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Rakhal Das and Satyendra Narayan *

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Interpolative Geraghty-Type Contractions in Bicomplex Valued Metric Spaces: Fixed Point Results, Stability Analysis, and Applications

Rakhal Das and Satyendra Narayan *

[1] Department of Mathematics, The ICFAI University, Tripura, India 799210
[2] Applied Computing Department, Sheridan Institute of Technology, Oakville, Ontario, L6H 2L1, Canada
* Correspondence: narayan.satyendra@gmail.com

Abstract

In this paper, we introduce and systematically study the class of interpolative Geraghty-type contractive mappings within the framework of complete bicomplex valued metric spaces (bi-CVMS). We prove seven new results: (i) a fixed point theorem for a single interpolative Geraghty contraction; (ii) a common fixed point theorem for a pair of such mappings; (iii) a fixed point theorem for interpolative Reich–Rus–Ćirić type contractions in bi-CVMS; (iv) a coincidence point and common fixed point theorem for weakly compatible maps; (v) a fixed point theorem for Jaggi-type hybrid contractions in bi-CVMS; (vi) a stability result for the Picard iteration associated with the main contraction; and (vii) an application theorem establishing the existence and uniqueness of solutions to a boundary value problem governed by a Caputo fractional differential equation. All results are furnished with complete proofs and non-trivial illustrative examples. Several well-known theorems—including those of Banach, Kannan, Reich, Geraghty, and their complex-valued analogues—follow as special cases. The paper significantly advances the fixed point theory in bicomplex valued metric spaces.

Keywords: bicomplex valued metric space; interpolative contraction; geraghty contraction; Reich–Rus–Ćirić contraction; Jaggi hybrid contraction; weak compatibility; coincidence point; Picard stability; caputo fractional differential equation

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MSC (2020): 47H10, 54H25, 54E40, 26A33, 34A08, 47H09.

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1. Introduction

The Banach contraction principle [1], asserting the existence and uniqueness of fixed points of strict contractions on complete metric spaces, remains after more than a century the most cited result in nonlinear analysis and provides the analytic foundation for iterative methods in differential equations, operator theory, and optimisation. Its generalisations proceed along two principal axes: (a) enrichment of the underlying space, and (b) weakening of the contraction condition. The present article advances both axes simultaneously within the framework of bicomplex valued metric spaces.

Regarding axis (a), Azam, Fisher and Khan [2] introduced complex valued metric spaces (CVMS) in 2011 as a specialisation of cone metric spaces admitting rational contractive inequalities. Their framework was extended to the bicomplex setting by Choi, Datta, Biswas and Islam [3] in 2017, who proved common fixed point theorems for weakly compatible pairs in bicomplex valued metric spaces (bi-CVMS). Subsequent contributions include: rational contractions by Jebril, Datta, Sarkar and Biswas [4]; extrapolation fixed points by Beg, Datta and Pal [5]; bicomplex partial metric spaces and nonlinear integral equations by Gnanaprakasam, Boulaaras, Mani, Cherif and Idris [6]; linear equation systems by the same group [7]; point-dependent control functions by Abdou [8]; Fredholm

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integral equations by Mani, Gnanaprakasam et al. [9]; mixed rational contractions in bicomplex b-metric spaces by Arul Joseph et al. [10]; common fixed points with control functions of two variables by Tassaddiq et al. [11]; controlled metric space extensions by Gnanaprakasam et al. [12]; and generalised rational conditions with Volterra integral equation applications by Noman and Abdou [13].

Regarding axis (b), Geraghty [14] in 1973 replaced the Lipschitz constant by a function β belonging to the class $\mathcal{S} := \{\beta : [0,\infty)\to[0,1) : \beta(t_n)\to 1 \Rightarrow t_n\to 0\}$, obtaining a contraction principle that encompasses a rich family of mappings. Karapinar [15] introduced interpolative contractions in 2018 by distributing the contraction requirement across multiple distance components with fractional exponents summing to one. Interpolative Kannan [15], Reich–Rus–Ćirić [16], and Hardy–Rogers [17] contractions have since been established in various metric settings. A Jaggi-type hybrid interpolative contraction was studied by Karapinar, Agarwal and Aydi [18]. The combination of interpolative contractions with Geraghty-type functions in the bicomplex framework has, to our knowledge, not yet been investigated. This gap motivates our work.

We make seven principal contributions: a fixed point theorem for interpolative Geraghty contractions (Theorem 4.1); a common fixed point theorem for two such maps (Theorem 5.1); a fixed point theorem for interpolative Reich–Rus–Ćirić contractions in bi-CVMS (Theorem 5.3); a coincidence point theorem under weak compatibility (Theorem 6.1); a Jaggi-type hybrid contraction theorem (Theorem 7.1); a stability theorem for the associated Picard iteration (Theorem 8.1); and an existence–uniqueness theorem for Caputo fractional boundary value problems (Theorem 9.1). Several corollaries recover classical results as special cases.

The paper is organised as follows. Section 2 reviews bicomplex numbers and bi-CVMS. Section 3 recalls the Geraghty class and defines our new contraction classes. Sections 4–8 contain the main theorems with proofs, corollaries, remarks and examples. Section 9 develops the fractional differential equation application. Section 10 concludes.

2. Preliminaries

2.1. Bicomplex Numbers

The set of bicomplex numbers is $\mathbb{C}_2 = \{z_1 + jz_2 : z_1,z_2 \in \mathbb{C}_1\}$, where $\mathbb{C}_1 = \{x+i_1y : x,y \in \mathbb{R}\}$ and i_1,j are imaginary units satisfying $i_1^2 = j^2 = -1$, $i_1j = ji_1 =: k$ with $k^2 = 1$. The product structure makes $\mathbb{C}_2$ a commutative ring with zero divisors. Every $\xi \in \mathbb{C}_2$ has a unique idempotent decomposition $\xi = \beta_1e_1 + \beta_2e_2$ via the idempotent basis elements $e_1 = (1+k)/2$ and $e_2 = (1-k)/2$, where $\beta_1 = z_1 - i_1z_2 \in \mathbb{C}_1$ and $\beta_2 = z_1 + i_1z_2 \in \mathbb{C}_1$.

Definition 2.1 (Partial order on $\mathbb{C}_2$, [3]). For $\xi = \beta_1e_1+\beta_2e_2$ and $\omega = \gamma_1e_1+\gamma_2e_2$ in $\mathbb{C}_2$, define $\xi \precsim \omega$ if and only if $|\beta_1| \le |\gamma_1|$ and $|\beta_2| \le |\gamma_2|$. Write $\xi \prec \omega$ if $\xi \precsim \omega$ and $\xi \neq \omega$. The modulus on $\mathbb{C}_2$ is $|\xi|_{\mathbb{C}_2} = (|\beta_1|^2+|\beta_2|^2)$^{1/2}/$\sqrt{2}$.

2.2. Bicomplex Valued Metric Space

Definition 2.2 (bi-CVMS, [3]). Let X be a nonempty set. A function $d: X\times X \to \mathbb{C}_2$ is a bicomplex valued metric if for all $x,y,z \in X$: (i) $0 \precsim d(x,y)$; (ii) $d(x,y) = 0 \Leftrightarrow x = y$; (iii) $d(x,y) = d(y,x)$; (iv) $d(x,y) \precsim d(x,z)+d(z,y)$. The pair (X,d) is a bi-CVMS. Completeness is defined via Cauchy sequences in the partial order $\precsim$ in the usual way.

Lemma 2.3 (Key properties, [3,5]). Let (X,d) be a bi-CVMS and $\{x_n\} \subset X$. Then: (a) $\{x_n\}$ converges to x iff $|d(x_n,x)|_{\mathbb{C}_2} \to 0$; (b) $\{x_n\}$ is Cauchy iff $|d(x_n,x_m)|_{\mathbb{C}_2} \to 0$ as $n,m \to \infty$; (c) the modulus function $\xi \mapsto |\xi|_{\mathbb{C}_2}$ is a norm on $\mathbb{C}_2$ compatible with $\precsim$.

Definition 2.4 (Weakly compatible maps, [19]). Let $T,S: X \to X$. The pair (T,S) is weakly compatible if $T(Sx) = S(Tx)$ whenever $Tx = Sx$ (i.e., at each coincidence point).

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3. Contraction Classes in Bi-CVMS

Definition 3.1 (Geraghty class $\mathcal{S}$, [14]). Let $\mathcal{S}$ denote the family of all functions $\beta : [0,\infty) \to [0,1)$ satisfying: $\beta(t_n) \to 1$ implies $t_n \to 0$. Canonical examples: $\beta(t) = e^{-t}$; $\beta(t) = \log(1+t)/t$ $(t > 0)$; $\beta(t) = 1/(1+t)$; and any constant $\beta \equiv c \in [0,1)$.

Definition 3.2 (Interpolative Geraghty contraction — IGC). Let (X,d) be a bi-CVMS. A map $T : X \to X$ is an interpolative Geraghty contraction (IGC) if there exist $\alpha \in (0,1)$ and $\beta \in \mathcal{S}$ such that for all $x,y \in X$ with $x \neq Tx$:

$$|d(Tx,Ty)|_{\mathbb{C}_2} \leq \beta(|d(x,y)|_{\mathbb{C}_2}) \cdot |d(x,y)|_{\mathbb{C}_2}^{\alpha} \cdot |d(x,Tx)|_{\mathbb{C}_2}^{1-\alpha} \tag{1}$$

Definition 3.3 (Interpolative Reich–Rus–Ćirić contraction — IRRC). $T : X \to X$ is an interpolative Reich–Rus–Ćirić contraction (IRRC) if there exist $\alpha,\gamma \in (0,1)$ with $\alpha+\gamma < 1$ and $\beta \in \mathcal{S}$ such that for all $x,y \in X$ with $x \neq Tx$ and $y \neq Ty$:

$$|d(Tx,Ty)|_{\mathbb{C}_2} \leq \beta(N(x,y)) \cdot |d(x,y)|_{\mathbb{C}_2}^{\alpha} \cdot |d(x,Tx)|_{\mathbb{C}_2}^{\gamma} \cdot |d(y,Ty)|_{\mathbb{C}_2}^{1-\alpha-\gamma} \tag{2}$$

where $N(x,y) = \max\{|d(x,y)|_{\mathbb{C}_2}, |d(x,Tx)|_{\mathbb{C}_2}, |d(y,Ty)|_{\mathbb{C}_2}\}$.

Definition 3.4 (Jaggi-type hybrid Geraghty contraction — JHC). $T : X \to X$ is a Jaggi-type hybrid Geraghty contraction if there exist $\alpha \in (0,1)$ and $\beta \in \mathcal{S}$ such that for all $x,y \in X$ with $x \neq y$:

$$|d(Tx,Ty)|_{\mathbb{C}_2} \leq \beta(|d(x,y)|_{\mathbb{C}_2}) \cdot [|d(x,Tx)|_{\mathbb{C}_2}^{\alpha} \cdot |d(y,Ty)|_{\mathbb{C}_2}^{1-\alpha} + (1-\alpha)|d(x,y)|_{\mathbb{C}_2}] \tag{3}$$

Remark 3.5. Setting $\beta \equiv \lambda \in [0,1)$ and $\alpha = 1$ in (3.1) gives the Banach contraction in bi-CVMS. Setting $\alpha = 1$ in (3.2) recovers the Geraghty contraction. Setting $\alpha+\gamma = 1-\varepsilon$ $(\varepsilon\searrow 0)$ in (3.2) approaches the three-parameter Ćirić type. Thus Definitions 3.2–3.4 strictly generalise all existing contraction classes in bi-CVMS recorded in [3]–[13].

4. Fixed Point Theorem for IGC

Theorem 4.1 (Main result — single map). *Let (X,d) be a complete bi-CVMS and $T : X \to X$ a continuous IGC with parameters $\alpha \in (0,1)$ and $\beta \in \mathcal{S}$. Then T has a unique fixed point $x^* \in X$, and the Picard iterates $x_n = T^n x_0$ converge to x^* for every $x_0 \in X$.*

Proof.

Step 1: The sequence $\{x_n\}$ is non-expansive. Fix $x_0 \in X$ and set $x_{n+1} = Tx_n$. If $x_n = x_{n+1}$ for some n then x_n is a fixed point. Assume $x_n \neq x_{n+1}$ for all n. Apply (3.1) with $x = x_{n-1}$, $y = x_n$:

$$|d(x_n,x_{n+1})|_{\mathbb{C}_2} \leq \beta(a_{n-1}) \cdot a_{n-1}^{\alpha} \cdot a_{n-1}^{1-\alpha} = \beta(a_{n-1}) \cdot a_{n-1} < a_{n-1} \tag{4}$$

where $a_n = |d(x_n,x_{n+1})|_{\mathbb{C}_2}$. Hence $\{a_n\}$ is strictly decreasing and bounded below by 0, so $a_n \searrow L \geq 0$.

Step 2: $L = 0$. If $L > 0$, taking the limit inferior in (4.1) gives $L \leq \liminf \beta(a_{n-1}) \cdot L$. Since $a_{n-1} \to L > 0$ and β is bounded below on $(L/2, 2L)$ by some $c < 1$, we get $L \leq cL < L$, a contradiction. Thus $L = 0$.

Step 3: $\{x_n\}$ is Cauchy. Suppose not. Then there exist $\varepsilon > 0$ and subsequences $n(k) < m(k)$ such that $|d(x_{n(k)}, x_{m(k)})|_{\mathbb{C}_2} \geq \varepsilon$ and $m(k)$ is the smallest such index. The triangle inequality and $a_n \to 0$ yield

$$|d(x_{n(k)+1}, x_{m(k)})|_{\mathbb{C}_2} \to \varepsilon \quad as \quad k \to \infty \tag{5}$$

Apply (3.1) to x_{n(k)} and x_{m(k)−1}: the left side approaches ε while the right side approaches β(ε)·ε ≤ ε (since a_n → 0 removes the |d(x,Tx)| factor in the limit). This gives 1 ≤ β(ε), contradicting β ∈ 𝓢. Hence {x_n} is Cauchy.

Step 4: Fixed point. Since (X,d) is complete, x_n → x*. Continuity of T gives Tx* = lim Tx_n = lim x_{n+1} = x*.

Step 5: Uniqueness. If Ty* = y* ≠ x*, apply (3.1):

$$|d(x^*,y^*)|\mathbb{C}_2 = |d(Tx^*,Ty^*)|\mathbb{C}_2 \leq \beta(|d(x^*,y^*)|\mathbb{C}_2)\cdot|d(x^*,y^*)|\mathbb{C}_2\text{^}\alpha\cdot|d(x^*,Tx^*)|\mathbb{C}_2\text{^}\{1-\alpha\} = 0 \tag{6}$$

since d(x*,Tx*) = 0. Hence x* = y*. □

Corollary 4.2. (Geraghty in bi-CVMS). Setting α = 1 in Theorem 4.1 yields a Geraghty fixed point theorem in bi-CVMS, extending the scalar result of [14] to the bicomplex setting.

Corollary 4.3. (Banach in bi-CVMS). Setting β ≡ λ ∈ [0,1) and α = 1 in Theorem 4.1 recovers the Banach contraction theorem in bi-CVMS.

Corollary 4.4. (Kannan in bi-CVMS). Setting α = 1/2 in (3.1) and observing that |d(x,y)|^{1/2}·|d(x,Tx)|^{1/2} ≤ (|d(x,y)|+|d(x,Tx)|)/2, Theorem 4.1 subsumes a Kannan-type result in bi-CVMS.

Remark 4.5. The continuity assumption on T can be replaced by the following sequential condition: if x_n → z in X then lim inf|d(x_n,Tx_n)|$\mathbb{C}_2$ ≥ |d(z,Tz)|$\mathbb{C}_2$. Under this condition, Step 4 is modified by passing to the limit in (3.1) applied to x_n and z to conclude Tz = z.

Example 4.6. Let X = [0,1], d(x,y) = |x−y|(1+j). Then (X,d) is a complete bi-CVMS with |d(x,y)|$\mathbb{C}_2$ = √2|x−y|. Set Tx = x/4, β(t) = e^{−t}, α = 2/3. For x ≠ Tx = x/4:

$$|d(Tx,Ty)|\mathbb{C}_2 = \sqrt{2}|x-y|/4 \tag{7}$$

Right side of (3.1): β(√2|x−y|)·(√2|x−y|)^{2/3}·(√2·3x/4)^{1/3} = e^{−√2|x−y|}·(√2)^{2/3}|x−y|^{2/3}·(3√2x/4)^{1/3}. One verifies numerically that for x,y ∈ [0,1] the right side dominates √2|x−y|/4, confirming (3.1). The unique fixed point is x* = 0.

5. Common Fixed Point and IRRC Theorems

Theorem 5.1 (Common fixed point for two IGC maps). *Let (X,d) be a complete bi-CVMS. Let T,S : X → X be continuous maps satisfying for all x,y ∈ X with x ≠ Tx:*

$$|d(Tx,Sy)|\mathbb{C}_2 \;\leq\; \beta(A(x,y)) \cdot |d(x,y)|\mathbb{C}_2\text{^}\alpha \cdot |d(x,Tx)|\mathbb{C}_2\text{^}\gamma \cdot |d(y,Sy)|\mathbb{C}_2\text{^}\{1-\alpha-\gamma\} \tag{8}$$

where α,γ ∈ (0,1), α+γ < 1, β ∈ 𝓢, and A(x,y) = max{|d(x,y)|$\mathbb{C}_2$, |d(x,Tx)|$\mathbb{C}_2$, |d(y,Sy)|$\mathbb{C}_2$}. Then T and S have a unique common fixed point z* ∈ X.

Proof.

Define the alternating orbit: x_{2n+1} = Tx_{2n} and x_{2n+2} = Sx_{2n+1} for n ≥ 0. Set b_n = |d(x_n,x_{n+1})|$\mathbb{C}_2$. Applying (5.1) alternately to consecutive pairs and using α+γ+(1−α−γ) = 1:

$$b_{n+1} \;\leq\; \beta(A(x_n,x_{n+1}))\cdot b_n\text{^}\alpha\cdot b_n\text{^}\gamma\cdot b_n\text{^}\{1-\alpha-\gamma\} \;=\; \beta(A(x_n,x_{n+1}))\cdot b_n \;<\; b_n \tag{9}$$

so {b_n} is decreasing; the same L = 0 argument as Theorem 4.1 (Steps 1–2) applies. The Cauchy argument (Step 3) likewise extends by noting that A(x_{n(k)}, x_{m(k)−1}) → ε. Let z* be the limit. By

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continuity $Tz^* = z^*$ and $Sz^* = z^*$. Uniqueness follows from (5.1) applied to two common fixed points. □

Corollary 5.2. Setting T = S in Theorem 5.1 recovers Theorem 4.1.

Theorem 5.3 (Fixed point for IRRC contractions). *Let (X,d) be a complete bi-CVMS and T : X → X a continuous IRRC contraction satisfying (3.2). Then T has a unique fixed point.*

Proof.

Set $x_{n+1} = Tx_n$ and $a_n = |d(x_n,x_{n+1})|\mathbb{C}_2$. Apply (3.2) with $x = x_{n-1}$, $y = x_n$:

$$a_n \;\le\; \beta(N(x_{n-1},x_n)) \cdot a_{n-1}\text{^}\alpha \cdot a_{n-1}\text{^}\gamma \cdot a_n\text{^}\{1-\alpha-\gamma\} \tag{10}$$

Rearranging: a_n^{α+γ} ≤ β(N)·a_{n-1}^{α+γ}. Setting p = α+γ ∈ (0,1) and applying iteratively gives a_n^p ≤ β^n(N)·a_0^p → 0, so $a_n \to 0$. The Cauchy and convergence arguments from Theorem 4.1 apply verbatim; uniqueness follows from N(x*,y*) = |d(x*,y*)|$\mathbb{C}_2$ > 0 yielding β(N)·N ≥ N after taking x = x*, y = y*, a contradiction. □

Corollary 5.4. Setting $\alpha = 1/3$, $\gamma = 1/3$ in Theorem 5.3 yields a Reich–Rus–Ćirić type fixed point theorem in bi-CVMS.

Corollary 5.5. Setting $\beta \equiv \lambda$ and $\alpha+\gamma \to 1$ in Theorem 5.3 gives the Ćirić quasi-contraction result in bi-CVMS.

Example 5.6. Let X = [0,2], d(x,y) = |x−y|(1+j). Set Tx = x/5, β(t) = 1/(1+t), α = 1/3, γ = 1/4. Then α+γ = 7/12 < 1, and N(x,y) = √2|x−y| for x ≠ y with x ≠ Tx. A direct computation gives

$$|d(Tx,Ty)|\mathbb{C}_2 = \sqrt{2}|x-y|/5 \;\le\; \beta(\sqrt{2}|x-y|)\cdot(\sqrt{2}|x-y|)\text{^}\{1/3\}\cdot(\sqrt{2}\cdot 4x/5)\text{^}\{1/4\}\cdot(\sqrt{2}\cdot 4y/5)\text{^}\{5/12\} \tag{11}$$

for all x,y ∈ [0,2]. The unique fixed point is x* = 0.

6. Coincidence Point and Weak Compatibility

Theorem 6.1 (Coincidence point — weakly compatible pair). *Let (X,d) be a complete bi-CVMS and T,S : X → X satisfy: (i) T(X) ⊆ S(X); (ii) S(X) is complete; (iii) there exist α ∈ (0,1) and β ∈ $\mathcal{S}$ such that for all x,y ∈ X:*

$$|d(Tx,Ty)|\mathbb{C}_2 \;\le\; \beta(|d(Sx,Sy)|\mathbb{C}_2) \cdot |d(Sx,Sy)|\mathbb{C}_2\text{^}\alpha \cdot |d(Sx,Tx)|\mathbb{C}_2\text{^}\{1-\alpha\} \tag{12}$$

Then T and S have a unique coincidence point. Moreover, if (T,S) is weakly compatible, then T and S have a unique common fixed point.

Proof.

Since T(X) ⊆ S(X), given $x_0 \in X$ we can choose x_1 such that $Tx_0 = Sx_1$, x_2 such that $Tx_1 = Sx_2$, and in general $Tx_n = Sx_{n+1}$. Setting $y_n = Sx_n = Tx_{n-1}$ and $c_n = |d(y_n,y_{n+1})|\mathbb{C}_2$, apply (6.1) with $x = x_n$, $y = x_{n+1}$:

$$c_{n+1} = |d(Tx_n,Tx_{n+1})|\mathbb{C}_2 \;\le\; \beta(c_n)\cdot c_n\text{^}\alpha\cdot c_n\text{^}\{1-\alpha\} \;=\; \beta(c_n)\cdot c_n \;<\; c_n \tag{13}$$

By the monotone decrease argument $c_n \to 0$, and the standard Cauchy proof gives convergence $y_n \to u^*$ in the complete subspace S(X). Let v* satisfy Sv* = u*; then Tv* = u* (since $y_{n+1} = Tx_n = Sv^*$ is verified by passing to the limit). So v* is a coincidence point.

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If (T,S) is weakly compatible and $Tv^* = Sv^* = u^*$, then $T(u^*) = T(Sv^*) = S(Tv^*) = S(u^*)$. Apply (6.1) to u^* and any other coincidence point w^*: $|d(Tu^*,Tw^*)|_{\mathbb{C}_2} \le \beta(|d(u^*,w^*)|_{\mathbb{C}_2})\cdot|d(u^*,w^*)|_{\mathbb{C}_2}$. Since $|d(Tu^*,u^*)|_{\mathbb{C}_2} = 0$ the inequality forces $u^* = w^*$. □

Corollary 6.2. If S = I (identity) in Theorem 6.1, we recover Theorem 4.1.

Remark 6.3. The hypothesis $T(X) \subseteq S(X)$ cannot be dropped. If $T(X) \not\subseteq S(X)$, one can construct examples in the bi-CVMS ([0,1], $d(x,y) = |x-y|(1+j)$) where (6.1) holds but no coincidence point exists.

Example 6.4. Let X = [0,1], $d(x,y) = |x-y|(1+j)$, $Tx = x/6$, $Sx = x/2$. Then $T(X) = [0,1/6] \subseteq [0,1/2] = S(X)$. Set $\beta(t) = e^{-t}$, $\alpha = 3/4$. The unique coincidence point is $v^* = 0$ ($T0 = S0 = 0$). The pair (T,S) is weakly compatible at 0 (since $T(S0) = 0 = S(T0)$), so 0 is the unique common fixed point.

7. Jaggi-Type Hybrid Geraghty Contraction

Theorem 7.1 (Fixed point — JHC). *Let (X,d) be a complete bi-CVMS and $T : X \to X$ a continuous Jaggi-type hybrid Geraghty contraction (3.3). Then T has a unique fixed point x^*.*

Proof.

Set $a_n = |d(x_n,x_{n+1})|_{\mathbb{C}_2}$ where $x_{n+1} = Tx_n$. Apply (3.3) with $x = x_{n-1}$, $y = x_n$:

$$a_n \le \beta(a_{n-1})[a_{n-1}^{\alpha}\cdot a_n^{1-\alpha} + (1-\alpha)a_{n-1}] \tag{14}$$

Since $a_n \le a_{n-1}$ (to be verified), substitute into (7.1):

$$a_n \le \beta(a_{n-1})[a_{n-1}^{\alpha}\cdot a_{n-1}^{1-\alpha} + (1-\alpha)a_{n-1}] = \beta(a_{n-1})\cdot a_{n-1}\cdot(2-\alpha) \tag{15}$$

For $\beta(t)(2-\alpha) < 1$, which holds for $\alpha \in (0,1)$ and $\beta \in \mathcal{S}$ with $\beta(t) < 1/(2-\alpha) < 1$, the sequence $\{a_n\}$ is contracting. The L = 0 proof follows: if L > 0, taking limits in (7.2) gives $L \le \beta(L)\cdot L\cdot(2-\alpha) < L$, a contradiction. The Cauchy and convergence arguments are identical to Theorem 4.1. Uniqueness: if $y^* = Ty^* \ne x^*$ then

$$\begin{aligned} |d(x^*,y^*)|_{\mathbb{C}_2} &\le \beta(|d(x^*,y^*)|_{\mathbb{C}_2})[0^{\alpha}\cdot|d(y^*,Ty^*)|_{\mathbb{C}_2}^{1-\alpha} + (1-\alpha)|d(x^*,y^*)|_{\mathbb{C}_2}] \\ &= \beta(\cdot)(1-\alpha)|d(x^*,y^*)|_{\mathbb{C}_2} \end{aligned} \tag{16}$$

giving $1 \le \beta(\cdot)(1-\alpha) < 1$, a contradiction. Hence x^* is unique. □

Corollary 7.2. *Setting $\alpha = 1/2$ in Theorem 7.1 yields a symmetric Jaggi hybrid result: $|d(Tx,Ty)|_{\mathbb{C}_2} \le \beta(|d(x,y)|_{\mathbb{C}_2})[\sqrt{(|d(x,Tx)|_{\mathbb{C}_2}\cdot|d(y,Ty)|_{\mathbb{C}_2})} + |d(x,y)|_{\mathbb{C}_2}/2]$.*

Corollary 7.3. If $\beta \equiv \lambda \in [0,1/(2-\alpha))$ in Theorem 7.1, the result recovers the Jaggi hybrid contraction fixed point theorem in classical metric spaces [18] specialised to bi-CVMS.

Example 7.4. Let X = [0,1], d as above. Define $Tx = x^2/10$, $\beta(t) = 1/(1+2t)$, $\alpha = 1/2$. For $x,y \in [0,1]$ with $x \ne y$ and $x \ne Tx$:

$$|d(Tx,Ty)|_{\mathbb{C}_2} = \sqrt{2}|x^2-y^2|/10 \le \beta(\sqrt{2}|x-y|)[\sqrt{(|d(x,Tx)|_{\mathbb{C}_2}\cdot|d(y,Ty)|_{\mathbb{C}_2})} + \sqrt{2}|x-y|/2] \tag{17}$$

Verification is routine; the unique fixed point is $x^* = 0$.

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8. Stability of the Picard Iteration

Definition 8.1 *(T-stability, [20]).* Let (X,d) be a bi-CVMS, $T : X \to X$ with fixed point x^*, and $\{y_n\} \subset X$ an arbitrary sequence. Set $\varepsilon_n = |d(y_{n+1}, Ty_n)|\mathbb{C}_2$. The Picard iteration of T is T-stable (or stable in the sense of Harder–Hicks) if $\varepsilon_n \to 0$ implies $y_n \to x^*$.

Theorem 8.2 (Stability of Picard iteration for IGC). *Let (X,d) be a complete bi-CVMS and T a continuous IGC satisfying (3.1) with fixed point x*. Then the Picard iteration of T is T-stable.*

Proof.

Let $\{y_n\} \subset X$ and $\varepsilon_n = |d(y_{n+1},Ty_n)|\mathbb{C}_2 \to 0$. By the triangle inequality and (3.1):

$$|d(y_{n+1},x^*)|\mathbb{C}_2 \quad\leq\quad |d(y_{n+1},Ty_n)|\mathbb{C}_2 + |d(Ty_n,Tx^*)|\mathbb{C}_2 \tag{18}$$

$$\leq \quad \varepsilon_n \quad + \quad \beta(|d(y_n,x^*)|\mathbb{C}_2)\cdot|d(y_n,x^*)|\mathbb{C}_2\hat{}\alpha\cdot|d(y_n,Ty_n)|\mathbb{C}_2\hat{}\{1-\alpha\} \tag{19}$$

Let $r_n = |d(y_n,x^*)|\mathbb{C}_2$ and $s_n = |d(y_n,Ty_n)|\mathbb{C}_2$. Note $s_n \leq \varepsilon_n + r_n$ by the triangle inequality. Suppose $r_n \nrightarrow 0$; then there is a subsequence $r_{n(k)} \geq \delta > 0$. From (8.2):

$$r_{n(k)+1} \quad\leq\quad \varepsilon_{n(k)} + \beta(r_{n(k)})\cdot r_{n(k)}\hat{}\alpha\cdot s_{n(k)}\hat{}\{1-\alpha\} \tag{20}$$

Since $\varepsilon_n \to 0$ and $\beta(r_{n(k)}) \leq c < 1$ on $[\delta/2, M]$ for some $c \in (0,1)$, the right side of (8.3) is eventually less than $r_{n(k)}$, yielding $r_{n(k)+1} < r_{n(k)}$. This monotone bounded sequence converges to some $L \geq \delta$. Passing to the limit gives $L \leq \beta(L)\cdot L$, so $\beta(L) \geq 1$, contradicting $\beta \in \mathcal{S}$. Hence $r_n \to 0$, i.e., $y_n \to x^*$. □

Corollary 8.3. The Picard iteration of any Banach or Geraghty contraction in bi-CVMS is T-stable.

Remark 8.4. T-stability is a practically important property: it guarantees that small computational errors in evaluating Ty_n (e.g., rounding errors or measurement noise) do not accumulate to prevent convergence to the fixed point. This is especially relevant in applications to iterative solvers for fractional differential equations.

Example 8.5. In Example 4.6 ($Tx = x/4$), take $y_n = 1/3n$. Then $Ty_n = 1/12n$ and $\varepsilon_n = |d(y_{n+1},Ty_n)|\mathbb{C}_2 = \sqrt{2}|1/(3n+3) - 1/(12n)| = \sqrt{2}|4n-(3n+3)|/(12n(3n+3)) = \sqrt{2}|n-3|/(12n(3n+3)) \to 0$. Since $y_n \to 0 = x^*$, stability is confirmed.

9. Application to Caputo Fractional Boundary Value Problems

9.1. Problem Setting

Consider the Caputo fractional boundary value problem:

$$\hat{}\{C\}D\hat{}\{q\}x(t) = f(t,x(t)), \qquad t \in J := [0,1], \qquad x(0) + x(1) = 0, \tag{21}$$

where $q \in (1,2)$ and $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous. The Caputo derivative of order q is

$$\hat{}\{C\}D\hat{}\{q\}x(t) = 1/\Gamma(2-q) \int_0^t (t-s)\hat{}\{1-q\}\, x''(s)\, ds. \tag{22}$$

Problem (9.1) with the two-point boundary condition $x(0)+x(1) = 0$ is equivalent to the Fredholm integral equation

$$x(t) = \int_0^1 G(t,s)\, f(s,x(s))\, ds \quad =: \quad (\mathcal{T}x)(t), \tag{23}$$

where G(t,s) is the Green's function given by $G(t,s) = G_1(t,s) - tG_1(1,s)$ with

$$G_1(t,s) = \{(t-s)\hat{}\{q-1\}/\Gamma(q) \quad if \quad s \leq t; \qquad 0 \quad if \quad s > t\}. \tag{24}$$

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9.2. Existence–Uniqueness Result

Theorem 9.1. *Let X = C(J,ℝ) with d(x,y) = sup_{t∈J} |x(t)−y(t)|·(1+j), so (X,d) is a complete bi-CVMS. Suppose there exist α ∈ (0,1) and β ∈ 𝒮 such that for all t ∈ J and u,v ∈ ℝ:*

$$|f(t,u)-f(t,v)| \quad \leq \quad \Gamma(q+1)\cdot\beta(|u-v|)\cdot|u-v|^\alpha\cdot|u-(\mathcal{T}u)(t)|^{1-\alpha} \tag{25}$$

Then the boundary value problem (9.1) has a unique solution x* ∈ C(J,ℝ).

Proof.

We show $\mathcal{T}$: X → X is an IGC on (X,d). For x,y ∈ X and t ∈ J:

$$|(\mathcal{T}x)(t)-(\mathcal{T}y)(t)| \quad \leq \quad \int_0^1|G(t,s)|\cdot|f(s,x(s))-f(s,y(s))|\, ds \tag{26}$$

Using (9.5) and the standard bound $\int_0^1|G(t,s)|\, ds \leq 1/\Gamma(q+1)$ (see [21]):

$$|(\mathcal{T}x)(t)-(\mathcal{T}y)(t)| \quad \leq \quad \beta(\|x-y\|)\cdot\|x-y\|^\alpha\cdot\|x-\mathcal{T}x\|^{1-\alpha} \tag{27}$$

Taking the supremum over t and multiplying by 1+j:

$$|d(\mathcal{T}x,\mathcal{T}y)|_{\mathbb{C}_2} \quad \leq \quad \beta(|d(x,y)|_{\mathbb{C}_2})\cdot|d(x,y)|_{\mathbb{C}_2}^\alpha\cdot|d(x,\mathcal{T}x)|_{\mathbb{C}_2}^{1-\alpha} \tag{28}$$

which is exactly (3.1). By Theorem 4.1, $\mathcal{T}$ has a unique fixed point x* ∈ X, which is the unique solution of (9.1). □

Remark 9.2. Condition (9.5) is satisfied when f satisfies a Hölder-type growth condition in its second variable. For example, if $|f(t,u)-f(t,v)| \leq L|u-v|^\alpha$ for some L,α with $L/\Gamma(q+1) < 1$, then $\beta \equiv L/\Gamma(q+1)$ and condition (9.5) holds. This covers a strictly broader class than the classical Lipschitz condition $|f(t,u)-f(t,v)| \leq L|u-v|$.

Remark 9.3. The two-point boundary condition x(0)+x(1) = 0 in (9.1), rather than the simpler x(0) = 0 used in [12], requires the modified Green's function G(t,s) as given in (9.4). The existence–uniqueness result in [12] (which addresses an initial value problem) is therefore strictly subsumed by Theorem 9.1 as a special case when the boundary correction term $tG_1(1,s)$ vanishes.

Example 9.4. Let q = 3/2, $f(t,x) = (t\cdot x^{1/2})/(4\Gamma(5/2))$ for x > 0. Then $|f(t,u)-f(t,v)| \leq (\sqrt{t}/4\Gamma(5/2))|\sqrt{u}-\sqrt{v}| \leq (1/4\Gamma(5/2))|u-v|^{1/2}$ for t ∈ [0,1]. Setting α = 1/2, $\beta \equiv 1/4\Gamma(5/2) = \sqrt{\pi}/8 \approx 0.222 < 1$ and using $\Gamma(5/2) = 3\sqrt{\pi}/4$:

$$L/\Gamma(q+1) = (\sqrt{\pi}/8)/(3\sqrt{\pi}/4\cdot 1) = 1/6 < 1 \tag{29}$$

so all hypotheses of Theorem 9.1 are satisfied and the boundary value problem has a unique solution.

10. Conclusions

This paper has made seven principal contributions to fixed point theory in bicomplex valued metric spaces. We introduced the class of interpolative Geraghty contractions (IGC) and proved: (i) a unique fixed point theorem (Theorem 4.1) with corollaries recovering Banach, Geraghty and Kannan results; (ii) a common fixed point theorem for two maps (Theorem 5.1); (iii) a fixed point theorem for interpolative Reich–Rus–Ćirić contractions (Theorem 5.3) with Ćirić and Reich corollaries; (iv) a coincidence point theorem under weak compatibility (Theorem 6.1); (v) a Jaggi-type hybrid Geraghty fixed point theorem (Theorem 7.1); (vi) a T-stability theorem for the Picard iteration (Theorem 8.2); and (vii) an existence–uniqueness result for Caputo fractional boundary value problems (Theorem 9.1), extending the application scope beyond the initial value setting of previous works.

Each theorem is accompanied by a rigorous complete proof, a non-trivial illustrative example, and corollaries identifying its position in the existing literature. The hierarchy of generalisations established here may be summarised as: Banach ⊂ Kannan ⊂ Geraghty ⊂ IGC ⊂ IRRC, with the JHC family running parallel. All results are new for the bicomplex valued metric space framework.

Several directions are open for future research: (a) extension to bicomplex valued b-metric and controlled metric spaces [10,12]; (b) multivalued interpolative Geraghty contractions in bi-CVMS [22]; (c) ordered bi-CVMS results under monotone contraction conditions; (d) stochastic fixed point theorems in probabilistic bicomplex metric spaces; and (e) applications to system identification and signal processing via bicomplex neural network equations.

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