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Article

A Fractional (q, q') Non-Extensive Information Dimension of Complex Networks

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Simple Summary: The fractional (q, q')-information dimension of complex networks was introduced. Also, a dual version of the (q, q')-entropy, called (q, q')-extropy, is proposed. The experiments reveal that the fractional (q, q')-information dimension is less than the classical one (based on the Shannon entropy) for real-world and synthetic networks

Abstract: This article introduces a new fractional approach to the concept of information dimension of complex networks, based on a (q, q')-entropy proposed in the literature. The q parameter measures how far is the number of subsystems (for a given size ε) from the mean number of overall sizes. The q' (interaction index) measures when the interactions between subsystems are greater ($q' > 1$), lesser ($q' < 1$) or equal to the interactions into these subsystems. The computations of the proposed information dimension are carried out on several real-world and synthetic complex networks. The results from the proposed information dimension are compared with those from the information dimension based on the Shannon entropy.

Keywords: complex networks; measures of information; fractional order entropy

1. Introduction

Entropy is a crucial measure of the uncertainty of the state in physical systems that would be needed to specify the state of disorder, randomness, or uncertainty in the micro-structure of the system. Due to this fact, researchers in many scientific fields have continually extended, interpreted, and applied the notion of entropy (introduced by Clausius [1] in thermodynamics).

Several generalizations of celebrated Shannon entropy, originally related to the processes information [2], have been introduced in the literature. For a deeper gather entropy measures, the reader is referred to [3–8].

Given a probability distribution $P = \{p_1, p_2, \ldots, p_N\}$ under a probability space $X = \{x_1, x_2, \ldots, x_N\}$, under P (see [9]) is generated as:

$$I = \lim_{t \to -1} \frac{d}{dt} \sum_{i=1}^{N} p_i^{-t} = -\sum_{i=1}^{N} p_i \ln p_i, \tag{1}$$

where N is the total number of (microscopic) possibilities p_i and $\sum_{i=1}^{N} p_i = 1$.

Similarly, Tsallis entropy (also called q-entropy) [10–12] is generated by the same procedure but using Jackson's q-derivative operator $D_q^t f(t) = \dfrac{f(qt) - f(t)}{(q-1)t}$, $t \neq 0$, [13], see also [9,14,15] and it is given by

$$I_T = \lim_{t \to -1} D_q^t \sum_{i=1}^{N} p_i^{-t} = -\sum_{i=1}^{N} p_i ln_q p_i, \tag{2}$$

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where $q-$logarithm is defined by

$$ln_q(p_i) = \frac{p_i^{1-q} - 1}{1-q}, \tag{3}$$

$(p_i > 0, q \in \mathbb{R}, q \neq 1, ln_1 p_i = lnp_i)$.

Tsallis entropy is connected to the Shannon entropy through the limit

$$\lim_{q \to 1} I_T = I, \tag{4}$$

which being the reason it is considered one parameter extension of Shannon entropy.

Several entropy measures have been revealed following the same procedure above and using appropriate fractional order differentiation operators on the generative function $\sum_{i=1}^{N} p_i^{-t}$ concerning variable t and then letting $t \to -1$, see for instance [16–23].

A new measure of information, called extropy, has been introduced by Lad, Sanfilippo and Agrò [24] as the dual version of Shannon entropy. In the literature, this measure of uncertainty has received considerable attention in the last years [25–27]. The entropy and the extropy of a binary distribution $(N = 2)$ are identical.

In recent years, complex networks and systems have been extensively studied since they are applied to describe a wide range of systems in many disciplines to solve practical problems [28].

In the study of the structure of complex networks has been considered the method of fractional order information dimension by combining the fractional order entropy and information dimension, see for instance [29,30] and the references given there.

This article proposes a fractional two-parameter non-extensive information dimension of complex networks based on fractional order entropy proposed in [31]. This new information dimension is computed on real-world and synthetic complex networks. The contents of the sections of the paper, besides this introduction, are described as follows. Section 2 introduces the reader to a fractional entropy measure and the information dimension of complex networks. Then, the new definitions of fractional information dimension are introduced in Section 3. Section 4 focuses on applying the new measure to several real complex networks. Finally, the findings of this study and the conclusion are given in Section 4.

2. Preliminaries

2.1. Fractional $(q, q')-$entropy

Following the same produce to obtain Shannon and Tsallis entropies, a generalized nonextensive two-parameter entropy, named fractional $(q, q')-$entropy, is developed in [31] and obtained by the action of a derivative operator already proposed by Chakrabarti and Jagannathan [32]:

$$I_{q,q'} := \lim_{t \to -1} D^t_{q,q'} \sum_{i=1}^{N} p_i^{-t} = \sum_{i=1}^{N} \frac{p_i^{q'} - p_i^{q}}{q - q'}, \tag{5}$$

where $D^t_{q,q'}$ of a function f is given by $D^t_{q,q'} f(t) = \frac{f(qt) - f(q't)}{(q-q')t}$.

Following the general idea that extropy is the complementary dual version of entropy, we present the $(q, q')-$extropy for a discrete random variable X as

$$J_{q,q'} = \sum_{i=1}^{N} \frac{(1-p_i)^{q'} - (1-p_i)^{q}}{q - q'}. \tag{6}$$

An easy computation shows that Eq. (5) can be expressed in terms of Tsallis entropy

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$$I_{q,q'} = \frac{(1-q')I_T - (1-q)I_T}{q-q'}. \quad (7)$$

The $I_{q,q'} \geq 0, \forall q, q'$ and $I_{q,q'} = \dfrac{W^{1-q} - W^{1-q'}}{q'-q}$ for $p_i = 1/W, \forall i$. Consider a system composed of two independent subsystems, A and B, with factorized probabilities $p_{i,A}$ and $p_{i,B}$ then

$$I_{q,q'} = I^A_{q,q'} + I^B_{q,q'} + (1-q)I^A_{q,q'}I^B_{q,1} + (1-q')I^B_{q,q'}I^A_{q,1}, \quad (8)$$

where, $I(q,1)$ entropy resembles Tsallis entropy Eq. (2) and $I(1,1)$ the Shannon entropy Eq. (1). Thus, $I(q,q')$ is non-additive for $q, q' \neq 1$.

2.2. *Information dimensions of complex networks*

The information dimensions measuring the topological complexity of a given network are sketched briefly. Let us consider the Shannon entropy Eq. (1); a definition of the information dimension is introduced in [33] as follows

$$d_I = -\lim_{\varepsilon \to 0} \frac{I(\varepsilon)}{\ln \varepsilon} = \lim_{\varepsilon \to 0} \frac{\sum_{i=1}^{N_b} p_i(\varepsilon) \ln p_i(\varepsilon)}{\ln \varepsilon}, \quad (9)$$

where $p_i(\varepsilon) = \dfrac{n_i(\varepsilon)}{n}$, $n_i(\varepsilon)$ is the mass of the ith box of size ε, n is the number of nodes of complex networks, and N_b is the minimum number of boxes to cover the network. The reader is referred to [34,35] for in-depth details on obtaining N_b.

Applying Eq. 9, we can assert that

$$I(\varepsilon) \sim -d_I \ln \varepsilon + \beta, \quad (10)$$

for some constant β, where ε is diameter of the boxes to perform the covering of the network.

3. Fractional (q,q') information dimension of complex

Now we proceed to the primary goal of this article, which is to introduce the fractional (q,q') information dimension of complex network denoted by $d_{q,q'}$ as follows:

$$d_{qq'} = -\lim_{\varepsilon \to 0} \frac{I_{q,q'}(\varepsilon)}{\ln \varepsilon} = \lim_{\varepsilon \to 0} \frac{\frac{\sum_{i=1}^{N_b} p_i^{q'}(\varepsilon) - p_i^{q}(\varepsilon)}{q-q'}}{\ln \varepsilon}, \quad (11)$$

where $p_i(\varepsilon) = \dfrac{n_i(\varepsilon)}{n}$, $n_i(\varepsilon)$ is the mass of the ith box of size ε, n is the number of nodes of the network, and N_b is the minimum number of boxes to cover the network. The parameters q, q' depend on the minimal covering of the network; thus, the maximal entropy minimal covering principle was adopted as in previous research on complex network [29,30,36–38] to the computation for $\varepsilon = [2, \Delta]$, where Δ means the diameter of the network.

For some constant β, the Eq. (12) is deduced from Eq. (11)

$$I(\varepsilon) \sim -d_{q,q'} \ln \varepsilon + \beta, \quad (12)$$

3.1. *Computation of q, q'*

The computation of parameter q relies on the idea that considers the networks as a system that can be divided into several subsystems. This division is based on the formation of minimum boxes by the box-covering algorithm. Hence, the number of subsystems equals the number of boxes N_b for a given size ε.

For a given box size ε, the value of q is determined as the average of q_ε, denoted by $\overline{q}_\varepsilon$, where

$$q_\varepsilon := \frac{(\Delta - 1)N_b(\varepsilon)}{\sum_{\varepsilon=2}^{\Delta} N_b(\varepsilon)}, \tag{13}$$

Note that $q_\varepsilon = (q_2, q_3, ..., q_\Delta)$.

This approximation measures how far the number of subsystems (for a given size ε) is from the mean number of overall sizes, which is the baseline.

Now, to quantify the interactions among the elements that conform the subsystems (nodes) and among these subsystems (boxes), the α and β were introduced in [30]:

$$\alpha_{\varepsilon,i} = 1 - \frac{|S_i| indeg(S_i)}{n \sum_{i=1}^{N_b} indeg(S_i)}, \tag{14}$$

$$\beta_{\varepsilon,i} = 1 - \frac{outdeg(S_i)\varepsilon}{\Delta \sum_{i=1}^{N_b} outdeg(S_i)}, \tag{15}$$

where $|S_i|$ is the number of nodes in S_i, n is the number of nodes of the network, $indeg(G_i)$ are the edges among the nodes that are in S_i, $outdeg(S_i)$ are the edges among the sub-networks S_i, ε is the diameter of the box to compute the sub-network S_i and Δ is the diameter of the network.

Finally, q' is defined by

$$q' = \frac{\overline{\beta}_{\varepsilon,i}}{\overline{\alpha}_{\varepsilon,i}} \tag{16}$$

where $\overline{\beta}_{\varepsilon,i}$, $\overline{\alpha}_{\varepsilon,i}$ are the mean of $\beta_{\varepsilon,i}$ and $\alpha_{\varepsilon,i}$, respectively, since they are vectors $(a_{\varepsilon,1}, a_{\varepsilon,2}, ..., a_{\varepsilon,N_b})$. The Eq. (16) is the interaction index [30] that shows if β is equal to ($q' = 1$), greater ($q' > 1$) or lesser ($q' < 1$) than α; hence, it reflects which type of interactions is stronger: inner subsystems interaction (α), outer them (β) or if both are balanced.

Figure 1 show an example α, β computations. Once a box covering is performed, using the approach in [35], for $\varepsilon = 2$, see Figure 1a), the re-normalization agglutinates the nodes into the boxes in a super node (subsystem) S_1, S_2 as shown in Figure 1b). Since $\Delta = 2$, in the example, $q = \overline{q}_\varepsilon = q_2 = 1$. Furthermore, $indeg(S_1) = 3$, $indeg(S_2) = 1$ and $outdeg(S_1) = 1$, $outdeg(S_2) = 1$ which is the degree of the nodes of the re-normalized network in Figure 1b).

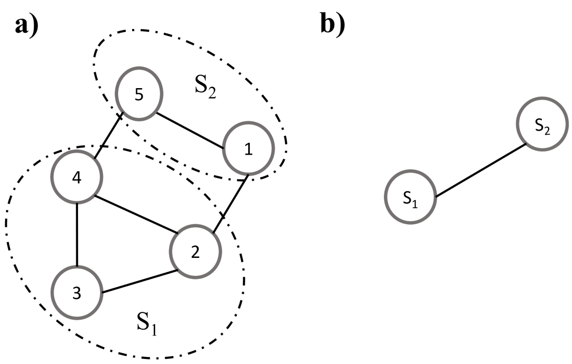

Figure 1. The box covering of a) network and b) network re-normalization for $\varepsilon = 2$.

Since $n = 5$, the result of Eq. (14) are $\alpha_{2,S_1} = 0.55$, $\alpha_{2,S_2} = 0.9$ and $\beta_{2,S_1} = 0.5$, $\beta_{2,S_2} = 0.5$ from Eq. (15); thus, $q' = 0.689$. The networks included have a diameter greater than the example shown above, so the steps to estimate q_ε and q'_ε are repeated by each box size ε resulting in two vectors that are averaged to obtain q and q'.

4. Results

4.1. Real-world networks

The fractional (q, q') information dimension Eq.(11) and the classical information dimension Eq.(9) were computed on 28 real-world networks that were gathered from [30,39], see Table 1 for number of nodes, edges, diameter and source. These networks cover several domains, such as biological, social, technological, and communications, so they are representative.

Table 1. Diameter, number of nodes, source, d_I and $d_{q,q'}$ of real-world networks.

Network	Full name	Source	Diameter	Nodes	Edges
ACF	American college football	[30]	4	115	613
BCEPG	Bio-CE-PG	[30]	8	1692	47309
BGP	Bio-grid-plant	[30]	26	1272	2726
BGW	Bio-grid-worm	[30]	12	16259	762774
CEN	C. elegant neural network	[30]	5	297	2148
CNC	Ca-netscience	[30]	17	379	914
COL	SocfbColgate88	[39]	6	3482	155044
DRO	Drosophilamedulla1	[39]	6	1770	33635
DS	Dolphins social network	[30]	8	62	159
ECC	E. coli cellular network	[30]	18	2859	6890
EM	Email	[30]	8	1133	5451
IOF	Infopenflights	[39]	14	2905	30442
JM	Jazz-musician	[30]	6	198	2742
JUN	Jung2015	[39]	16	2989	31548
LAS	Lada Adamic's network	[30]	8	350	3492
LDU	Labanderiadunne	[39]	6	700	6444
MAR	Marvel	[39]	11	19365	96616
MIT	SocfbMIT	[39]	8	6402	251230
PAIR	Pairdoc	[39]	14	8914	25514
PG	Power grid network	[30]	46	4941	6594
PGP	Techpgp	[39]	24	10680	24340
POW	Powerbcspwr10	[39]	49	5300	13571
PRI	SocfbPrinceton12	[39]	9	6575	293307
TC	Topology of communications	[30]	7	174	557
USAA	USA airport network	[30]	7	500	2980
WHO	TechWHOIS	[39]	8	7476	56943
YEAST	Protein interaction	[30]	11	2223	7046
ZCK	Zachary's karate club	[30]	5	34	78

Next, the models of Eq.(10) and Eq.(12) –that corresponds to the classical information dimension and the fractional (q, q') information model, respectively– were approximated using Nonlinear Regression [40] in MATLAB R2022a. The best model is selected by summed Bayesian information criterion with bonuses (SBICR)[41]. The SBICR penalize the complex models (that were estimated

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independently) and the size of the data sets employed to approximate the parameters; hence, the model with the largest SBICR score must be selected.

Table 2 shows the fit of information model Eq.(10) and fractional (q, q') model Eq.(12) on the information and fractional (q, q') information, respectively. The results of SBICR, d_I, $d_{q,q'}$, q, q' computations are also shown. The column $SBICR_I$ and $SBICR_{q,q'}$ show that the Eq.(12) is better than Eq.(10) for all networks except for PG and POW (in bold). Additionally, $q > 1$ means that the number of subsystems for a given ε is higher than the baseline (mean subsystems found for all ε). On the other hand, for 12 networks, the interaction between subsystems $(q' > 1)$(in bold) is stronger and for 16 networks, the interaction between the elements of the subsystems named inner interactions $(q' < 1)$ are higher than those between subsystems (outer interactions).

Table 2. The SBICR of information model Eq.(10) and the fractional (q, q') information model Eq.(12), d_I, $d_{q,q'}$ and the q, q' values.

Network	$SBICR_I$	$SBICR_{(q,q')}$	d_I	$d_{q,q'}$	q	q'
ACF	-10.135	-7.348	1.913	.93	2.976	.442
BCEPG	-35.38	-20.988	1.828	1.004	6.109	**1.814**
BGP	-95.919	-95.442	1.591	1.037	3.558	.817
BGW	-64.525	-42.927	1.949	1.006	2.437	**1.219**
CEN	-13.897	-12.103	1.822	.988	3.253	.805
CNC	-58.166	-49.747	1.835	1.014	3.606	.733
COL	-25.187	-18.343	2.679	1.001	3.655	**1.364**
DRO	-23.34	-18.202	1.443	.998	5.856	.662
DS	-17.868	-14.616	1.498	.989	3.959	.788
ECC	-87.426	-66.706	1.625	1.029	5.044	**1.442**
EM	-34.459	-31.15	1.547	1.001	4.663	.915
IOF	-65.328	-51.656	1.736	1.028	4.846	**1.112**
JM	-19.449	-13.532	2.805	.986	2.659	.726
JUN	-63.643	-58.511	2.485	1.006	5.95	**1.105**
LAS	-30.269	-21.898	2.103	1.02	3.459	.365
LDU	-20.75	-20.091	1.85	.998	2.879	.864
MAR	-52.612	-46.428	1.644	1.003	3.976	.9
MIT	-39.208	-25.351	2.295	1.006	4.353	**1.165**
PAIR	-68.947	-57.065	1.582	1.004	2.726	.67
PG	**-186.322**	-199.049	1.463	.999	5.221	**1.324**
PGP	-117.159	-100.827	1.573	1.013	2.93	**1.666**
POW	**-186.721**	-206.691	1.589	.994	5.176	.489
PRI	-45.866	-27.325	2.533	1.016	6.338	**1.234**
TC	-22.759	-22.025	1.57	.991	5.494	**1.143**
USAA	-26.655	-18.497	1.678	.996	2.358	.626
WHO	-36.125	-29.873	1.675	1.002	4.751	**1.28**
YEAST	-48.576	-43.622	1.533	1.006	6.5	.594
ZCK	-9.77	-3.672	1.5	.95	5.094	.696

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Figure 2 a) show the SocfbPrinceton12 network, were fractional (q,q') model (dotted line) is closer to fractional (q,q') information (+) than information model (solid line) to information(*). It is rather difficult to appreciate in Figure 2 b), so the SBICR is a valuable tool. The opposite occurs in Figure 2 c) where the information model is better than the fractional (q,q') model since the value of $SBICR_I$ is higher than the value $SBICR_{(q,q')}$ for Power grid network (PG), see Table 2.

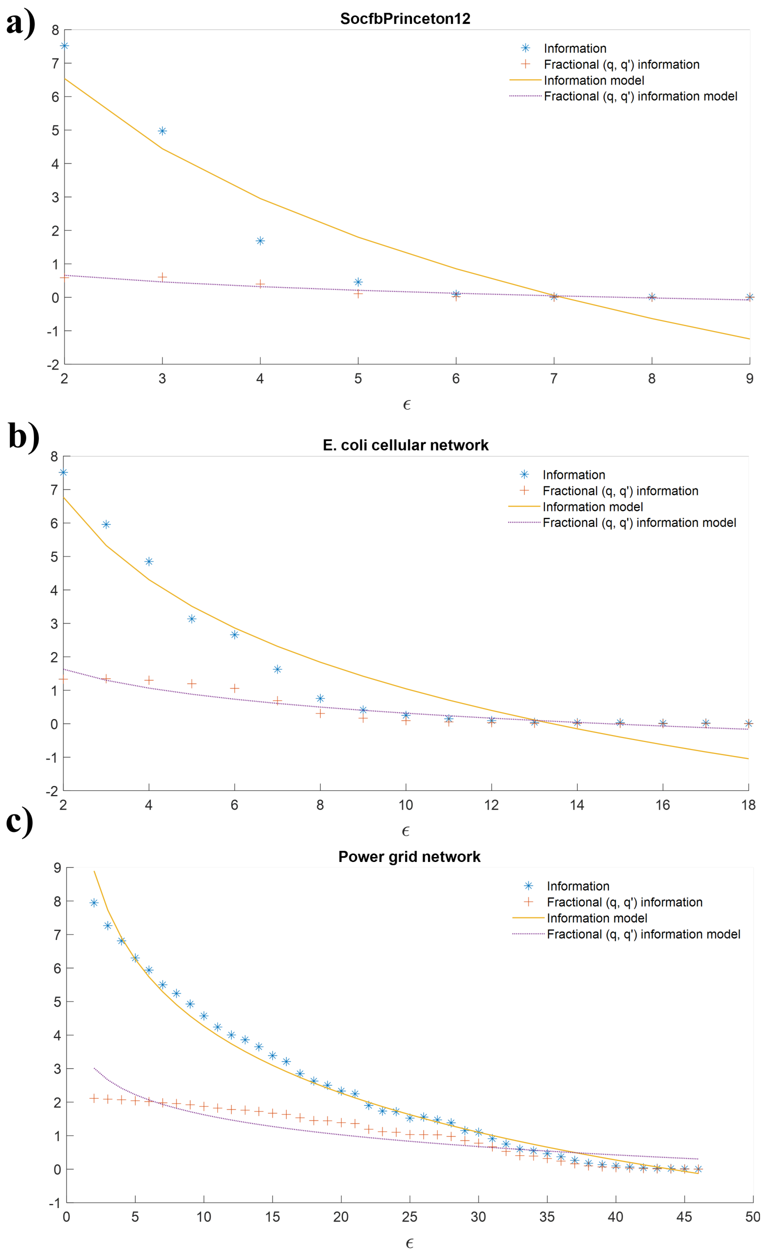

Figure 2. The box covering of a) network and b) network renormalization for $\varepsilon = 2$.

4.2. Synthetic networks

The same produce was followed on networks generated by Barabasi-Albert (BA) [42], Song, Havlin and Makse (SHM) [43], and Watts and Strogatz (WS) models [44]. First, d_I, $d_{q,q'}$ were computed and then the best models of Eq.(10) and Eq.(12) were chosen based on the SBICR. The BA networks were 225, 211 for SHM and 216 for WS. Table 3 summarise the nodes, edges, d_I, $d_{q,q'}$ and the information model selected between Eq.(10) and Eq.(12). See the supplementary material for the details on the

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parameters of each model to generate the networks as well as the specific $SBICR_I$, $SBICR_{(q,q')}$, d_I, $d_{q,q'}$, q and q'.

Table 3. The nodes (max-min), edges (max-min), d_I (max-min), $d_{q,q'}$ (max-min) and the percentage of information model for synthetic networks. I=Eq.(10) and $q, q' =$ Eq.(12)

Network model	Nodes	Edges	d_I	$d_{q,q'}$	Model
BA	2000-4500	2685-40455	3.93-7.41	.86-1.73	q, q'(100%)
SHM	10-36480	9-880475	.83-12.69	.01-2.43	q, q'(70.83%)-I(29.17%)
WS	2000-4000	4000-40000	.96-7.36	.01-1.54	q, q'(100%)

A remarkable finding on real and synthetic networks is that $d_{q,q'} < d_I$. The fractional (q, q') model fitted better for all BA and WS networks and about 71% of the SHM. Table 4 summarises the parameter of the SHM model that produces $29\%(153)$ networks for which the information model fits better; see the supplementary material for the meaning of each parameter. The values of the SHM parameters are influenced by the assortativity ($MODE = 1$) and hub repulsion ($MODE = 2$), so the only conditions that intersect on $MODE = 1$ and $MODE = 2$ were $G = 2$ $M = 3$ $IB = 0$ $BB = .4$, $G = 3$ $M = 2$ $IB = 0$ $BB = 1$, and $G = 3$ $M = 2$ $IB = .4$ $BB = .8$.

Table 4. The parameter of SHM model that produce networks for which the information model fitted better

G	M	IB	BB	$MODE$
2	2	0	.4	1
2	2	.4	.4	1
2	3	0	.4	1
2	3	.4	$[0, 1]$	1
2	4	0	.8	1
3	2	0	$[0, 1]$	1
3	2	.4	$[.2, .8]$	1
3	3	$[0, .4]$	$\leq .8$	1
3	4	0	1	1
4	$[2, 3]$	0	.4	1
2	2	0	$[0, .2, .8]$	2
2	2	.4	$[0, 1]$	2
2	2	1	$\leq .8$	2
2	3	0	$\leq .4$	2
2	3	.4	.2	2
3	2	0	$[2, .6, 1]$	2
3	2	.4	.8	2
3	4	0	.4	2
4	2	0	$\leq .2$	2
4	2	.4	$[.4, .2]$	2
4	3	0	.8	2

Additionally, for BA, the average node degree (ad) equal to 1 produces networks with stronger outer interactions than inner one $q' > 1$. It occurs no matter what the number of initial nodes ($n0$) and

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total nodes(n) values were chosen; see Table S1 of the supplementary material. On the other hand, for SHM three networks: SHM_G-3 M-4 IB-0.40 BB-0.00 MODE-2, SHM_G-4 M-3 IB-0.00 BB-0.40 MODE-2, SHM_G-4 M-3 IB-0.40 BB-0.00 MODE-2 and for WS a network: WS-2000-2-0.400000 obtained $q' > 1$, see Tables S2 and S3. These results suggest that the fractional (q, q') information dimension captures the complexity of the network topology since the SHM model tunes the links between nodes into the boxes (IB) and the connections between boxes through BB. This capability is not in BA and WS models.

5. Conclusion

This article introduced complex networks' fractional $(q, q')-$information dimension. The rationale of the definition is that the network can be divided into several subsystems. Hence, q measures how far the number of subsystems (for a given size ε) is from the mean number of overall sizes, which is the baseline. On the other hand, q' (interaction index) measures if the interactions between subsystems are greater($q' > 1$), lesser ($q' < 1$) or equal to the interactions into these subsystems ($q' = 1$).

Starting from the experimental results on real networks and synthetic networks, a glance at the interaction of the subsystems shows that clear interconnection patterns emerge, especially in the networks generated by the SHM model, where its parameters play a crucial role in obtaining networks for which the information model best fit. The initial nodes parameter of BA model generated networks where the outer interactions are stronger than inner ones $q' > 1$, no matter what value takes the remaining parameters. Finally, our experiments reveal that $d_{q,q'} < d_I$ in both types of networks.

We have enough evidence to state that the fractional $(q, q')-$information dimension of complex networks based on $(q, q')-$extropy seems to be a complementary dual statistical index of the fractional $(q, q')-$information dimension. It is an exciting area for future research to prove to what extent the new formulations will be helpful.

Supplementary Materials: The following supporting information can be downloaded at the website of this paper posted on Preprints.org, Table S1: The SBICR of information model Eq.(8) and the fractional (q, q') information model Eq.(10), d_I, $d_{q,q'}$ and the q, q' values of BA networks; Table S2: The SBICR of information model Eq.(8) and the fractional (q, q') information model Eq.(10), d_I, $d_{q,q'}$ and the q, q' values of SHM networks; Table S3: The SBICR of information model Eq.(8) and the fractional (q, q') information model Eq.(10), d_I, $d_{q,q'}$ and the q, q' values of WS networks.

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Abbreviations

The following abbreviations are used in this manuscript:

SBICR	Bayesian Information Criterion with Bonuses
BA	Barabasi-Albert
SHM	Song, Havlin and Makse
WS	Watts and Strogatz

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