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Uniform-Window Shell Kernels for the Syracuse Map: Renewal Structure and a Primitive-Frequency Obstruction

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Abstract

We study shell kernels for the odd-to-odd Syracuse dynamics generated by uniformly distributed initial windows. For backstepped first-passage shells, we prove short-time localization, derive an exact inverse-affine representation of the fixed-time kernel, and reduce the shell-slice discrepancy to weighted primitive-frequency correlations. We also prove a quantitative boundary-layer estimate and identify a formal renewal model for the corresponding shell mechanism. On the arithmetic side, we obtain an exact block decomposition for the primitive-frequency transfer operator, prove that no naive operator gap is available, and reduce the unresolved step to explicit incomplete principal-unit exponential sums modulo powers of 3. Thus the paper is unconditional up to a final primitive-frequency estimate, which is formulated explicitly.

Keywords: discrete chaotic dynamics; operator theory; Syracuse map; affine path representation; renewal structure; primitive-frequency method; Collatz conjecture

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1. Introduction

The Collatz map

$$\mathrm{Col}(N) = \begin{cases} 3N+1, & N \text{ odd}, \\ N/2, & N \text{ even}, \end{cases} \qquad N \in \mathbb{N}+1, \tag{1}$$

and its odd-to-odd acceleration, the Syracuse map, remain analytically resistant despite the elementary form of their local rule [1,2]. The strongest current almost-all theorem of this type is Tao's logarithmic-density result, where for every function $f(N) \to \infty$, one has $\mathrm{Col}_{\min}(N) \le f(N)$ for logarithmically almost all N [3]. Later work of Inselmann established natural-density information at the coarser scale $T_{\min}(m) \le m^{\varepsilon}$ for every fixed $\varepsilon > 0$ [4]. In parallel, Rees initiated a push-forward program for geometric distributions under Collatz iteration [5], while Chang obtained an orbit-level structural reduction in a compressed odd-to-odd framework [6].

The present paper studies a different object, shell kernels generated by uniform odd initial windows. The point of departure is that Tao's logarithmic framework is naturally adapted to multiplicative transport, whereas uniform windows interact with the affine structure of the Syracuse map in a different way. Our aim is therefore not to claim a new unconditional density theorem for Syracuse orbits. Instead, we develop an exact shell-kernel formalism for uniform windows, isolate the geometric and arithmetic contributions to that kernel, and identify the final unresolved step as a primitive-frequency estimate for an explicit family of incomplete principal-unit exponential sums modulo powers of 3.

More precisely, we fix a threshold x, choose a short uniform odd window

$$\mathcal{W}_y := (2\mathbb{N}+1) \cap [y, y^{\alpha}], \tag{2}$$

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with $\alpha > 1$ close to 1, and study the orbit of a uniformly distributed $U_y \in \mathcal{W}_y$ near its first passage below x. Retreating a logarithmic number of steps from the first-passage location produces a shell point at scale

$$M_x := (4/3)^{m_0} x, \qquad m_0 = \lfloor \beta \log x \rfloor, \tag{3}$$

where $\beta > 0$ is small. The shell mechanism generated in this way is the principal object of the paper.

The analysis has four unconditional components. First, we prove short-time localization for uniform windows. The relevant first-passage times lie, with exponentially high probability, in an admissible logarithmic band, and the corresponding backstepped orbit points lie in a shell centered at M_x. Second, we derive an exact inverse-affine representation of the fixed-time shell kernel. This rewrites the shell contribution as a path sum over Syracuse valuation vectors and separates the Archimedean window constraint from the arithmetic condition modulo powers of 3. Third, we prove two quantitative reduction results. The first is a boundary-layer estimate showing that the region where the exact affine window constraint differs from its coarse phase approximation contributes only a power-saving fraction of the total shell mass. The second is an exact interval-averaged arithmetic reduction, after averaging over shell slices, the remaining discrepancy is controlled by weighted primitive-frequency correlations. Importantly, this step is stated with the primitive-frequency envelope left explicit, so no unresolved Fourier estimate is used implicitly. Fourth, we analyze the weighted primitive-frequency operator itself. We derive an exact block decomposition, prove that no naive operator gap is available, and reduce the remaining arithmetic step to explicit incomplete principal-unit exponential sums on the 3-adic principal-unit subgroup. In particular, the paper identifies the final obstruction not as a qualitative mixing question, but as a concrete short-window Fourier-mass problem for an explicit oscillatory family. This last point is central. The argument developed here does *not* prove an unconditional shell asymptotic for the actual Syracuse dynamics. What it proves is a reduction theory precise enough to isolate the final arithmetic barrier. Thus the paper should be read as a completed reduction program rather than as a final asymptotic theorem.

1.1. Main Unconditional Results

We summarize the unconditional outputs established in the paper.

Theorem 1. *Short-time localization Let $y \in \{x^\alpha, x^{\alpha^2}\}$, with $\alpha > 1$ sufficiently close to 1 and $\beta > 0$ sufficiently small. Then the first-passage time $T_x(U_y)$ satisfies*

$$\mathbb{P}\Big(T_x(U_y) - m_0 \in \mathcal{J}_{x,y} \text{ and } \mathrm{Syr}^{T_x(U_y)-m_0}(U_y) \in \Sigma_x\Big) = 1 - O\Big(e^{-c\log^{0.2} x}\Big), \tag{4}$$

for a suitable admissible time band $\mathcal{J}_{x,y}$ and shell Σ_x.

Proposition 2 (Exact inverse-affine shell kernel)**.** *The fixed-time shell kernel admits an exact pathwise representation obtained by inverting the affine Syracuse path map. In particular, the shell contribution of a uniform initial window can be written exactly as a sum over Syracuse valuation paths and shell points.*

Proposition 3 (Shell-slice arithmetic reduction)**.** *After averaging over a shell slice, the arithmetic discrepancy is reduced to a weighted primitive-frequency envelope*

$$D_r(n) := \sup_{t\in[-\pi,\pi]} \sup_{\xi\in(\mathbb{Z}/3^r\mathbb{Z})^\times} \Big|\mathbb{E}\Big[e^{itA_n} e_{3^r}(\xi(B_n \bmod 3^r))\Big]\Big|, \tag{5}$$

where A_n is the centered path weight and B_n is the Syracuse offset random variable. No unresolved primitive-frequency estimate is used at this stage; the envelope $D_r(n)$ remains explicit.

Proposition 4 (Primitive-frequency operator and obstruction)**.** *The weighted primitive-frequency recursion admits an exact block decomposition on a 3-adic frequency tree. The corresponding principal operator has no naive spectral gap. The remaining arithmetic step is thereby reduced to short-window Fourier mass for explicit incomplete principal-unit exponential sums modulo powers of 3.*

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The renewal calculation associated with the formal shell model is included separately and is not used in the main unconditional theorem chain. Its role is explanatory, it describes the model main term that would arise in the absence of the unresolved primitive-frequency discrepancy.

1.2. Relation to Existing Work

The present paper is complementary to the current analytic literature rather than competitive with it in direct theorem strength. Tao's theorem remains the strongest logarithmic-density result [3], and Inselmann's theorem remains the strongest currently available natural-density statement at the coarse scale m^{ε} [4]. Rees's program emphasizes the dyadic push-forward side [5], while Chang's reduction isolates an orbit-level one-bit obstruction [6]. The contribution here is to show that, for shell transport generated by uniform windows, the last unresolved step can be formulated as an explicit primitive-frequency problem on the 3-adic side.

In particular, the paper sharpens the remaining arithmetic obstruction beyond the level of a generic mixing conjecture. The final gap is encoded by an explicit family of incomplete exponential sums with non-stationary 3-adic phase on the principal-unit subgroup. To the best of our knowledge, this exact shell-level obstruction does not appear in the existing literature in this form.

1.3. Structure of the Paper

Section 2 records the affine formalism for Syracuse paths, the shell notation, and the weighted primitive-frequency quantities that will govern the arithmetic discrepancy. Section 3 proves short-time localization, derives the exact inverse-affine shell-kernel formula, and reduces the shell-slice discrepancy to the primitive-frequency envelope. Section 4 develops the weighted primitive-frequency operator, proves the exact block decomposition, rules out a naive operator-gap argument, and reduces the remaining arithmetic step to incomplete principal-unit sums. A formal renewal calculation for the corresponding model is included separately in Appendix A and is not used in the proofs of the main unconditional results.

2. Preliminaries

2.1. Syracuse Dynamics and Affine Paths

Let

$$2\mathbb{N}+1 = \{1, 3, 5, \dots\} \tag{6}$$

denote the odd positive integers. The Syracuse map

$$\mathrm{Syr} : 2\mathbb{N}+1 \to 2\mathbb{N}+1 \tag{7}$$

is defined by

$$\mathrm{Syr}(N) := \frac{3N+1}{2^{\nu_2(3N+1)}}, \qquad N \in 2\mathbb{N}+1, \tag{8}$$

where ν_2 is the 2-adic valuation. Thus $\mathrm{Syr}(N)$ is the largest odd divisor of $3N+1$.

For $a \in \mathbb{N}+1$, define the affine map

$$\mathrm{Aff}_a(x) := \frac{3x+1}{2^a}. \tag{9}$$

For a path

$$\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{N}+1)^n, \tag{10}$$

write

$$\mathrm{Aff}_{\mathbf{a}} := \mathrm{Aff}_{a_n} \circ \cdots \circ \mathrm{Aff}_{a_1}. \tag{11}$$

A direct calculation gives

$$\mathrm{Aff}_{\mathbf{a}}(x) = 3^n 2^{-|\mathbf{a}|} x + F_n(\mathbf{a}), \tag{12}$$

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where
$$|\mathbf{a}| := a_1 + \cdots + a_n \tag{13}$$
and
$$F_n(\mathbf{a}) := \sum_{m=1}^{n} 3^{n-m} 2^{-a[m,n]}, \qquad a[j,k] := \sum_{i=j}^{k} a_i. \tag{14}$$
The function F_n is Tao's n-step Syracuse offset map [3].

For $N \in 2\mathbb{N}+1$, define the n-path of N by
$$\mathbf{a}^{(n)}(N) := \big(\nu_2(3N+1), \nu_2(3\,\mathrm{Syr}(N)+1), \ldots, \nu_2(3\,\mathrm{Syr}^{n-1}(N)+1)\big). \tag{15}$$
Then
$$\mathrm{Syr}^n(N) = \mathrm{Aff}_{\mathbf{a}^{(n)}(N)}(N). \tag{16}$$
We shall use the following uniqueness statement repeatedly.

Lemma 5 (Path uniqueness). *Let $N \in 2\mathbb{N}+1$ and $n \geq 0$. Then $\mathbf{a}^{(n)}(N)$ is the unique path $\mathbf{a} \in (\mathbb{N}+1)^n$ such that $\mathrm{Aff}_{\mathbf{a}}(N)$ is an odd integer.*

Proof. This is [3, Lemma 2.1]. □

The explicit form (14) also yields the elementary bound
$$0 \leq F_n(\mathbf{a}) \leq 3^n \qquad (\mathbf{a} \in (\mathbb{N}+1)^n), \tag{17}$$
which we use repeatedly to control affine inversions in the shell regime.

2.2. Uniform Windows, Shells, and Coarse Phase

Fix $\alpha > 1$ sufficiently close to 1, and let $\beta > 0$ be sufficiently small depending on α. For large x, define
$$m_0 := \lfloor \beta \log x \rfloor, \qquad M_x := (4/3)^{m_0} x, \qquad L_x := \log^{0.7} x. \tag{18}$$
We write
$$\Sigma_x := (2\mathbb{N}+1) \cap [M_x e^{-L_x},\, M_x e^{L_x}] \tag{19}$$
for the shell at scale x.

For
$$y \in \{x^{\alpha}, x^{\alpha^2}\}, \tag{20}$$
define the odd window
$$\mathcal{W}_y := (2\mathbb{N}+1) \cap [y, y^{\alpha}], \tag{21}$$
and let U_y be the uniform random variable on $\mathcal{W}_y$.

The drift parameter
$$\mu := \log(4/3) \tag{22}$$
governs the first-passage scale. For a fixed large constant $C > 0$, define the admissible time band
$$\mathcal{J}_{x,y} := \left[\frac{\log(y/x)}{\mu} - m_0 - C\log^{0.6} x,\ \frac{\log(y^{\alpha}/x)}{\mu} - m_0 + C\log^{0.6} x\right] \cap \mathbb{N}. \tag{23}$$
For $n \geq 0$ and $s \in \mathbb{Z}$, define the centered affine phase
$$R_{n,s} := n\log(4/3) + s\log 2. \tag{24}$$
For an odd shell point M, define the coarse phase
$$\Gamma_{n,s}(M) := R_{n,s} + \log M. \tag{25}$$

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The source phase window is

$$J_y := [\log y, \alpha \log y]. \tag{26}$$

The corresponding phase slice in the shell is

$$I_{x,y,n,s} := (2\mathbb{N}+1) \cap \Sigma_x \cap [ye^{-R_{n,s}}, y^\alpha e^{-R_{n,s}}]. \tag{27}$$

2.3. Inverse-Affine Notation

For a path $\mathbf{a} \in (\mathbb{N}+1)^n$, define its centered weight by

$$A(\mathbf{a}) := |\mathbf{a}| - 2n. \tag{28}$$

If

$$M = \mathrm{Aff}_{\mathbf{a}}(N) = 3^n 2^{-|\mathbf{a}|} N + F_n(\mathbf{a}), \tag{29}$$

then

$$N = \frac{2^{|\mathbf{a}|}(M - F_n(\mathbf{a}))}{3^n}. \tag{30}$$

Accordingly, for odd M we define the inverse-affine point

$$N_{\mathbf{a}}(M) := \frac{2^{|\mathbf{a}|}(M - F_n(\mathbf{a}))}{3^n}. \tag{31}$$

The exact shell kernel will be built by summing over those pairs $(\mathbf{a}, M)$ for which $N_{\mathbf{a}}(M)$ falls in the source window $\mathcal{W}_y$.

2.4. Arithmetic Path Counts

For $s \in \mathbb{Z}$ and $M \in \mathbb{Z}/3^n\mathbb{Z}$, define

$$C_{n,s}(M) := \#\{\mathbf{a} \in (\mathbb{N}+1)^n : A(\mathbf{a}) = s,\ F_n(\mathbf{a}) \equiv M \pmod{3^n}\}. \tag{32}$$

Also define

$$N_n(s) := \#\{\mathbf{a} \in (\mathbb{N}+1)^n : A(\mathbf{a}) = s\}. \tag{33}$$

Let

$$a_1, \ldots, a_n \overset{\text{iid}}{\sim} \mathrm{Geom}(2), \tag{34}$$

and define

$$A_n := a_1 + \cdots + a_n - 2n, \qquad B_n := F_n(a_1, \ldots, a_n) \pmod{3^n}. \tag{35}$$

Since every path with $A(\mathbf{a}) = s$ has probability 2^{-2n-s} under the iid geometric law,

$$2^{-2n-s} C_{n,s}(M) = \mathbb{P}(A_n = s,\ B_n \equiv M \pmod{3^n}). \tag{36}$$

For $t \in [-\pi, \pi]$ and $\xi \in \mathbb{Z}/3^n\mathbb{Z}$, define the weighted Fourier coefficient

$$\Psi_n(t, \xi) := \mathbb{E}\Big[e^{itA_n} e_{3^n}(\xi B_n)\Big]. \tag{37}$$

For $1 \le r \le n$, define the weighted primitive-frequency envelope

$$D_r(n) := \sup_{t \in [-\pi,\pi]} \sup_{\xi \in (\mathbb{Z}/3^r\mathbb{Z})^\times} \left| \mathbb{E}\Big[e^{itA_n} e_{3^r}\big(\xi (B_n \bmod 3^r)\big)\Big] \right|. \tag{38}$$

The key point is that $D_r(n)$ will remain explicit throughout the unconditional reduction. In particular, the main theorem chain does not assume any unresolved primitive-frequency decay beyond what is proved.

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2.5. Standing Parameter Condition

We assume throughout that $\alpha > 1$ and $\beta > 0$ have been fixed so that there exists $c_* > 0$ with

$$\sup_{\substack{y\in\{x^\alpha, x^{\alpha^2}\} \\ n\in\mathcal{J}_{x,y}}} 3^{2n} \le M_x x^{-2c_*} \tag{39}$$

for all sufficiently large x.

This is achieved by first taking $\alpha > 1$ sufficiently close to 1 and then choosing $\beta > 0$ sufficiently small. The inequality (39) is used only as an upper-bound mechanism in the shell regime; in particular, we never replace shell quantities by asymptotic constants at the scale M_x^{-1} without stating the corresponding normalization explicitly.

2.6. Bibliographic Convention

We use Tao's affine-path formalism and path-uniqueness lemma throughout [3]. When discussing coarse natural-density or structural reductions, we refer to [4–6]. Prime-power exponential-sum heuristics and p-adic stationary-phase considerations are invoked only at the level of motivation for the final unresolved arithmetic estimate; see, for example, [7,8].

3. Uniform-Window Shell Kernels

This section develops the shell-kernel formalism associated with uniformly distributed odd initial windows. The first result is a short-time localization theorem for the backstepped first-passage shell. The second is an exact inverse-affine representation of the fixed-time shell kernel. The section then reduces the shell-slice discrepancy to the weighted primitive-frequency envelope $D_r(n)$ and isolates the boundary layer.

3.1. Setup and Localization

Fix $\alpha > 1$ sufficiently close to 1, and let $\beta > 0$ be sufficiently small depending on α. For large x, define

$$m_0 := \lfloor \beta \log x \rfloor, \qquad M_x := (4/3)^{m_0} x, \qquad L_x := \log^{0.7} x. \tag{40}$$

We write

$$\Sigma_x := (2\mathbb{N}+1) \cap [M_x e^{-L_x}, M_x e^{L_x}] \tag{41}$$

for the shell at scale x. For

$$y \in \{x^\alpha, x^{\alpha^2}\}, \tag{42}$$

define the odd window

$$\mathcal{W}_y := (2\mathbb{N}+1) \cap [y, y^\alpha], \tag{43}$$

and let U_y denote the uniform random variable on $\mathcal{W}_y$.

We set

$$\mu := \log(4/3). \tag{44}$$

For a sufficiently large absolute constant $C > 0$, define the admissible time band

$$\mathcal{J}_{x,y} := \left[\frac{\log(y/x)}{\mu} - m_0 - C\log^{0.6} x, \ \frac{\log(y^\alpha/x)}{\mu} - m_0 + C\log^{0.6} x \right] \cap \mathbb{N}. \tag{45}$$

Throughout Sections 3 and 4, we assume that α and β have been fixed so that there exists $c_* > 0$ with

$$\sup_{\substack{y\in\{x^\alpha, x^{\alpha^2}\} \\ n\in\mathcal{J}_{x,y}}} 3^{2n} \le M_x x^{-2c_*} \tag{46}$$

for all sufficiently large x.

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For an odd initial value N, define the first-passage time

$$T_x(N) := \inf\{m \in \mathbb{N} : \mathrm{Syr}^m(N) \le x\}. \tag{47}$$

We also write

$$A_n^{\mathrm{orb}}(N) := \left|\mathbf{a}^{(n)}(N)\right| - 2n \tag{48}$$

for the centered weight of the actual Syracuse n-path of N. This notation is reserved for genuine orbits. The iid geometric model variables are introduced later and denoted separately.

Lemma 6 (Quantitative valuation law for uniform windows)**.** *Let $n \ge 1$, let $n_0 \ge 3n$, and let U be uniformly distributed on a set of odd integers*

$$\mathcal{W} = (2\mathbb{N}+1) \cap [Y_0, Y_1]$$

with $|\mathcal{W}| \gg 2^{n_0}$. Then

$$\mathrm{d_{TV}}(U \bmod 2^{n_0}, \mathrm{Unif}((2\mathbb{Z}+1)/2^{n_0}\mathbb{Z})) \ll 2^{-n_0}. \tag{49}$$

Consequently, Tao's valuation-distribution theorem implies

$$\mathrm{d_{TV}}\Big(\mathbf{a}^{(n)}(U), \mathrm{Geom}(2)^n\Big) \ll e^{-cn} \tag{50}$$

for some $c > 0$.

Proof. Because $\mathcal{W}$ is an interval of odd integers of length $\gg 2^{n_0}$, each odd residue class modulo 2^{n_0} is represented either

$$\left\lfloor \frac{|\mathcal{W}|}{2^{n_0-1}} \right\rfloor \quad \text{or} \quad \left\lceil \frac{|\mathcal{W}|}{2^{n_0-1}} \right\rceil$$

times. Hence the discrepancy between the induced residue distribution and the uniform distribution on odd residue classes is $O(2^{-n_0})$, which gives (49). Applying Tao's valuation-distribution theorem with this modulus input yields (50). □

Theorem 7 (Short-time localization)**.** *Assume $\alpha > 1$ is sufficiently close to 1, and choose $c_1 > 0$ such that*

$$\frac{\alpha^3 - 1}{\log(4/3)} < c_1 < \min\left\{\frac{\alpha^2}{3\log 2}, \frac{1}{\log 3}\right\}. \tag{51}$$

Let

$$n_1 := \lfloor c_1 \log x \rfloor. \tag{52}$$

Then, for $\beta > 0$ sufficiently small depending on α and c_1, one has

$$\mathcal{J}_{x,y} \subset [0, n_1] \tag{53}$$

for all $y \in \{x^\alpha, x^{\alpha^2}\}$ and all sufficiently large x, and

$$\mathbb{P}\Big(T_x(U_y) - m_0 \in \mathcal{J}_{x,y} \text{ and } \mathrm{Syr}^{T_x(U_y)-m_0}(U_y) \in \Sigma_x\Big) = 1 - O\Big(e^{-c\log^{0.2} x}\Big) \tag{54}$$

for some $c > 0$.

Proof. The inequality $3c_1 \log 2 < \alpha^2$ implies

$$2^{3n_1} \ll |\mathcal{W}_y| \tag{55}$$

uniformly in $y \in \{x^\alpha, x^{\alpha^2}\}$. Hence Lemma 6 with $n = n_1$ and $n_0 = 3n_1$ gives

$$d_{\mathrm{TV}}\Big(\mathbf{a}^{(n_1)}(U_y),\ \mathrm{Geom}(2)^{n_1}\Big) \ll e^{-cn_1}. \tag{56}$$

Define the good event

$$\Omega_x := \Big\{ \max_{0 \le m \le n_1} \Big| A_m^{\mathrm{orb}}(U_y) \Big| \le \log^{0.6} x \Big\}. \tag{57}$$

Because Ω_x depends only on the full n_1-path, (56) transfers it to the product model $\mathrm{Geom}(2)^{n_1}$. The Chernoff bound for partial sums of $\mathrm{Geom}(2) - 2$, followed by a union bound over $0 \le m \le n_1$, gives

$$\mathbb{P}(\Omega_x^c) \ll e^{-c \log^{0.2} x}. \tag{58}$$

On Ω_x, Tao's affine formula implies that for every $0 \le m \le n_1$,

$$\mathrm{Syr}^m(U_y) = (3/4)^m 2^{-A_m^{\mathrm{orb}}(U_y)} U_y + F_m\big(\mathbf{a}^{(m)}(U_y)\big) = e^{O(\log^{0.6} x)} (3/4)^m U_y + O(3^m). \tag{59}$$

Since $c_1 < 1/\log 3$, one has

$$3^{n_1} = o(x). \tag{60}$$

Thus the additive term in (59) is negligible on the first-passage scale. Because $c_1 > (\alpha^3 - 1)/\log(4/3)$, one also has

$$(3/4)^{n_1} U_y \ll x \tag{61}$$

uniformly for $y \in \{x^\alpha, x^{\alpha^2}\}$, so $T_x(U_y) \le n_1$ on Ω_x once x is large.

Applying (59) at times $T_x(U_y)$ and $T_x(U_y) - 1$, and using

$$\mathrm{Syr}^{T_x(U_y)}(U_y) \le x < \mathrm{Syr}^{T_x(U_y)-1}(U_y), \tag{62}$$

we obtain

$$\left| T_x(U_y) - \frac{\log(U_y/x)}{\mu} \right| \le C \log^{0.6} x \tag{63}$$

for a sufficiently large constant C. Subtracting m_0 yields

$$T_x(U_y) - m_0 \in \mathcal{J}_{x,y}.$$

A further application of (59) at time $T_x(U_y) - m_0$ gives

$$\mathrm{Syr}^{T_x(U_y)-m_0}(U_y) = e^{O(\log^{0.6} x)} M_x, \tag{64}$$

hence membership in Σ_x. Combining this with (58) proves (54). $\square$

3.2. Exact Inverse-Affine Shell Kernel

For $n \ge 0$, $s \in \mathbb{Z}$, and $M \in \Sigma_x$, define the fixed-time shell kernel

$$K_{x,y,n}(s, M) := \frac{1}{|\mathcal{W}_y|} \sum_{\substack{\mathbf{a} \in (\mathbb{N}+1)^n \\ |\mathbf{a}| - 2n = s}} 1_{\{N_{\mathbf{a}}(M) \in \mathcal{W}_y\}}, \tag{65}$$

where

$$N_{\mathbf{a}}(M) := \frac{2^{|\mathbf{a}|}(M - F_n(\mathbf{a}))}{3^n}. \tag{66}$$

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Proposition 8 (Exact inverse-affine shell kernel). *For every bounded test function $f : \mathbb{Z} \times (2\mathbb{N}+1) \to \mathbb{C}$,*

$$\mathbb{E}\Big[f\Big(A_n^{\mathrm{orb}}(U_y), \operatorname{Syr}^n(U_y)\Big) 1_{\Sigma_x}(\operatorname{Syr}^n(U_y))\Big] = \sum_{s\in\mathbb{Z}} \sum_{M\in\Sigma_x} f(s, M)\, K_{x,y,n}(s, M). \tag{67}$$

Equivalently,

$$\mathbb{E}\Big[f\Big(A_n^{\mathrm{orb}}(U_y), \operatorname{Syr}^n(U_y)\Big) 1_{\Sigma_x}(\operatorname{Syr}^n(U_y))\Big] = \frac{1}{|\mathcal{W}_y|} \sum_{\mathbf{a}\in(\mathbb{N}+1)^n} \sum_{M\in\Sigma_x} f(|\mathbf{a}| - 2n, M)\, 1_{\{N_{\mathbf{a}}(M)\in\mathcal{W}_y\}}. \tag{68}$$

Proof. Expand the expectation on the left-hand side of (68) as a uniform average over $N \in \mathcal{W}_y$. For each such N, let

$$\mathbf{a} := \mathbf{a}^{(n)}(N), \qquad M := \operatorname{Syr}^n(N). \tag{69}$$

Then the affine path formula gives

$$M = \operatorname{Aff}_{\mathbf{a}}(N), \qquad N = N_{\mathbf{a}}(M), \qquad A_n^{\mathrm{orb}}(N) = |\mathbf{a}| - 2n. \tag{70}$$

Thus each N contributes exactly one term on the right-hand side.

Conversely, if $N_{\mathbf{a}}(M) \in \mathcal{W}_y$, then

$$\operatorname{Aff}_{\mathbf{a}}(N_{\mathbf{a}}(M)) = M \tag{71}$$

is an odd integer. By the uniqueness characterization of Syracuse paths from the preliminaries, $\mathbf{a}$ is then the actual n-path of $N_{\mathbf{a}}(M)$. Hence the correspondence is bijective, and (68) follows. Formula (67) is the same identity rewritten in terms of the kernel (65). □

3.3. Arithmetic Counts and the Weighted Envelope

We now introduce the model random variables used to average the arithmetic path counts. Let

$$a_1, \ldots, a_n \overset{\mathrm{iid}}{\sim} \operatorname{Geom}(2), \tag{72}$$

and define the model centered weight and offset by

$$A_n^{\mathrm{mod}} := a_1 + \cdots + a_n - 2n, \qquad B_n^{\mathrm{mod}} := F_n(a_1, \ldots, a_n) \bmod 3^n. \tag{73}$$

For $s \in \mathbb{Z}$ and $M \in \mathbb{Z}/3^n\mathbb{Z}$, define

$$C_{n,s}(M) := \#\{\mathbf{a} \in (\mathbb{N}+1)^n : |\mathbf{a}| - 2n = s,\ F_n(\mathbf{a}) \equiv M \pmod{3^n}\}, \tag{74}$$

and

$$N_n(s) := \#\{\mathbf{a} \in (\mathbb{N}+1)^n : |\mathbf{a}| - 2n = s\}. \tag{75}$$

Since every path with $|\mathbf{a}| - 2n = s$ has probability 2^{-2n-s} under the model law,

$$2^{-2n-s} C_{n,s}(M) = \mathbb{P}\Big(A_n^{\mathrm{mod}} = s,\ B_n^{\mathrm{mod}} \equiv M \pmod{3^n}\Big). \tag{76}$$

For $t \in [-\pi, \pi]$ and $\xi \in \mathbb{Z}/3^n\mathbb{Z}$, define

$$\Psi_n(t, \xi) := \mathbb{E}\Big[e^{itA_n^{\mathrm{mod}}} e_{3^n}(\xi B_n^{\mathrm{mod}})\Big]. \tag{77}$$

For $1 \le r \le n$, define the weighted primitive-frequency envelope

$$D_r(n) := \sup_{t\in[-\pi,\pi]} \sup_{\xi\in(\mathbb{Z}/3^r\mathbb{Z})^\times} \Big|\mathbb{E}\Big[e^{itA_n^{\mathrm{mod}}} e_{3^r}\big(\xi (B_n^{\mathrm{mod}} \bmod 3^r)\big)\Big]\Big|. \tag{78}$$

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Proposition 9 (Fourier inversion for the weighted count). *For every $s \in \mathbb{Z}$ and $M \in \mathbb{Z}/3^n\mathbb{Z}$,*

$$2^{-2n-s} C_{n,s}(M) = \frac{1}{2\pi\, 3^n} \sum_{\xi \in \mathbb{Z}/3^n\mathbb{Z}} \int_{-\pi}^{\pi} e^{-its}\, e_{3^n}(-\xi M)\, \Psi_n(t,\xi)\, dt. \tag{79}$$

Proof. By definition,

$$2^{-2n-s} C_{n,s}(M) = \mathbb{P}\Big(A_n^{\mathrm{mod}} = s,\ B_n^{\mathrm{mod}} \equiv M \pmod{3^n} \Big). \tag{80}$$

Expand both indicators by exact Fourier inversion. For the integer condition,

$$1_{\{A_n^{\mathrm{mod}} = s\}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it(A_n^{\mathrm{mod}} - s)}\, dt. \tag{81}$$

For the residue-class condition modulo 3^n,

$$1_{\{B_n^{\mathrm{mod}} \equiv M \pmod{3^n}\}} = \frac{1}{3^n} \sum_{\xi \in \mathbb{Z}/3^n\mathbb{Z}} e_{3^n}\big(\xi (B_n^{\mathrm{mod}} - M)\big). \tag{82}$$

Multiplying (81) and (82), taking expectations, and using

$$\Psi_n(t,\xi) = \mathbb{E}\Big[e^{itA_n^{\mathrm{mod}}}\, e_{3^n}(\xi B_n^{\mathrm{mod}}) \Big], \tag{83}$$

we obtain (79). □

Proposition 10 (Exact interval-averaged arithmetic reduction). *Let $I \subset \mathbb{Z}$ be an interval of odd integers. Then*

$$\sum_{M \in I} \frac{1}{|\mathcal{W}_y|} C_{n,s}(M) = \frac{N_n(s)}{3^n |\mathcal{W}_y|} \#I + O\left(\frac{2^{2n+s}}{|\mathcal{W}_y|} \sum_{r=1}^{n} 3^{2r-n} D_r(n) \right), \tag{84}$$

uniformly in admissible x, in $y \in \{x^\alpha, x^{\alpha^2}\}$, in $n \in \mathcal{J}_{x,y}$, in $s \in \mathbb{Z}$, and in odd intervals I.

Proof. Summing (79) over $M \in I$ and multiplying by $2^{2n+s}/|\mathcal{W}_y|$ gives

$$\sum_{M \in I} \frac{1}{|\mathcal{W}_y|} C_{n,s}(M) = \frac{2^{2n+s}}{|\mathcal{W}_y|\, 2\pi\, 3^n} \sum_{\xi \in \mathbb{Z}/3^n\mathbb{Z}} \int_{-\pi}^{\pi} e^{-its} \Psi_n(t,\xi)\, S_I(\xi)\, dt, \tag{85}$$

where

$$S_I(\xi) := \sum_{M \in I} e_{3^n}(-\xi M). \tag{86}$$

The zero mode $\xi = 0$ contributes

$$\frac{2^{2n+s}}{|\mathcal{W}_y|\, 3^n} \mathbb{P}(A_n^{\mathrm{mod}} = s)\, \#I = \frac{N_n(s)}{3^n |\mathcal{W}_y|} \#I, \tag{87}$$

which is the main term.

Now let $\xi \neq 0$, and let $r = r(\xi)$ denote its exact 3-adic scale, so

$$\xi = 3^{n-r} \xi', \qquad \xi' \in (\mathbb{Z}/3^r\mathbb{Z})^\times. \tag{88}$$

Then

$$e_{3^n}(\xi B_n^{\mathrm{mod}}) = e_{3^r}\big(\xi' (B_n^{\mathrm{mod}} \bmod 3^r)\big), \tag{89}$$

and hence

$$|\Psi_n(t,\xi)| \le D_r(n). \tag{90}$$

Because I is an interval of consecutive odd integers, $S_I(\xi)$ is a geometric sum with ratio $e_{3^n}(-2\xi)$, which has exact order 3^r. Therefore

$$|S_I(\xi)| \ll 3^r. \tag{91}$$

There are $O(3^r)$ frequencies of exact scale r, so the total nonzero contribution in (85) is

$$O\left(\frac{2^{2n+s}}{|\mathcal{W}_y|\, 3^n} \sum_{r=1}^{n} 3^{2r} D_r(n)\right), \tag{92}$$

which is exactly (84). □

3.4. Shell Normalization, Boundary Layer, and Shell Slices

For $n \geq 0$ and $s \in \mathbb{Z}$, define

$$R_{n,s} := n \log(4/3) + s \log 2, \tag{93}$$

and for $M \in 2\mathbb{N} + 1$,

$$\Gamma_{n,s}(M) := R_{n,s} + \log M. \tag{94}$$

Define the phase interval

$$I_{x,y,n,s} := (2\mathbb{N} + 1) \cap \Sigma_x \cap [ye^{-R_{n,s}}, y^{\alpha} e^{-R_{n,s}}]. \tag{95}$$

Lemma 11 (Shell-regime normalization)**.** *Assume that $I_{x,y,n,s} \neq \varnothing$. Then for every $M \in I_{x,y,n,s}$,*

$$\frac{2^{2n+s}}{|\mathcal{W}_y|\, 3^n} = \frac{e^{R_{n,s}}}{|\mathcal{W}_y|} \ll \frac{1}{M} \ll \frac{e^{L_x}}{M_x}. \tag{96}$$

The implied constants are uniform in admissible x, in $y \in \{x^{\alpha}, x^{\alpha^2}\}$, in $n \in \mathcal{J}_{x,y}$, and in $s \in \mathbb{Z}$.

Proof. By definition,

$$e^{R_{n,s}} = \frac{2^{2n+s}}{3^n}, \tag{97}$$

which gives the identity in (96). If $M \in I_{x,y,n,s}$, then

$$ye^{-R_{n,s}} \leq M \leq y^{\alpha} e^{-R_{n,s}}, \tag{98}$$

so

$$e^{R_{n,s}} \leq \frac{y^{\alpha}}{M}. \tag{99}$$

Since $|\mathcal{W}_y| \asymp y^{\alpha}$, this yields

$$\frac{e^{R_{n,s}}}{|\mathcal{W}_y|} \ll \frac{1}{M}. \tag{100}$$

Finally, $M \in \Sigma_x$ implies

$$M_x e^{-L_x} \leq M \leq M_x e^{L_x}, \tag{101}$$

and therefore

$$\frac{1}{M} \leq \frac{e^{L_x}}{M_x}. \tag{102}$$

Combining (97), (100), and (102) proves the lemma. □

Lemma 12 (Shell-ratio bound)**.** *There exists a constant $c_{\mathrm{rat}} > 0$, depending only on the fixed shell parameters, such that for all sufficiently large x, for every admissible $y \in \{x^{\alpha}, x^{\alpha^2}\}$, for every $n \in \mathcal{J}_{x,y}$, and for every $M \in \Sigma_x$, one has*

$$\frac{3^n}{M} \ll x^{-c_{\mathrm{rat}}}. \tag{103}$$

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Consequently, for every path $\mathbf{a} \in (\mathbb{N}+1)^n$,

$$\frac{F_n(\mathbf{a})}{M} \ll x^{-c_{\mathrm{rat}}}, \tag{104}$$

since $0 \le F_n(\mathbf{a}) \le 3^n$.

Proof. Because $M \in \Sigma_x$, one has

$$M \ge M_x e^{-L_x}, \tag{105}$$

and therefore

$$\frac{3^n}{M} \le e^{L_x} \frac{3^n}{M_x}. \tag{106}$$

Now the standing shell assumption gives

$$3^{2n} \le M_x x^{-2c_*}. \tag{107}$$

Since $3^n \ge 1$, dividing (107) by $3^n M_x$ yields

$$\frac{3^n}{M_x} \le x^{-2c_*}. \tag{108}$$

Substituting (108) into (106), we obtain

$$\frac{3^n}{M} \le e^{L_x} x^{-2c_*}. \tag{109}$$

Since $L_x = \log^{0.7} x$, one has $e^{L_x} = x^{o(1)}$. Hence there exists a fixed constant $c_{\mathrm{rat}} \in (0, 2c_*)$ such that

$$e^{L_x} x^{-2c_*} \ll x^{-c_{\mathrm{rat}}} \tag{110}$$

for all sufficiently large x. This proves (103). The bound (104) follows immediately from $0 \le F_n(\mathbf{a}) \le 3^n$. □

From now on, fix $c_0 > 0$ and $C_0 > 0$ so that

$$0 < c_0 < c_{\mathrm{rat}}. \tag{111}$$

Define the source phase window

$$J_y := [\log y,\, \alpha \log y], \tag{112}$$

and

$$\partial J_y^{(x)} := \{u \in \mathbb{R} : \mathrm{dist}\big(u, \{\log y, \alpha \log y\}\big) \le C_0 x^{-c_0}\}, \tag{113}$$

as well as the boundary slice

$$\partial I_{x,y,n,s}^{(x)} := (2\mathbb{N}+1) \cap \Sigma_x \cap \{M : \mathrm{dist}\big(\Gamma_{n,s}(M), \{\log y, \alpha \log y\}\big) \le C_0 x^{-c_0}\}. \tag{114}$$

Theorem 13 (Boundary-layer negligibility)**.** *There exists* $c > 0$ *such that*

$$\sup_{y \in \{x^\alpha, x^{\alpha^2}\}} \sup_{n \in \mathcal{J}_{x,y}} \sum_{s \in \mathbb{Z}} \sum_{M \in \Sigma_x} 1_{\partial J_y^{(x)}}\big(\Gamma_{n,s}(M)\big) K_{x,y,n}(s, M) \ll x^{-c}. \tag{115}$$

Proof. By the exact inverse-affine shell-kernel formula,

$$\sum_{s \in \mathbb{Z}} \sum_{M \in \Sigma_x} 1_{\partial J_y^{(x)}}\big(\Gamma_{n,s}(M)\big) K_{x,y,n}(s, M) = \mathbb{P}\Big(\mathrm{Syr}^n(U_y) \in \Sigma_x,\, \Gamma_{n, A_n^{\mathrm{orb}}(U_y)}\big(\mathrm{Syr}^n(U_y)\big) \in \partial J_y^{(x)}\Big). \tag{116}$$

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Set

$$N := U_y, \qquad M := \mathrm{Syr}^n(N), \qquad B := F_n(\mathbf{a}^{(n)}(N)). \tag{117}$$

By the affine path formula and the definition of $A_n^{\mathrm{orb}}(N)$,

$$N = (4/3)^n 2^{A_n^{\mathrm{orb}}(N)}(M - B), \tag{118}$$

hence

$$\Gamma_{n,A_n^{\mathrm{orb}}(N)}(M) = \log N + \log \frac{M}{M - B}. \tag{119}$$

Now

$$0 \le B \le 3^n, \tag{120}$$

so Lemma 12 gives

$$\frac{B}{M} \ll x^{-c_{\mathrm{rat}}}. \tag{121}$$

Therefore

$$\log \frac{M}{M - B} = O(x^{-c_{\mathrm{rat}}}), \tag{122}$$

since $B/M = o(1)$.

If the event in (116) occurs, then (119) and (122) force $\log N$ to lie within $O(x^{-c_0}) + O(x^{-c_{\mathrm{rat}}})$ of one of the two endpoints $\log y$ or $\alpha \log y$. Because $c_0 < c_{\mathrm{rat}}$, this is $O(x^{-c})$ for some fixed $c > 0$. Thus N belongs to one of two endpoint intervals of total length $O(y^\alpha x^{-c})$ inside $[y, y^\alpha]$. Since U_y is uniform on $\mathcal{W}_y$ and $|\mathcal{W}_y| \asymp y^\alpha$, the probability of this event is $O(x^{-c})$. This proves (115). □

For admissible x, y, n and $s \in \mathbb{Z}$, define the shell-slice mass

$$\mathcal{K}_{x,y,n}(s) := \sum_{M \in \Sigma_x} K_{x,y,n}(s, M). \tag{123}$$

Proposition 14 (Shell-slice decomposition). *For every admissible x, y, n, s,*

$$\mathcal{K}_{x,y,n}(s) = \frac{N_n(s)}{3^n |\mathcal{W}_y|} \# I_{x,y,n,s} + O\left(\frac{N_n(s)}{3^n |\mathcal{W}_y|} \# \partial I^{(x)}_{x,y,n,s} + \frac{2^{2n+s}}{|\mathcal{W}_y|} \sum_{r=1}^{n} 3^{2r-n} D_r(n) \right). \tag{124}$$

Proof. By definition of the shell kernel,

$$\mathcal{K}_{x,y,n}(s) = \frac{1}{|\mathcal{W}_y|} \sum_{M \in \Sigma_x} \sum_{\substack{\mathbf{a} \in (\mathbb{N}+1)^n \\ |\mathbf{a}| - 2n = s}} 1_{\{N_{\mathbf{a}}(M) \in \mathcal{W}_y\}}. \tag{125}$$

Fix $M \in \Sigma_x$ and a path $\mathbf{a}$ with $|\mathbf{a}| - 2n = s$. Then

$$N_{\mathbf{a}}(M) = \frac{2^{|\mathbf{a}|}(M - F_n(\mathbf{a}))}{3^n} = e^{R_{n,s}}(M - F_n(\mathbf{a})). \tag{126}$$

Taking logarithms gives

$$\log N_{\mathbf{a}}(M) = \Gamma_{n,s}(M) + \log\left(1 - \frac{F_n(\mathbf{a})}{M}\right). \tag{127}$$

Because

$$0 \le F_n(\mathbf{a}) \le 3^n, \tag{128}$$

Lemma 12 gives

$$\frac{F_n(\mathbf{a})}{M} \ll x^{-c_{\mathrm{rat}}}, \tag{129}$$

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hence
$$\log N_{\mathbf{a}}(M) = \Gamma_{n,s}(M) + O(x^{-c_{\mathrm{rat}}}). \tag{130}$$

Therefore the exact affine window condition
$$N_{\mathbf{a}}(M) \in [y, y^{\alpha}] \tag{131}$$
agrees with the coarse phase condition
$$M \in I_{x,y,n,s} \tag{132}$$
unless $M \in \partial I^{(x)}_{x,y,n,s}$, because $c_0 < c_{\mathrm{rat}}$. Thus
$$\mathcal{K}_{x,y,n}(s) = \frac{1}{|\mathcal{W}_y|} \sum_{M \in I_{x,y,n,s}} C_{n,s}(M) + O\left(\frac{1}{|\mathcal{W}_y|} \sum_{M \in \partial I^{(x)}_{x,y,n,s}} C_{n,s}(M) \right). \tag{133}$$

Apply Proposition 10 first with $I = I_{x,y,n,s}$ and then with $I = \partial I^{(x)}_{x,y,n,s}$. This gives
$$\frac{1}{|\mathcal{W}_y|} \sum_{M \in I_{x,y,n,s}} C_{n,s}(M) = \frac{N_n(s)}{3^n |\mathcal{W}_y|} \# I_{x,y,n,s} + O\left(\frac{2^{2n+s}}{|\mathcal{W}_y|} \sum_{r=1}^{n} 3^{2r-n} D_r(n) \right), \tag{134}$$
and
$$\frac{1}{|\mathcal{W}_y|} \sum_{M \in \partial I^{(x)}_{x,y,n,s}} C_{n,s}(M) = \frac{N_n(s)}{3^n |\mathcal{W}_y|} \# \partial I^{(x)}_{x,y,n,s} + O\left(\frac{2^{2n+s}}{|\mathcal{W}_y|} \sum_{r=1}^{n} 3^{2r-n} D_r(n) \right). \tag{135}$$

Substituting (134) and (135) into (133) proves (124). □

Remark 15. Proposition 14 is a shell-*slice* statement. It does not imply a pointwise asymptotic at a fixed shell point M. No passage from averaged-in-M information to pointwise fixed-M shell mass is used in the unconditional theorem chain.

4. Primitive-Frequency Operator and the Remaining Arithmetic Obstruction

This section analyzes the weighted primitive-frequency quantities appearing in Proposition 14. The first result is an exact recursion for the weighted primitive frequencies. The second is an exact block decomposition of the corresponding transfer operator. The third is a negative statement, no naive operator gap is available. The section then reduces the unresolved arithmetic step to explicit incomplete principal-unit exponential sums and formalizes the exact shell-slice consequence of primitive-frequency decay.

4.1. Weighted Primitive-Frequency Recursion

For $r \geq 1$, let
$$\Omega_r := (\mathbb{Z}/3^r\mathbb{Z})^{\times}, \qquad N_r := |\Omega_r| = 2 \cdot 3^{r-1}. \tag{136}$$
Using the fact that 2 has order N_r modulo 3^r, identify
$$\Omega_r \cong \mathbb{Z}/N_r\mathbb{Z}, \qquad j \longleftrightarrow 2^j \pmod{3^r}. \tag{137}$$

Let Y_r denote Tao's Syracuse random variable modulo 3^r, and define
$$\Phi_r(t, \eta) := \mathbb{E}\left[e^{itA_r^{\mathrm{mod}}} e_{3^r}(\eta Y_r) \right], \qquad \eta \in \Omega_r. \tag{138}$$

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Proposition 16 (Primitive-frequency recursion). *For each $t \in [-\pi, \pi]$ and each $\xi \in \Omega_{r+1}$,*

$$\Phi_{r+1}(t, \xi) = \sum_{a \geq 1} 2^{-a} e^{it(a-2)} e_{3^{r+1}}(\xi 2^{-a}) \, \Phi_r(t, \pi_r(\xi 2^{-a})), \tag{139}$$

where $\pi_r : \Omega_{r+1} \to \Omega_r$ is reduction modulo 3^r.

Proof. Tao's Syracuse recursion gives

$$Y_{r+1} \equiv \frac{3Y_r + 1}{2^{a_{r+1}}} \pmod{3^{r+1}}. \tag{140}$$

Hence

$$e_{3^{r+1}}(\xi Y_{r+1}) = e_{3^{r+1}}(\xi 2^{-a_{r+1}}) \, e_{3^r}\big(\pi_r(\xi 2^{-a_{r+1}}) Y_r\big). \tag{141}$$

Also

$$A_{r+1}^{\mathrm{mod}} = A_r^{\mathrm{mod}} + (a_{r+1} - 2). \tag{142}$$

Condition on a_{r+1} and use independence of a_{r+1} and Y_r to obtain (139). □

For each $t \in [-\pi, \pi]$, define the operator

$$\mathcal{T}_{r,t} : \ell^2(\Omega_r) \to \ell^2(\Omega_{r+1}) \tag{143}$$

by

$$(\mathcal{T}_{r,t} f)(\xi) := \sum_{a \geq 1} 2^{-a} e^{it(a-2)} e_{3^{r+1}}(\xi 2^{-a}) \, f(\pi_r(\xi 2^{-a})), \qquad \xi \in \Omega_{r+1}. \tag{144}$$

Then

$$\Phi_{r+1}(t, \cdot) = \mathcal{T}_{r,t} \Phi_r(t, \cdot). \tag{145}$$

4.2. Exact Block Decomposition

Let

$$\omega_r := 2^{N_r} \pmod{3^{r+1}}. \tag{146}$$

Then ω_r has order 3, and every primitive frequency in Ω_{r+1} can be written uniquely as

$$\xi = 2^u \omega_r^{\ell}, \qquad u \in \mathbb{Z}/N_r\mathbb{Z}, \quad \ell \in \{0, 1, 2\}. \tag{147}$$

For $t \in [-\pi, \pi]$, define

$$q_r(t) := 2^{-N_r} e^{itN_r}. \tag{148}$$

For $d \in \mathbb{Z}/N_r\mathbb{Z}$, let $b(d) \in \{1, \ldots, N_r\}$ denote the positive representative of d. Set

$$a_{r,t}(d) := 2^{-b(d)} e^{it\, b(d)}. \tag{149}$$

For $\ell \in \{0, 1, 2\}$ and $j \in \mathbb{Z}/N_r\mathbb{Z}$, define

$$h_{r,\ell}(j) := e_{3^{r+1}}(\omega_r^{\ell} 2^j). \tag{150}$$

Proposition 17 (Exact block decomposition). *Under the identifications*

$$\Omega_r \cong \mathbb{Z}/N_r\mathbb{Z}, \qquad \Omega_{r+1} \cong \mathbb{Z}/N_r\mathbb{Z} \times \{0, 1, 2\}, \tag{151}$$

the operator $\mathcal{T}_{r,t}$ satisfies

$$(\mathcal{T}_{r,t} f)(u, \ell) = \frac{e^{-2it}}{1 - q_r(t)^3} \sum_{j \in \mathbb{Z}/N_r\mathbb{Z}} a_{r,t}(u - j) \Big(h_{r,\ell}(j) + q_r(t) h_{r,\ell-1}(j) + q_r(t)^2 h_{r,\ell-2}(j) \Big) f(j), \tag{152}$$

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with the index ℓ taken modulo 3.

Proof. Fix (u, ℓ) and write $\xi = 2^u \omega_r^\ell$. For a fixed $j \in \mathbb{Z}/N_r\mathbb{Z}$, the condition

$$\pi_r(\xi 2^{-a}) = 2^j \tag{153}$$

is equivalent to

$$u - a \equiv j \pmod{N_r}. \tag{154}$$

Thus the admissible exponents are exactly

$$a = b(u-j) + kN_r + 3qN_r, \qquad k \in \{0,1,2\}, \quad q \geq 0. \tag{155}$$

For such a,

$$2^{-a} = 2^{-b(u-j)}(2^{-N_r})^k(2^{-3N_r})^q. \tag{156}$$

Since

$$2^{-N_r} \equiv \omega_r^{-1} \pmod{3^{r+1}}, \tag{157}$$

we obtain

$$\xi 2^{-a} = 2^u \omega_r^\ell 2^{-b(u-j)} \omega_r^{-k} \equiv 2^j \omega_r^{\ell-k} \pmod{3^{r+1}}, \tag{158}$$

and therefore

$$e_{3^{r+1}}(\xi 2^{-a}) = h_{r,\ell-k}(j). \tag{159}$$

Also,

$$2^{-a} e^{it(a-2)} = e^{-2it}\, a_{r,t}(u-j)\, q_r(t)^k\, (q_r(t)^3)^q. \tag{160}$$

Substituting (155) into (144), summing first over $q \geq 0$ and then over $k = 0, 1, 2$, yields (152). □

Define the principal operator

$$(\mathcal{A}_{r,t} f)(u, \ell) := e^{-2it} \sum_{j \in \mathbb{Z}/N_r\mathbb{Z}} a_{r,t}(u-j)\, h_{r,\ell}(j)\, f(j). \tag{161}$$

Corollary 18 (Exponentially small remainder)**.**

$$\|\mathcal{T}_{r,t} - \mathcal{A}_{r,t}\|_{\ell^2(\Omega_r) \to \ell^2(\Omega_{r+1})} \ll 2^{-N_r} \tag{162}$$

uniformly in r and t.

Proof. From (152), one has

$$\frac{1}{1 - q_r(t)^3} = 1 + O(2^{-N_r}) \tag{163}$$

uniformly in t, since $|q_r(t)| = 2^{-N_r}$. The two lifted off-diagonal terms carry coefficients $q_r(t)$ and $q_r(t)^2$, so their total row and column mass is $O(2^{-N_r})$. Schur's test yields (162). □

4.3. No Naive Spectral Gap

Define the cyclic convolution operator

$$(C_{r,t} g)(u) := \sum_{j \in \mathbb{Z}/N_r\mathbb{Z}} a_{r,t}(u-j)\, g(j). \tag{164}$$

Proposition 19 (No naive operator gap)**.**

$$\|\mathcal{T}_{r,0}\|_{\ell^2(\Omega_r) \to \ell^2(\Omega_{r+1})} \geq 1 - O(2^{-N_r}). \tag{165}$$

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Proof. Since multiplication by $h_{r,\ell}$ is unitary in physical space,

$$\|\mathcal{A}_{r,t}\| \geq \|C_{r,t}\|. \tag{166}$$

The operator $C_{r,t}$ is diagonalized by the Fourier basis on $\mathbb{Z}/N_r\mathbb{Z}$. Its multiplier is

$$m_{r,t}(\theta) = \sum_{b=1}^{N_r} 2^{-b} e^{ib(t-\theta)} = \frac{\frac{1}{2}e^{i(t-\theta)}\left(1 - (\frac{1}{2}e^{i(t-\theta)})^{N_r}\right)}{1 - \frac{1}{2}e^{i(t-\theta)}}, \tag{167}$$

where $\theta \in \frac{2\pi}{N_r}\mathbb{Z}/2\pi\mathbb{Z}$. At $t = 0$ and $\theta = 0$,

$$m_{r,0}(0) = \sum_{b=1}^{N_r} 2^{-b} = 1 - 2^{-N_r}. \tag{168}$$

Hence

$$\|C_{r,0}\| \geq 1 - 2^{-N_r}. \tag{169}$$

Now apply Corollary 18. □

4.4. Reduction to Incomplete Principal-Unit Sums

Define the discrete Fourier transform of $h_{r,\ell}$ by

$$\widehat{h}_{r,\ell}(m) := \sum_{j=0}^{N_r-1} h_{r,\ell}(j) e^{-2\pi i m j/N_r}, \qquad m \in \mathbb{Z}/N_r\mathbb{Z}. \tag{170}$$

Proposition 20 (Reduction to incomplete principal-unit sums)**.** *For every $r \geq 1$, $\ell \in \{0,1,2\}$, and $m \in \mathbb{Z}/N_r\mathbb{Z}$,*

$$\widehat{h}_{r,\ell}(m) = \sum_{\varepsilon=0}^{1} e_{N_r}(-m\varepsilon)\, S_{r,\ell,\varepsilon}(m), \tag{171}$$

where

$$S_{r,\ell,\varepsilon}(m) := \sum_{u=0}^{3^{r-1}-1} e_{3^{r+1}}\Big(2^{\varepsilon}\omega_r^{\ell} 4^u - 9mu\Big). \tag{172}$$

Proof. Write

$$j = \varepsilon + 2u, \qquad \varepsilon \in \{0,1\}, \quad 0 \leq u \leq 3^{r-1} - 1. \tag{173}$$

Then

$$2^j = 2^{\varepsilon} 4^u, \qquad e^{-2\pi i m j/N_r} = e_{N_r}(-m\varepsilon) e_{N_r}(-2mu). \tag{174}$$

Since $N_r = 2 \cdot 3^{r-1}$, one has

$$e_{N_r}(-2mu) = e_{3^{r+1}}(-9mu). \tag{175}$$

Substituting (173)–(175) into (170) gives (171). □

Let $\log_3$ and $\exp_3$ denote the 3-adic logarithm and exponential on the principal-unit subgroup $1 + 3\mathbb{Z}_3$. Since $4 \in 1 + 3\mathbb{Z}_3$, define

$$\lambda := \log_3(4) \in 3\mathbb{Z}_3^{\times}. \tag{176}$$

Then

$$4^u = \exp_3(u\lambda) \qquad (u \in \mathbb{Z}_{\geq 0}), \tag{177}$$

so (172) may be rewritten as

$$S_{r,\ell,\varepsilon}(m) = \sum_{u=0}^{3^{r-1}-1} e_{3^{r+1}}(c_{\ell,\varepsilon} \exp_3(u\lambda) - 9mu), \qquad c_{\ell,\varepsilon} := 2^{\varepsilon}\omega_r^{\ell}. \tag{178}$$

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Proposition 21 (No stationary points). *Let*

$$\Phi_{c,m}(u) := c\ \exp_3(u\lambda) - 9mu, \qquad c \in \mathbb{Z}_3^{\times}. \tag{179}$$

Then for every integer m and every $u \in \mathbb{Z}_3$,

$$v_3\big(\Phi'_{c,m}(u)\big) = 1. \tag{180}$$

In particular, the phase is non-stationary modulo 9, and hence modulo every higher power of 3.

Proof. Differentiate (179):

$$\Phi'_{c,m}(u) = c\,\lambda\ \exp_3(u\lambda) - 9m. \tag{181}$$

Since $v_3(\lambda) = 1$, c is a unit, and $\exp_3(u\lambda) \in 1 + 3\mathbb{Z}_3$ is also a unit, the first term has valuation exactly 1. The second term has valuation at least 2. Thus the sum has valuation exactly 1, proving (180). □

4.5. Dangerous Band and Microlocal Leakage

Define

$$m_\infty(\phi) := \sum_{b=1}^{\infty} 2^{-b} e^{ib\phi} = \frac{\frac{1}{2}e^{i\phi}}{1 - \frac{1}{2}e^{i\phi}}. \tag{182}$$

Then

$$|m_\infty(\phi)|^2 = \frac{1}{1 + 8\sin^2(\phi/2)}. \tag{183}$$

For $0 < \eta < 1/10$, define the dangerous band

$$\mathcal{D}_{r,t}(\eta) := \left\{ m \in \mathbb{Z}/N_r\mathbb{Z} : \left| m_{r,t}\left(\frac{2\pi m}{N_r}\right)\right| > 1 - \eta \right\}, \tag{184}$$

where $m_{r,t}$ is the multiplier (167).

Proposition 22 (Dangerous-band geometry). *There exists an absolute constant $C > 0$ such that*

$$\mathcal{D}_{r,t}(\eta) \subset \left\{ m \in \mathbb{Z}/N_r\mathbb{Z} : \operatorname{dist}\left(\frac{2\pi m}{N_r}, t + 2\pi\mathbb{Z}\right) \le C\sqrt{\eta} \right\}, \tag{185}$$

and therefore

$$|\mathcal{D}_{r,t}(\eta)| \ll \sqrt{\eta}\, N_r + 1. \tag{186}$$

Proof. Because the finite geometric sum differs from the infinite one by a tail of size $O(2^{-N_r})$, one has

$$m_{r,t}(\theta) = m_\infty(t - \theta) + O(2^{-N_r}) \tag{187}$$

uniformly in r, t, θ. If

$$\left| m_{r,t}\left(\frac{2\pi m}{N_r}\right)\right| > 1 - \eta,$$

then for large r,

$$|m_\infty(t - 2\pi m/N_r)| > 1 - 2\eta.$$

Using (183), this implies

$$\sin^2\left(\frac{t - 2\pi m/N_r}{2}\right) \ll \eta, \tag{188}$$

which gives (185). Counting lattice points of spacing $2\pi/N_r$ yields (186). □

Let $M_{h_{r,\ell}}$ denote multiplication by $h_{r,\ell}$ in physical space, and let $P_{\mathcal{D}}$ denote Fourier projection to $\mathcal{D} \subset \mathbb{Z}/N_r\mathbb{Z}$.

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Proposition 23 (Microlocal leakage estimate). *For every $\mathcal{D} \subset \mathbb{Z}/N_r\mathbb{Z}$,*

$$\|P_{\mathcal{D}} M_{h_{r,\ell}} P_{\mathcal{D}}\|_{\ell^2\to\ell^2} \le N_r^{-1/2}\,\|\widehat{h}_{r,\ell}\|_{\ell^1(\mathcal{D}-\mathcal{D})}. \tag{189}$$

Proof. Under the unitary Fourier transform on $\mathbb{Z}/N_r\mathbb{Z}$, multiplication by $h_{r,\ell}$ becomes convolution with $N_r^{-1/2}\widehat{h}_{r,\ell}$. If both input and output are projected to $\mathcal{D}$, only differences in $\mathcal{D}-\mathcal{D}$ contribute. Young's inequality $\ell^1 * \ell^2 \to \ell^2$ then gives (189). □

Proposition 24 (Dangerous-band suppression under short-window Fourier mass). *Assume that there exist $\delta > 0$ and $\eta_0 > 0$ such that for every $0 < \eta \le \eta_0$,*

$$\sup_{r\ge 1}\ \sup_{\ell\in\{0,1,2\}}\ \sup_{\substack{J\subset\mathbb{Z}/N_r\mathbb{Z}\\ |J|\ll\sqrt{\eta}\,N_r}} N_r^{-1/2}\|\widehat{h}_{r,\ell}\|_{\ell^1(J)} \ll \eta^{1/2+\delta}. \tag{190}$$

Then, uniformly in $0 < \eta \le \eta_0$, in $r \ge 1$, in $t \in [-\pi,\pi]$, and in $\ell \in \{0,1,2\}$, one has

$$\|P_{\mathcal{D}_{r,t}(\eta)} M_{h_{r,\ell}}\|_{\ell^2\to\ell^2} + \|P_{\mathcal{D}_{r,t}(\eta)} M_{h_{r,\ell}} P_{\mathcal{D}_{r,t}(\eta)}\|_{\ell^2\to\ell^2} \ll \eta^{1/2+\delta}. \tag{191}$$

Proof. Under the unitary Fourier transform, the kernel of

$$P_{\mathcal{D}_{r,t}(\eta)} M_{h_{r,\ell}} \tag{192}$$

is

$$K(m,m') = N_r^{-1/2}\widehat{h}_{r,\ell}(m-m')\,1_{\mathcal{D}_{r,t}(\eta)}(m). \tag{193}$$

By Schur's test,

$$\|P_{\mathcal{D}_{r,t}(\eta)} M_{h_{r,\ell}}\|_{\ell^2\to\ell^2} \le \sup_{m'} \sum_{m\in\mathcal{D}_{r,t}(\eta)} N_r^{-1/2}\left|\widehat{h}_{r,\ell}(m-m')\right|. \tag{194}$$

For fixed m', the set

$$\{m-m' : m \in \mathcal{D}_{r,t}(\eta)\}$$

is a translate of $\mathcal{D}_{r,t}(\eta)$. By Proposition 22, this set has cardinality $O(\sqrt{\eta}\,N_r)$, so (190) implies

$$\|P_{\mathcal{D}_{r,t}(\eta)} M_{h_{r,\ell}}\|_{\ell^2\to\ell^2} \ll \eta^{1/2+\delta}. \tag{195}$$

The second bound in (191) follows from

$$\|P_{\mathcal{D}_{r,t}(\eta)} M_{h_{r,\ell}} P_{\mathcal{D}_{r,t}(\eta)}\|_{\ell^2\to\ell^2} \le \|P_{\mathcal{D}_{r,t}(\eta)} M_{h_{r,\ell}}\|_{\ell^2\to\ell^2}. \tag{196}$$

□

4.6. Bridge Back to the Shell-Slice Envelope

We now relate the Section 4 quantities to the envelope $D_r(n)$ from Section 3.

Define

$$\mathcal{H}_r(t) := \sup_{\eta\in\Omega_r} |\Phi_r(t,\eta)|, \qquad \mathcal{H}_r := \sup_{t\in[-\pi,\pi]} \mathcal{H}_r(t). \tag{197}$$

Also define

$$\chi(t) := \mathbb{E}\Big[e^{it(\mathrm{Geom}(2)-2)}\Big] = \frac{e^{-2it}}{2-e^{it}}. \tag{198}$$

Since

$$|\chi(t)|^2 = \frac{1}{5-4\cos t} \le 1, \tag{199}$$

one has $|\chi(t)| \le 1$ for all $t \in [-\pi,\pi]$.

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Proposition 25 (Exact factorization of the weighted envelope)**.** *For every $1 \le r \le n$, every $t \in [-\pi, \pi]$, and every $\xi \in (\mathbb{Z}/3^r\mathbb{Z})^\times$,*

$$\mathbb{E}\Big[e^{itA_n^{\mathrm{mod}}} e_{3^r}\big(\xi(B_n^{\mathrm{mod}} \bmod 3^r)\big)\Big] = \chi(t)^{n-r}\,\Phi_r(t,\xi). \tag{200}$$

Consequently,

$$D_r(n) \le \mathcal{H}_r. \tag{201}$$

Proof. From the explicit formula for F_n,

$$B_n^{\mathrm{mod}} = \sum_{m=1}^{n} 3^{n-m} 2^{-a[m,n]} \pmod{3^n}. \tag{202}$$

Reducing modulo 3^r, only the last r coordinates survive:

$$B_n^{\mathrm{mod}} \bmod 3^r = \sum_{j=0}^{r-1} 3^j 2^{-a[n-j,n]} \pmod{3^r}. \tag{203}$$

If we set

$$c_j := a_{n-j+1}, \qquad 1 \le j \le r, \tag{204}$$

then the right-hand side of (203) becomes

$$\sum_{j=0}^{r-1} 3^j 2^{-c[1,j+1]}, \tag{205}$$

which has exactly the same distribution as Y_r modulo 3^r.

Next decompose

$$A_n^{\mathrm{mod}} = \sum_{j=1}^{n-r}(a_j - 2) + \sum_{j=n-r+1}^{n}(a_j - 2) =: U_{n-r} + V_r. \tag{206}$$

The two terms are independent, and V_r has the same law as A_r^{mod}. Therefore

$$\mathbb{E}\Big[e^{itA_n^{\mathrm{mod}}} e_{3^r}\big(\xi(B_n^{\mathrm{mod}} \bmod 3^r)\big)\Big] = \mathbb{E}\Big[e^{itU_{n-r}}\Big]\,\mathbb{E}\Big[e^{itA_r^{\mathrm{mod}}} e_{3^r}(\xi Y_r)\Big]. \tag{207}$$

Since U_{n-r} is a sum of $n-r$ iid copies of $\mathrm{Geom}(2) - 2$,

$$\mathbb{E}\Big[e^{itU_{n-r}}\Big] = \chi(t)^{n-r}. \tag{208}$$

The second factor in (207) is $\Phi_r(t,\xi)$, proving (200). The bound (201) follows from $|\chi(t)| \le 1$. □

4.7. Formal Conditional Consequence for Shell Slices

The next theorem records the exact shell-slice consequence of primitive-frequency decay. This is the terminal conditional implication of the reduction.

Theorem 26 (Conditional shell-slice asymptotic)**.** *Assume that for every $A > 0$,*

$$\mathcal{H}_r \ll_A 3^{-r} r^{-A} \qquad (r \ge 1). \tag{209}$$

Then, uniformly in admissible x, in $y \in \{x^\alpha, x^{\alpha^2}\}$, in $n \in \mathcal{J}_{x,y}$, and in $s \in \mathbb{Z}$ such that $I_{x,y,n,s} \ne \varnothing$,

$$\mathcal{K}_{x,y,n}(s) = \frac{N_n(s)}{3^n|\mathcal{W}_y|}\,\#I_{x,y,n,s} + O\bigg(\frac{N_n(s)}{3^n|\mathcal{W}_y|}\,\#\partial I^{(x)}_{x,y,n,s} + \frac{e^{L_x}}{M_x}\,n^{-A}\bigg). \tag{210}$$

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In particular, whenever the boundary contribution is negligible in the regime under consideration, the shell-slice mass is asymptotic to the main term

$$\frac{N_n(s)}{3^n|\mathcal{W}_y|}\#I_{x,y,n,s}. \tag{211}$$

Proof. By Proposition 25,

$$D_r(n) \le \mathcal{H}_r. \tag{212}$$

Hence the hypothesis (209) implies

$$D_r(n) \ll_A 3^{-r}r^{-A} \qquad (1 \le r \le n). \tag{213}$$

Insert (213) into Proposition 14. The primitive-frequency error term becomes

$$\frac{2^{2n+s}}{|\mathcal{W}_y|}\sum_{r=1}^{n}3^{2r-n}D_r(n) \ll_A \frac{2^{2n+s}}{|\mathcal{W}_y|}\sum_{r=1}^{n}3^{r-n}r^{-A}. \tag{214}$$

Since

$$\sum_{r=1}^{n}3^{r-n}r^{-A} \ll_A n^{-A}, \tag{215}$$

we obtain

$$\frac{2^{2n+s}}{|\mathcal{W}_y|}\sum_{r=1}^{n}3^{2r-n}D_r(n) \ll_A \frac{2^{2n+s}}{|\mathcal{W}_y|\,3^n}n^{-A}. \tag{216}$$

Now apply Lemma 11, which is available because $I_{x,y,n,s} \neq \varnothing$, to conclude that

$$\frac{2^{2n+s}}{|\mathcal{W}_y|\,3^n} \ll \frac{e^{L_x}}{M_x}. \tag{217}$$

Therefore

$$\frac{2^{2n+s}}{|\mathcal{W}_y|}\sum_{r=1}^{n}3^{2r-n}D_r(n) \ll_A \frac{e^{L_x}}{M_x}n^{-A}. \tag{218}$$

Substituting (218) into (124) proves (210). □

Remark 27 (Status of the remaining implication). The unconditional theorem chain of the paper ends with Proposition 25 and Theorem 26. Thus the only missing step for the shell-slice asymptotic is the envelope decay (209).

The short-window Fourier-mass estimate (190) is the explicit Section 4 input identified here as the natural mechanism for suppressing the dangerous band. Proposition 24 records its exact operator-level consequence. What is not yet proved is the remaining propagation step from the short-window estimate for $\widehat{h}_{r,\ell}$ to the envelope decay (209). The present paper therefore isolates the final arithmetic obstruction sharply, but does not claim to close that last implication.

5. Conclusions

The paper develops a shell-kernel framework for the odd-to-odd Syracuse dynamics generated by uniformly distributed initial windows. Within that framework, four points are established unconditionally.

First, the relevant first-passage times are localized in a logarithmic band, and the corresponding backstepped orbit points lie in a narrow shell with exponentially high probability. Second, the fixed-time shell kernel admits an exact inverse-affine representation in terms of Syracuse valuation paths and affine offsets. Third, after passing to shell slices, the arithmetic discrepancy is reduced to an explicit weighted primitive-frequency envelope, while the boundary layer contributes only a power-saving error. Fourth, the primitive-frequency recursion admits an exact block decomposition, and the principal operator is shown not to possess a naive spectral gap.

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These results clarify the structure of the obstruction. The renewal mechanism is not the unresolved step, nor is the shell geometry. The remaining difficulty is arithmetic and occurs at the primitive-frequency level. More precisely, the argument reduces the last unresolved step to short-window Fourier mass for the oscillatory factors

$$h_{r,\ell}(j) = e_{3^{r+1}}(\omega_r^{\ell} 2^j),$$

or equivalently to the incomplete principal-unit exponential sums

$$S_{r,\ell,\varepsilon}(m) = \sum_{u=0}^{3^{r-1}-1} e_{3^{r+1}}\Big(2^{\varepsilon}\omega_r^{\ell} 4^u - 9mu\Big).$$

The corresponding phase is non-stationary in the 3-adic sense, but that fact alone does not yield the cancellation required to close the shell analysis. Accordingly, the paper should be read as a completed reduction theorem rather than as a final shell asymptotic.

From the perspective of the existing literature, the contribution of the paper is therefore structural. Relative to the logarithmic-density theory of Tao, the coarse natural-density estimates of Inselmann, the push-forward program of Rees, and the orbit-level reduction of Chang, the present work identifies the remaining shell-level obstruction in a more explicit arithmetic form. In particular, the final gap is reduced to a concrete primitive-frequency estimate rather than left as a qualitative mixing principle.

The natural next step is now clear. One must establish a short-window Fourier-mass bound for $\widehat{h}_{r,\ell}$, or an equivalent estimate for the sums $S_{r,\ell,\varepsilon}(m)$, strong enough to control the dangerous frequency band of the primitive operator. Any such estimate would feed directly into the shell-slice reduction proved here. In that sense, the paper isolates the precise arithmetic theorem that separates the current uniform-window shell analysis from a full shell asymptotic.

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Appendix A. A Formal Renewal Model for the Shell Mechanism

This appendix records the asymptotic of the formal renewal model obtained by discarding the unresolved primitive-frequency discrepancy. The results below are not used in the proofs of the main unconditional theorems.

Let

$$X_j := a_j \log 2 - \log 3, \qquad S_n := X_1 + \cdots + X_n, \tag{A1}$$

where $a_1, a_2, \ldots$ are iid $\mathrm{Geom}(2)$ random variables. Then

$$\mathbb{E}[X_j] = 2\log 2 - \log 3 = \mu > 0.$$

Theorem A1 (Weighted renewal asymptotic)**.** *Let $u \to \infty$ and let $h = h(u) \to \infty$. Then*

$$\sum_{n\geq 0} \mathbb{E}\Big[e^{S_n} 1_{[u,u+h]}(S_n)\Big] = \frac{e^{u+h} - e^u}{\mu} + o(e^{u+h}). \tag{A2}$$

Proof. Let

$$U(B) := \sum_{n\geq 0} \mathbb{P}(S_n \in B)$$

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be the renewal measure. The increment law is nonarithmetic: if its support lay in a lattice, then both $\log 2$ and $\log 3$ would belong to a common additive lattice, forcing $\log 2/\log 3 \in \mathbb{Q}$, which is impossible. Hence Blackwell's renewal theorem applies:

$$U([t, t+\delta]) \to \frac{\delta}{\mu} \qquad (t \to \infty)$$

for every fixed $\delta > 0$ [9].

Now

$$\sum_{n \ge 0} \mathbb{E}\Big[e^{S_n} 1_{[u,u+h]}(S_n)\Big] = \int_{[u,u+h]} e^v\, U(dv).$$

Partition $[u, u+h]$ into intervals of length δ and use the elementary bounds

$$e^a U([a,b]) \le \int_{[a,b]} e^v\, U(dv) \le e^b U([a,b])$$

on each interval. Summing the resulting geometric series and applying Blackwell's theorem on each fixed-length subinterval gives

$$\int_{[u,u+h]} e^v\, U(dv) = \frac{e^{u+h} - e^u}{\mu} + o(e^{u+h}),$$

because the final remainder interval is negligible when $h \to \infty$. $\square$

Remark A2. Theorem A1 is a statement about the formal renewal model only. It is included for perspective and is not used in the proofs of the main unconditional shell-kernel results.

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