigma_2^*)_{c,q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)}$ given by (16), and the fourth expression of (17), it follows that

$$\begin{aligned} & T_{q,A}^{(1)}\left((F_{\sigma_1} * F_{\sigma_2})_q^{(C_1, C_2; A_1, A_2)}\right)(y) \\ &= \int_{\mathbb{B}^*} \exp(i(g, y)) d\left((\sigma_1^* * \sigma_2^*)_{c,q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)}\right)_A^A(g) \end{aligned} \quad (21)$$

where $((\sigma_1^* * \sigma_2^*)_{c,q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)})_A^A$ is the complex measure on $\mathcal{B}(\mathbb{B}^*)$ given by

$$\begin{aligned} & \left((\sigma_1^* * \sigma_2^*)_{c,q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)}\right)_A^A(U) \\ &= \int_U \exp\left(-\frac{i}{2q}(g, AA^*g)\right) d(\sigma_1^* * \sigma_2^*)_{c,q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)}(g) \end{aligned} \quad (22)$$

for $U \in \mathcal{B}(\mathbb{B}^*)$. Thus, using (21), (22), (16), and (15), it follows that

$$\begin{aligned} & T_{q,A}^{(1)}\left((F_{\sigma_1} * F_{\sigma_2})_q^{(C_1, C_2; A_1, A_2)}\right)(y) \\ &= \int_{\mathbb{B}^*} \exp\left(i(g, y) - \frac{i}{2q}(g, AA^*g)\right) d(\sigma_1^* * \sigma_2^*)_{c,q}^{(C_1, C_2; A_1, A_2; \sigma_1^*, \sigma_2^*)}(g) \\ &= \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} \exp\left(i(C_1^*g_1 + C_2^*g_2, y) - \frac{i}{2q}(C_1^*g_1 + C_2^*g_2, AA^*(C_1^*g_1 + C_2^*g_2))\right) d(\sigma_1^* * \sigma_2^*)_q^{(A_1, A_2; \sigma_1^*, \sigma_2^*)}(g) \\ &= \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} \exp\left(i(C_1^*g_1 + C_2^*g_2, y) - \frac{i}{2q}\left[(C_1^*g_1 + C_2^*g_2, AA^*(C_1^*g_1 + C_2^*g_2)) + |A_1^*g_1 - A_2^*g_2|^2\right]\right) d\sigma_1^*(g_1) d\sigma_2^*(g_2) \end{aligned}$$

for s-a.e. $y \in \mathbb{B}$. Next, in view of (7), we observe that

$$\begin{aligned} & (C_1^*g_1 + C_2^*g_2, AA^*(C_1^*g_1 + C_2^*g_2)) + |A_1^*g_1 - A_2^*g_2|^2 \\ &= ((C_1A)^*g_1 + (C_2A)^*g_2, (C_1A)^*g_1 + (C_2A)^*g_2) \\ &\quad + (A_1^*g_1 - A_2^*g_2, A_1^*g_1 - A_2^*g_2) \\ &= (g_1, C_1A(C_1A)^*g_1) + (g_2, C_2A(C_2A)^*g_2) + 2((C_1A)^*g_1, (C_2A)^*g_2) \\ &\quad + (g_1, A_1A_1^*g_1) + (g_2, A_2A_2^*g_2) - 2(A_1^*g_1, A_2^*g_2) \\ &= (g_1, \{C_1A(C_1A)^* + A_1A_1^*\}g_1) + (g_2, \{C_2A(C_2A)^* + A_2A_2^*\}g_2) \\ &\quad + 2(g_1, \{C_1A(C_2A)^* - A_1A_2^*\}g_2). \end{aligned}$$

But, by the hypothesis, we obtain that

$$\begin{aligned} & (C_1^*g_1 + C_2^*g_2, AA^*(C_1^*g_1 + C_2^*g_2)) + |A_1^*g_1 - A_2^*g_2|^2 \\ &= (g_1, \mathfrak{S}(C_1A, A_1)\mathfrak{S}(C_1A, A_1)^*g_1) + (g_2, \mathfrak{S}(C_2A, A_2)\mathfrak{S}(C_2A, A_2)^*g_2). \end{aligned}$$

Using this and the Fubini theorem, we have that

$$\begin{aligned} & T_{q,A}^{(1)} \left((F_{\sigma_1} * F_{\sigma_2})_q^{(C_1, C_2; A_1, A_2)} \right) (y) \\ &= \int_{\mathbb{B}^*} \exp \left(i(C_1^* g_1, y) - \frac{i}{2q} (g_1, \mathfrak{S}(C_1 A, A_1) \mathfrak{S}(C_1 A, A_1)^* g_1) \right) d\sigma_1^*(g_1) \\ & \quad \times \int_{\mathbb{B}^*} \exp \left(i(C_2^* g_2, y) - \frac{i}{2q} (g_2, \mathfrak{S}(C_2 A, A_2) \mathfrak{S}(C_2 A, A_2)^* g_2) \right) d\sigma_2^*(g_2) \\ &= T_{q, \mathfrak{S}(C_1 A, A_1)}^{(1)}(F_{\sigma_1})(C_1 y) T_{q, \mathfrak{S}(C_2 A, A_2)}^{(1)}(F_{\sigma_2})(C_2 y) \end{aligned}$$

for s-a.e. $y \in \mathbb{B}$. $\square$

Our next theorem tells us that under same hypothesis given in Theorem 4, a CP of FFTs of functionals in the class $\mathcal{F}(\mathbb{B}^*)$ equals a FFT of product of the functionals.

Theorem 5. Let $F_{\sigma_1}, F_{\sigma_2}, C_1, C_2, A_1$ and A_2 be as in Theorem 4. Then, under same hypothesis as for Theorem 4, it follows that for all $q \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} & \left(T_{q, \mathfrak{S}(C_1 A, A_1)}^{(1)}(F_{\sigma_1}) * T_{q, \mathfrak{S}(C_2 A, A_2)}^{(1)}(F_{\sigma_2}) \right)_{-q}^{(C_1, C_2; A_1, A_2)} (y) \\ &= T_{q,A}^{(1)} \left(F_{\sigma_1}(C_1 \cdot) F_{\sigma_2}(C_2 \cdot) \right) (y) \end{aligned} \quad (23)$$

for s-a.e. $y \in \mathbb{B}$, where $\mathfrak{S}(A, A_1)$ and $\mathfrak{S}(A, A_2)$ are the BLOPs in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$ which satisfy the relation (19) and (20) respectively.

Proof. By Theorems 1 and 3, the FFTs and the CP in the left-hand side of (23) exist. By Lemma 1 and the fact that the class $\mathcal{F}(\mathbb{B}^*)$ is a Banach algebra, the FFT in the right-hand side of (23) also exists. Thus equality in (23) is what needs to be shown.

Applying (12), (18) with $F_{\sigma_1}, F_{\sigma_2}$, and q replaced with $T_{q, \mathfrak{S}(C_1 A, A_1)}^{(1)}(F_{\sigma_1}), T_{q, \mathfrak{S}(C_2 A, A_2)}^{(1)}(F_{\sigma_2})$ and $-q$ respectively, and (12) again, it follows that for s-a.e. $y \in \mathbb{B}$,

$$\begin{aligned} & \left(T_{q, \mathfrak{S}(C_1 A, A_1)}^{(1)}(F_{\sigma_1}) * T_{q, \mathfrak{S}(C_2 A, A_2)}^{(1)}(F_{\sigma_2}) \right)_{-q}^{(C_1, C_2; A_1, A_2)} (y) \\ &= T_{q,A}^{(1)} \left(T_{-q,A}^{(1)} \left(\left(T_{q, \mathfrak{S}(C_1 A, A_1)}^{(1)}(F_{\sigma_1}) * T_{q, \mathfrak{S}(C_2 A, A_2)}^{(1)}(F_{\sigma_2}) \right)_{-q}^{(C_1, C_2; A_1, A_2)} \right) \right) (y) \\ &= T_{q,A}^{(1)} \left(T_{-q, \mathfrak{S}(C_1 A, A_1)}^{(1)} \left(T_{q, \mathfrak{S}(C_1 A, A_1)}^{(1)}(F_{\sigma_1}) \right) (C_1 \cdot) \right. \\ & \quad \times \left. T_{-q, \mathfrak{S}(C_2 A, A_2)}^{(1)} \left(T_{q, \mathfrak{S}(C_2 A, A_2)}^{(1)}(F_{\sigma_2}) \right) (C_2 \cdot) \right) (y) \\ &= T_{q,A}^{(1)} \left(F_{\sigma_1}(C_1 \cdot) F_{\sigma_2}(C_2 \cdot) \right) (y) \end{aligned}$$

as desired. $\square$

Example 1. The FFT $T_q^{(1)} \equiv T_{q,A}^{(1)}$ with $A = I$ (the identity operator on $\mathbb{B}$) and the CP $(\cdot * \cdot)_q \equiv (\cdot * \cdot)_q^{(C_1, C_2; A_1, A_2)}$ with $C_1 = C_2 = A_1 = A_2 = \frac{1}{\sqrt{2}}I$ was developed in [7–10, 12–14]. Under these setting, equations (18) and (23) yield the formulas

$$T_q^{(1)}((F_{\sigma_1} * F_{\sigma_2})_q)(y) = T_q^{(1)}(F_{\sigma_1})\left(\frac{y}{\sqrt{2}}\right) T_q^{(1)}(F_{\sigma_2})\left(\frac{y}{\sqrt{2}}\right)$$

and

$$(T_q^{(1)}(F_{\sigma_1}) * T_q^{(1)}(F_{\sigma_2}))_{-q}(y) = T_q^{(1)}\left(F_{\sigma_1}\left(\frac{\cdot}{\sqrt{2}}\right) F_{\sigma_2}\left(\frac{\cdot}{\sqrt{2}}\right)\right)(y)$$

for s-a.e. $y \in \mathbb{B}$, respectively.

Our next theorem also enables us that under another condition on BLOPs in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$, a CP of FFTs equals a product of FFTs.

Theorem 6. Let F_{σ_1} and F_{σ_2} be functionals in $\mathcal{F}(\mathbb{B}^*)$, let C_1, C_2, G_1, G_2, A_3 , and A_4 be operators in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$. Suppose that

$$\det \begin{pmatrix} A_3 & O \\ O & A_4^* \end{pmatrix} = O.$$

Then for all real $q \in \mathbb{R} \setminus \{0\}$, it follows that

$$\begin{aligned} & \left(T_{q,G_1}^{(1)}(F_{\sigma_1}) * T_{q,G_2}^{(1)}(F_{\sigma_2}) \right)_q^{(C_1, C_2; A_3, A_4)}(y) \\ &= T_{q, \mathfrak{S}(G_1, A_3)}^{(1)}(F_{\sigma_1})(C_1 y) T_{q, \mathfrak{S}(G_2, A_4)}^{(1)}(F_{\sigma_2})(C_2 y) \end{aligned}$$

for s-a.e. $y \in \mathbb{B}$, where $\mathfrak{S}(G_1, A_3)$ and $\mathfrak{S}(G_2, A_4)$ are operators in $\mathcal{L}(\mathbb{B}) \cap \mathcal{L}(\mathbb{H})$ which satisfy the relation

$$\mathfrak{S}(G_1, A_3) \mathfrak{S}(G_1, A_3)^* = G_1 G_1^* + A_3 A_3^*$$

and

$$\mathfrak{S}(G_2, A_4) \mathfrak{S}(G_2, A_4)^* = G_2 G_2^* + A_4 A_4^*$$

respectively.

Proof. Applying (10) and (11), we obtain that for each $j \in \{1, 2\}$,

$$T_{q,G_j}^{(1)}(F_{\sigma_j})(y) = \int_{\mathbb{B}^*} \exp(i(g, y)) d(\sigma_j^*)_{t,q}^{G_j}(g)$$

for s-a.e. $y \in \mathbb{B}$, where $(\sigma_j^*)_{t,q}^{G_j}$ is the complex measure on $\mathbb{B}^*$ given by

$$(\sigma_j^*)_{t,q}^{G_j}(U) = \int_U \exp\left(-\frac{i}{2q}(g, G_j G_j^* g)\right) d\sigma_j^*(g), \quad U \in \mathcal{B}(\mathbb{B}^*),$$

for each $j \in \{1, 2\}$. Using equation (13) with F_{σ_1} and F_{σ_2} replaced with $T_{q,G_1}^{(1)}(F_{\sigma_1})$ and $T_{q,G_2}^{(1)}(F_{\sigma_2})$, it follows that

$$\begin{aligned} & \left(T_{q,G_1}^{(1)}(F_{\sigma_1}) * T_{q,G_2}^{(1)}(F_{\sigma_2}) \right)_q^{(C_1, C_2; A_3, A_4)}(y) \\ &= \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} \exp\left(i(C_1^* g_1 + C_2^* g_2, y) - \frac{i}{2q}|A_3^* g_1 - A_4^* g_2|^2\right) d(\sigma_1^*)_{t,q}^{G_1}(g_1) d(\sigma_2^*)_{t,q}^{G_2}(g_2) \\ &= \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} \exp\left(i(C_1^* g_1 + C_2^* g_2, y) - \frac{i}{2q}|A_3^* g_1 - A_4^* g_2|^2\right. \\ & \quad \left. - \frac{i}{2q}(g_1, G_1 G_1^* g_1) - \frac{i}{2q}(g_2, G_2 G_2^* g_2)\right) d\sigma_1^*(g_1) d\sigma_2^*(g_2) \end{aligned}$$

for s-a.e. $y \in \mathbb{B}$. By the hypothesis and the Fubini theorem, we have that

$$\begin{aligned} & \left(T_{q,G_1}^{(1)}(F_{\sigma_1}) * T_{q,G_2}^{(1)}(F_{\sigma_2}) \right)_q^{(C_1, C_2; A_3, A_4)}(y) \\ &= \int_{\mathbb{B}^*} \int_{\mathbb{B}^*} \exp \left(i(C_1^* g_1 + C_2^* g_2, y) - \frac{i}{2q} (g_1, (G_1 G_1^* + A_3 A_3^*) g_1) \right. \\ & \quad \left. - \frac{i}{2q} (g_2, (G_2 G_2^* + A_4 A_4^*) g_2) \right) d\sigma_1^*(g_1) d\sigma_2^*(g_2) \\ &= \int_{\mathbb{B}^*} \exp \left(i(C_1^* g_1, y) - \frac{i}{2q} (g_1, (G_1 G_1^* + A_3 A_3^*) g_1) \right) d\sigma_1^*(g_1) \\ & \quad \times \int_{\mathbb{B}^*} \exp \left(i(C_2^* g_2, y) - \frac{i}{2q} (g_2, (G_2 G_2^* + A_4 A_4^*) g_2) \right) d\sigma_2^*(g_2) \\ &= T_{q, \mathfrak{S}(G_1, A_3)}^{(1)}(F_{\sigma_1})(C_1 y) T_{q, \mathfrak{S}(G_2, A_4)}^{(1)}(F_{\sigma_2})(C_2 y) \end{aligned}$$

for s-a.e. $y \in \mathbb{B}$. $\square$

7. Example of Bounded Linear Operators on the Classical Wiener Space and a Concluding Remark

Let $\mathbb{B} = C_0[0, T]$ be the one-parameter Wiener space (i.e., the Banach space of all real-valued continuous functions x on a time interval $[0, T]$ with $x(0) = 0$ with the supremum norm $\|x\| = \sup_{t \in [0, T]} |x(t)|$). The classical Wiener measure m_w is characterized by

$$m_w(\{x : x(t) \leq a\}) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^a \exp\left(-\frac{u^2}{2t}\right) du$$

for $t \in (0, T]$. Let $\mathbb{H} = C'_0[0, T]$ be the Cameron–Martin space in $\mathbb{B} = C_0[0, T]$, namely, the space of all functions $h \in C_0[0, T]$ such that h is absolutely continuous and the derivative $Dh \equiv dh/dt$ is of class $L_2[0, T]$. The inner product on $\mathbb{H} = C'_0[0, T]$ is given by

$$\langle h_1, h_2 \rangle_{C'_0} = \int_0^T Dh_1(t) Dh_2(t) dt.$$

It is well-known that the space $\mathbb{B}^* = C_0^*[0, T]$ can be identified as

$$C_0^*[0, T] = \{\theta \in C'_0[0, T] : D\theta \text{ is a right continuous function of bounded variation on } [0, T]\}.$$

Then we see that $(C_0[0, T])^* \subset (C'_0[0, T])^* \cong C'_0[0, T] \subset C_0[0, T]$ in the sense of triple (4) above and $C_0^*[0, T] \cong (C_0[0, T])^*$. The one-parameter Wiener space $(C'_0[0, T], C_0[0, T], m_w)$ is one of the most important examples of AWSs. For more details, see [23, 24, 27].

Let $\mathcal{W}(C_0[0, T])$ denote the class of all Wiener measurable subsets of $C_0[0, T]$. Let $\mathcal{B}(C_0[0, T])$ be the Borel σ -field on the metric space $(C_0[0, T], \|\cdot\|)$. It is well-known that the m_w -Carathéodory completion $\sigma(\mathcal{B}(C_0[0, T]))$ is equal to the σ -field $\mathcal{W}(C_0[0, T])$. Then $(C_0[0, T], \mathcal{W}(C_0[0, T]), m_w)$ is a complete measure space.

In [17, 18], FFTs associated with Gaussian processes $\mathcal{Z}_h$ for functionals on the one-parameter Wiener space $(C_0[0, T], \mathcal{W}(C_0[0, T]), m_w)$, and related topics were studied. For the definition of the processes $\mathcal{Z}_h$ on $C_0[0, T] \times [0, T]$, see equation (25) below. These results involving the FFT associated with the Gaussian process are extended in [16, 19, 20]. In this section we will verify that the Gaussian processes $\mathcal{Z}_h$ given by (25) are BLOPs on the Banach space $C_0[0, T]$.

Let U be the unitary operator from $L_2[0, T]$ onto $C'_0[0, T]$ given by $Uv(t) = \int_0^t v(s) ds$ for $v \in L_2[0, T]$. Let $BV_{rc}[0, T]$ be the space of real-valued functions on $[0, T]$ which are right continuous and are of bounded variation on $[0, T]$. Then it is well-known that $C_0^*[0, T] = \{Uv : v \in BV_{rc}[0, T]\}$

and $v(T) = 0$. For any $h \in C'_0[0, T]$ and $g \in C_0^*[0, T]$, let the operation $\odot$ between $C'_0[0, T]$ and $C_0^*[0, T]$ be defined by

$$h \odot g = U(DhDg) \quad (24)$$

where $DhDg$ denotes the pointwise multiplication of the functions Dh and Dg . Then $(C_0^*[0, T], \odot)$ is a commutative algebra with the identity $e : [0, T] \rightarrow \mathbb{R}$ given by $e(t) = t$.

Note that if $\{g_n\}_{n=1}^\infty$ is a complete orthonormal set of functions in $C'_0[0, T]$, each of whose derivatives is in $BV_{rc}[0, T]$. Then the sequence $\{Dg_n\}_{n=1}^\infty$ is a complete orthonormal set of functions in $L_2[0, T]$ and the stochastic linear functional $(h, x)^\sim$ defined by (5) on $C'_0[0, T] \times C_0[0, T]$ equals the Paley–Wiener–Zygmund(PWZ) stochastic integral $\int_0^T Dh(t)\tilde{d}x(t)$ for each $h \in C'_0[0, T]$ and m_w -a.e. $x \in C_0[0, T]$, see [27,30,31]. Furthermore, if $h \in C_0^*[0, T]$, the PWZ stochastic integral $(h, x)^\sim \equiv \int_0^T Dh(t)\tilde{d}x(t)$ is equal to the Riemann–Stieltjes integral $\int_0^T Dh(t)dx(t)$.

Given a function h in $C'_0[0, T]$, we consider the stochastic integral $\mathcal{Z}_h(x, t)$ given by

$$\mathcal{Z}_h(x, t) = \int_0^t \chi_{[0,t]}(\tau) Dh(\tau)\tilde{d}x(\tau) \equiv \int_0^t Dh(\tau)\tilde{d}x(\tau) \quad (25)$$

for $x \in C_0[0, T]$ and $t \in [0, T]$. Then the process $\mathcal{Z}_h$ on $C_0[0, T] \times [0, T]$ is a Gaussian process with mean zero and covariance function

$$\int_{C_0[0,T]} \mathcal{Z}_h(x, s)\mathcal{Z}_h(x, t)dm_w(x) = \int_0^{\min\{s,t\}} [Dh(\tau)]^2 d\tau.$$

For more details, see ([32], p.157). Furthermore one can see that

$$\int_{C_0[0,T]} \mathcal{Z}_{h_1}(x, s)\mathcal{Z}_{h_2}(x, t)dm_w(x) = \int_0^{\min\{s,t\}} Dh_1(\tau)Dh_2(\tau)d\tau.$$

Since the covariance function of $\mathcal{Z}_h(x, \cdot)$ is stochastically continuous ([33], Theorem 21.1), we may assume that almost every sample path of $\mathcal{Z}_h$ is in $C_0[0, T]$. But, if h is in the dual space $C_0^*[0, T]$ of $C_0[0, T]$, then for all $x \in C_0[0, T]$, $\mathcal{Z}_h(x, t)$ is continuous in $t \in [0, T]$ (i.e., $\mathcal{Z}_h$ is a continuous process on $C_0[0, T] \times [0, T]$). Hence $\mathcal{Z}_h(x, \cdot)$ is in $C_0[0, T]$ for any $h \in C_0^*[0, T]$ and $x \in C_0[0, T]$.

We note that the PWZ stochastic integral $\int_0^t Dg(s)d\mathcal{Z}_h(x, s)$ has the kernel exchange properties as follows: given functions g, k, k_1 and k_2 in $C_0^*[0, T]$,

$$\int_0^t Dg(s)d\mathcal{Z}_k(x, s) = \int_0^t Dg(s)Dk(s)dx(s) \quad (26)$$

and

$$\begin{aligned} \int_0^t Dg(s)d\mathcal{Z}_{k_2}(\mathcal{Z}_{k_1}(x, \cdot), s) &= \int_0^t Dg(s)Dk_1(s)Dk_2(s)dx(s) \\ &= \int_0^t Dg(s)d\mathcal{Z}_{k_1 \odot k_2}(x, s). \end{aligned} \quad (27)$$

Given $k \in C_0^*[0, T]$ and $x \in C_0[0, T]$, we denote the Stieltjes integral $\int_0^T Dk(t)dx(t)$ by the symbol (k, x) . Then the symbol $(\cdot, \cdot)$ on $C_0^*[0, T] \times C_0[0, T]$ is a bilinear form. Also, in view of the Riesz representation theorem, the symbol $(\cdot, \cdot)$ plays as the natural dual pairing characterized by equation (2) with $\mathbb{B} = C_0[0, T]$. In this case, equations (26) and (27) with t replaced with T , respectively, can be rewritten by

$$(g, \mathcal{Z}_k(x, \cdot)) = (g \odot k, x)$$

and

$$(g, \mathcal{Z}_{k_2}(\mathcal{Z}_{k_1}(x, \cdot), \cdot)) = (g \odot k_1 \odot k_2, x)$$

for all g, k, k_1 , and k_2 in $C_0^*[0, T]$, where $\odot$ is the operation between $C_0^*[0, T]$ and $C_0'[0, T]$ defined by equation (24). In [32,34], these rotation transformations with the processes $\mathcal{Z}_k$ replaced with the operators on AWSs are introduced via white noise setting on nuclear spaces.

The following lemma follows quite easily from the definition of the Stieltjes integral.

Lemma 2. For each $\gamma \in C_0'[0, T]$ and each $g \in C_0^*[0, T]$,

$$(\gamma, \mathcal{Z}_g(x, \cdot)) \equiv \int_0^T D\gamma(t) d\mathcal{Z}_g(x, t) = \int_0^T Dh(t) Dg(t) dx(t) \equiv (\gamma \odot g, x). \quad (28)$$

for m_W -a.e. $x \in C_0[0, T]$.

In the next theorem, we show that the Gaussian processes $\mathcal{Z}_g$ are BLOPs on the classical Wiener space $C_0[0, T]$.

Theorem 7. Given $g \in C_0^*[0, T]$, define an operator $A_g^w : C_0[0, T] \rightarrow C_0[0, T]$ by

$$A_g^w x = \mathcal{Z}_g(x, \cdot). \quad (29)$$

Then A_g^w is a BLOP on the Banach space $(C_0[0, T], \|\cdot\|)$ where $\|\cdot\|$ means the supremum norm.

Proof. Since Dg is of bounded variation on $[0, T]$, it follows by an integration by parts formula that

$$\begin{aligned} \|A_g^w x\| &= \sup_{t \in [0, T]} \left| \int_0^t Dg(\tau) dx(\tau) \right| \\ &= \sup_{t \in [0, T]} \left| Dg(t)x(t) - \int_0^t x(\tau) dDg(\tau) \right| \\ &\leq \sup_{t \in [0, T]} \left(|Dg(t)| |x(t)| + \left| \int_0^t x(\tau) dDg(\tau) \right| \right) \\ &\leq \|Dg\|_\infty \|x\| + \sup_{t \in [0, T]} \int_0^t |x(\tau)| d|Dg|(\tau) \\ &\leq \|Dg\|_\infty \|x\| + \|x\| \sup_{t \in [0, T]} \int_0^t d|Dg|(\tau) \\ &\leq (\|Dg\|_\infty + |Dg|(T)) \|x\| \end{aligned}$$

where $\|\cdot\|_\infty$ means the essential supremum norm and $|Dg|(\cdot)$ denotes the total variation function of $Dg(\cdot)$. Thus A_g^w is a BLOP on the Banach space $(C_0[0, T], \|\cdot\|)$. $\square$

Remark 2. Setting g to be the identity e in the algebra $(C_0^*[0, T], \odot)$, A_e^w equals the identity operator on $C_0[0, T]$.

In Theorem 8 below, we provide a Banach space adjoint operator of the Gaussian process $\mathcal{Z}_g$ on $C_0[0, T]$.

Theorem 8. Given a function g in $C_0^*[0, T]$, define an operator

$$\odot_g : C_0^*[0, T] \rightarrow C_0^*[0, T]$$

by $\odot_g \gamma = \gamma \odot g$. Then $\odot_g$ is the Banach space adjoint of the operator A_g^w given by (29), that is, $(A_g^w)^* = \odot_g$

Proof. We notice that Dg is of bounded variation on $[0, T]$. Using (29) and (28), it follows that $(\gamma, A_g^w x) = (\gamma \odot g, x) = (\odot_g \gamma, x)$ for any $\gamma \in C_0^*(Q)$. Thus, in view of (7), the adjoint operator of A_g^w , $(A_g^w)^* : C_0^*[0, T] \rightarrow C_0^*[0, T]$ is given by $(A_g^w)^* \gamma = \odot_g \gamma$ for each $g \in C_0^*[0, T]$. $\square$

Given any functions h, g_1, g_2, h_1 , and h_2 in $C_0^*[0, T]$, we denote the L_1 analytic FFT $T_{q, A_h^w}^{(1)}$ and the CP $(\cdot * \cdot)_q^{(A_{g_1}^w / \sqrt{2}, A_{g_2}^w / \sqrt{2}, A_{h_1}^w / \sqrt{2}, A_{h_2}^w / \sqrt{2})}$ by $T_{q, h}^{(1)}$ and

$$(\cdot * \cdot)_q^{(g_1, g_2, h_1, h_2)}, \quad (30)$$

respectively. In (30), choosing g_1 and g_2 to be the identity e in the commutative algebra $(C_0^*[0, T], \odot)$, the CP $(\cdot * \cdot)_q^{(e, e, h_1, h_2)}$, of s.i.m. functionals F and G on $C_0[0, T]$,

$$(F * G)_q^{(e, e, h_1, h_2)}(y) = \int_{C_0[0, T]} F\left(\frac{y + \mathcal{Z}_{h_1}(x, \cdot)}{\sqrt{2}}\right) G\left(\frac{y - \mathcal{Z}_{h_2}(x, \cdot)}{\sqrt{2}}\right) dm_w(x),$$

reduces to the CP studied in [17–20]. Also, the FFT $T_{q, h}^{(1)}$ equals the FFT in [16–20]. Thus the previous results in [16–20] are immediate corollaries of the results in this paper. Also, using the parts formulas [35,36] for multi-variables functions, we can show that the Gaussian processes $\mathcal{Y}_h$ defined on the Yeh–Wiener space $C_0([0, S] \times [0, T])$ (see [21]) are also BLOPs on $C_0([0, S] \times [0, T])$, and hence our results in this paper involve the results in [21]. Finally, in view of Remarks 1 and 2, the results in [7–9] are also immediate corollaries.

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