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Article

Mathematical Models for Removal of Pharmaceutical Pollutants in Rehabilitated Treatment Plants

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Abstract: This paper aims to investigate appropriate mathematical models devoted to optimization of some cleaning processes related to pharmaceutical contaminants removal. In our previous recent works has been reported that a possibility for removal from water sources of this type of micropollutants is to rehabilitate the existing cleaning plants by introducing efficient techniques such as adsorption on granulated active carbon filters, micro-, nano- or ultrafiltration. To have such processes under a better control and to pass from small to large scale treatment stations, specific mathematical models are necessary. Starting from Navier-Stokes equations and imposing proper boundary conditions, some mathematical physics problems are obtained for which adequate solving methods *via* variational methods and surjectivity results are proposed. The importance of these solution characterizations consists in their continuation in a numerical method and the possibility to visualize the result by using CFD program.

Keywords: mathematical model; Navier-Stokes equations; pharmaceutical pollutants; infiltration through membranes; flow through porous media; rehabilitation of existing cleaning plants

MSC: 35-11; 35A01; 35A15; 35D30; 76S05; 76-10

1. Introduction

Pharmaceutical pollutants (PhP) represent an enormous worldwide problem rising from one year to another due to the increasing consumption of medicines. Many recent studies debate on a lot of issues such as their trajectories, fate, harmful effects on humans and entire environment over all the world, impact and remedial solutions [1]. Despite this multitude of works and effervescent activities, many subjects are still open, like documented and comprehensive inventories for PhP in water sources in each country, to establish sampling methodologies, control parameters and indicators, the standardization of the acceptance limits for PhP concentrations in waters, to perform appropriate researches in order to establish the effects of the presence of these contaminants in drinkable water on humans and, in general, from aquatic systems on flora and fauna. One of the current problems raised in this regard is to search for appropriate cleaning or treatment methods to remove this kind of micropollutants which exhibit special characteristics, and the usual plants are improper for them [2].

In our recent studies we were interested to find out innovative technological solutions for removal of pharmaceutical contaminants from water sources and we found the rehabilitation of the existing cleaning / treatment plants as a viable depollution possibility [3]. In this cited paper it was obtained that such a rehabilitation of potable or wastewater treatment plants can be realized by completing the existing technologies with adsorption on activated carbon, micro-, ultra-, nanofiltration or reverse osmosis. Such an improve-ment by using these special kinds of membranes has been designed in paper [4] where calculation scheme and other technical details have been developed. In order to optimize the filtration of PhP through these media, which are solid or liquid membranes, we are interested to model the phenomena, to draw solutions for some mathematical physics problems in order to visualize in the end what happens to keep the processes under control.

Mathematical modeling of the movement through porous membranes is not a new subject, neither its usage in depollution studies, but its novelty consists here in the setting of the problem for PhP – microcontaminants that show special properties and characteristics. Among the important studies in this field of filtration through porous membranes we can mention the comprehensive paper [5], where a set of equations for fluid flow through synthetic membranes with extended applicability was obtained.

Concerning the fluid passage through porous media, the characterizing models conduct to mathematical physics problems with partial differential equations involving the p -Laplacian as many papers like [6–14] prove.

Related to the complexity of formulation of such a problem when chemical compounds are involved, one can mention the reference [15] where mathematical models for reactive flows through porous media are largely analyzed; moreover, interesting movements residing in evolution equation for the domains and stability problems appear in the paper [16].

Our attention was focused on the structure of the membrane that should retain a special kind of particles in which we are interested, since our goal is to improve the PhP retention efficiency of the membranes used in the cleaning process. On the other hand, we approached the engineering facade of this problem in [4] and the novelty here is to involve some methods and models to improve this membrane capability which is conferred by the structure of the pores able to retain such kind of micropollutants. However, to find mathematical models able to characterize the flow through porous membrane is not a new subject, being extensively discussed in the work [5], deeply analyzed in papers [17,18]. The issue of pore scaling is also discussed in paper [19], while a formulation of the problem can be consulted in [20] where the accent falls on different kinds of stability of the solution. Studies of chemical interactions with porous medium with a mathematical model developed for convection-diffusion were reported in [21] as also in paper [22], where reactive transport through porous membrane was developed.

For membranes modeling as also for flow through them, the p -Laplacian is the most indicated procedure as we can see in many studies like [14,23], while related to fractional porous medium equation, one can mention the articles [24–26].

In this paper, starting from the basis of the fluid flow through porous media models reported in [5] and [27–29], we propose an adaptation of it for a special type of membranes used in rehabilitation of existent treatment plants and thus arriving at some mathematical physics problems for which adequate solving methods are described.

2. Materials and Methods

Since we found as a viable solution for PhP removal from water sources the rehabilitation of the existing cleaning / treatment plants by using a special kind of membranes [3], we are interested in determination of a more precise image on the flow through them by obtaining an appropriate mathematical model. To find such a model, we start by writing the continuity equation for mass conservation and Navier-Stokes equations for the transport of momentum under special conditions and by taking into account the form and the dimension of the pore (knowing the type of PhP micropollutants from analysis bulletins, [4]), following the way proposed in paper [5].

Regarding the conditions that should be imposed, these are related to the membrane whose morphology follows the conditions to be stationary, condition what means that the motion of the membrane is negligible relative to the seepage velocity, and it should fulfill the isotropy assumption, i.e. this is a composite constructed by different types of porous structures which can be described as basic geometric pore structure models. For the fluid passing through the membrane, it is considered as a single phase, presenting constant density and viscosity. The properties of micro-flow, considering when the fluid traverses the pores of the membrane, this should be laminar, with no-slip boundary conditions imposed at fluid-solid interfaces. The condition set for macro-flow is to have small gradients for average velocity like in paper [5].

2.1. Basic Formulation of the Problem

To find the searched mathematical problem, we write the equations governing the flow of the liquid infill in the pores of the membrane, which are the continuity equation for mass conservation and Navier-Stokes, respectively:

$$\operatorname{div} v = 0, \quad (2.1)$$

$$\rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v + \nabla p - \rho g - \mu \Delta v = 0, \quad (2.2)$$

where:

- v is the field of fluid velocity within the volume V_f of the fluid filling the void within representative unit cell (notion proposed in [30]),
- p is the fluid pressure,
- g is the gravitational body force per unit mass,
- ρ is the fluid mass density,
- μ is the fluid dynamic viscosity.

It is impossible to write the movement in each pore; then one applies an averaging method through which all the elements involved in (2.1) and (2.2) are averaged on V_c – a control volume (the total volume of the representative unit cell, actually) as is established in [31]. So, we have the fluid part V_f of V_c on what we integrate the above two relations in order to average all their terms. The porosity, ϵ , is defined by the relation:

$$\epsilon = \frac{V_f}{V_c}. \quad (2.3)$$

We consider the specific discharge q representing the average volume of the fluid within the pore and introduce them in the equations (2.1) and (2.2) to average them. Then we have:

$$q = \frac{1}{V_c} \iiint_{V_f} v dV = \frac{\epsilon}{V_f} \iiint_{V_f} v dV. \quad (2.4)$$

The form of the continuity equation under volumetric phase averaging becomes [31]:

$$\operatorname{div} q = 0. \quad (2.5)$$

In a similar manner, the volumetrically averaged equation (2.2) is written as in [30], where the surface integral is evaluated by taking into account the real velocity gradients at the pore surface, this providing a quite accurate description for the microstructure of pores. This can be modeled by using methods similar to those from the work [30]. One can neglect the triple integral of the velocity dispersion, this being very small compared with the other terms, since we made the assumption that the velocity gradients are not high. In paper [30] the model of flow in porous structure can be found by considering a laminar flow in the pore sections. Under such a condition, the surface integral is approximated and the equation for the transport of the momentum, that can be applied at any local porosity ϵ and for each microscopic characteristic length d , in accordance with [32] as follows:

$$\rho \frac{\partial q}{\partial t} + \rho q \cdot \nabla \left(\frac{q}{\epsilon} \right) + \epsilon \nabla p_f - \epsilon \rho g - \mu \Delta q + \mu F q = 0, \quad (2.6)$$

where:

- p_f is the pressure of the fluid filling the void within representative unit cell;
- F is the microscopic shear factor.

2.2. Problem with p -Laplacian

As it was specified in the first section of this work, the filtration in porous medium equation, in its most general expression, involves the p -Laplacian, hence we have the equation:

$$\frac{\partial u}{\partial t} - \Delta_p A(u) = f. \quad (2.7)$$

where one can consider $A(u) = u^m$, having $m > 1$ for fractional porous medium equation and.

$0 < m < 1$ for fractional fast diffusion equation [33]. By applying the method of implicit time discretization as in [34] one obtains the nonlinear equation of elliptic type [33]:

$$-h\Delta_p v + B(v) = f, \quad (2.8)$$

where $h > 0$ is a constant, B is the inverse function of A which appears in the parabolic equation (2.7). Regarding the spatial definition sets for both equations (2.7) and (2.8), this is in the general case either $\mathbf{R}^N$ or $\Omega \subset \mathbf{R}^N$. For the problem considered in this paper, one takes into account the second case with special conditions for Ω related to the real situation. We emphasize that the mathematical physics problem is completed in a Dirichlet one with zero boundary conditions on $\partial\Omega$. Details related to the p -Laplacian can be consulted in [35].

3. Results

In this section, a series of results giving / characterizing solutions for the problem:

$$-h\Delta_p v + B(v) = f, \text{ on } \Omega, \quad (3.1)$$

$$v = 0 \text{ on } \partial\Omega \quad (3.2)$$

are presented. These propositions have been obtained by the author in papers [35,36].

3.1. Solutions via Surjectivity Approaches

3.1.1. Solving the Problem

$$(*) \begin{cases} -\lambda \Delta_p u = f(\cdot, u(\cdot)) + h, x \in \Omega, \lambda \in \mathbf{R} \\ u|_{\partial\Omega} = 0 \end{cases} \quad (3.3)$$

$$(3.4)$$

Proposition 1. Let Ω be an open bounded set of C^1 class from $\mathbf{R}^N$, $N \geq 2$, $p \in (1, +\infty)$, h from $W^{-1,p'}(\Omega)$ and $f: \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ a Carathéodory function with the properties:

$$1^0 f(x, -s) = -f(x, s) \quad \forall s \text{ from } \mathbf{R}, \forall x \text{ from } \Omega,$$

$$2^0 |f(x, s)| \leq c_1 |s|^{q-1} + \beta(x) \quad \forall s \text{ from } \mathbf{R}, \forall x \text{ from } \Omega \setminus A, \mu(A) = 0,$$

$$\text{where } c_1 \geq 0, q \in (1, p), \beta \in L^{q'}(\Omega), \frac{1}{q} + \frac{1}{q'} = 1.$$

Then, for any $\lambda \neq 0$, the problem $(*)$ has solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p'}(\Omega)$ [35].

Proposition 2. Let Ω be open bounded set of C^1 class from $\mathbf{R}^N$, $N \geq 2$, $p \in (1, +\infty)$, h from $W^{-1,p'}(\Omega)$ and $f: \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ Carathéodory function having the properties:

$$1^0 f(x, -s) = -f(x, s) \quad \forall x \text{ from } \Omega, \forall s \text{ from } \mathbf{R},$$

$$2^0 |f(x, s)| \leq c_1 |s|^{p-1} + \beta(x) \quad \forall s \text{ from } \mathbf{R}, \forall x \text{ from } \Omega \setminus A, \mu(A) = 0,$$

$$\text{where } c_1 \geq 0, \beta \in L^{p'}(\Omega), \frac{1}{p} + \frac{1}{p'} = 1.$$

Finally, let $i: W_0^{1,p}(\Omega) \rightarrow L^p(\Omega)$ be linear compact embedding. Then, for any λ , if

$$|\lambda| > c_1 \lambda_1^{-1}, \lambda_1 := \inf \left\{ \frac{\|u\|_{1,p}^p}{\|i(u)\|_{0,p}^p} : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}, \quad (3.5)$$

the problem $(*)$ has solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p'}(\Omega)$. [35]

Remark. For all the necessary elements – definitions and any other details – see the paper [35].

Solution for the problem (3.1) & (3.2). Since $h > 0$, one can divide (3.1) by h and one may apply Proposition 1 by taking $\lambda = 1$ ($\lambda \neq 0$ there) and let us take $f(\cdot, u(\cdot)) \equiv B(u)$ and f from (3.1) instead of h from the used result. The conditions required by this assertion are fulfilled. For the application of Proposition 2 one can correspondingly adapt the positive constant h .

3.1.2. Applications for Results of the Fredholm Alternative Type

For the same problem (*), the following results have been proven in [35].

Proposition 3. Let p be from $(1, +\infty)$ and $\lambda \neq 0$. If

$$\lambda(-\Delta_p u) = |u|^{p-2} u \quad (3.6)$$

has not a nonzero solution in $W_0^{1,p}(\Omega)$, then, for any h from $W^{-1,p'}(\Omega)$, the equation

$$\lambda(-\Delta_p u) = |u|^{p-2} u + h \quad (3.7)$$

has solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p'}(\Omega)$ [35].

Proposition 4. Let p be from $(1, +\infty)$ and $\lambda \neq 0$. If

$$\lambda(-\Delta_p u) = N_\lambda u \quad (3.8)$$

has no nonzero solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p'}(\Omega)$, then, for any h from $W^{-1,p'}(\Omega)$, the equation

$$\lambda(-\Delta_p u) = f(\cdot, u(\cdot)) + h, x \in \Omega \quad (3.9)$$

has a solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p'}(\Omega)$ [35].

Clarification. $N_\lambda: L^p(\Omega) \rightarrow L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $(N_\lambda u)x = f(x, u(x))$, the Nemytskii operator

with $f: \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ Carathéodory function which verifies

- 1^o $|f(x, s)| \leq c_1 |s|^{p-1} + \beta(x) \quad \forall s \in \mathbf{R}, \forall x \in \Omega \setminus A, \mu(A) = 0$, where $c_1 \geq 0, \beta \in L^{p'}(\Omega)$;
- 2^o f is odd and $(p-1)$ -homogeneous in the second variable.

Solution for the problem (3.1) & (3.2). When the conditions required by the above two propositions are fulfilled, then we can consider such a characterization for the existence of the solution of the problem which we study.

3.1.3. Applications for Results via Surjectivity at Different Homogeneity Degrees

In order to present the following two results, let us mention that the operator

$$N: L^q(\Omega) \rightarrow L^{q'}(\Omega), Nu = |u|^{q-2} u, \frac{1}{q} + \frac{1}{q'} = 1, \text{ with } q \in (1, p), \quad (3.10)$$

i is the canonical embedding and i' its adjoint.

Proposition 5. Under the above conditions, for any $\lambda \neq 0$ and for any h from $W^{-1,p'}(\Omega)$, there exists u_0 in $W_0^{1,p}(\Omega)$ such that

$$\lambda(-\Delta_p)u_0 = (i' \circ N \circ i)u_0 + h \quad [35]. \quad (3.11)$$

For the second proposition, replace the operator N from Proposition 5 by N_λ , Nemytskii operator, i.e. take $N: L^q(\Omega) \rightarrow L^{q'}(\Omega)$, $N = N_\lambda$ with $f: \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ odd Carathéodory function and $(q-1)$ -homogeneous in the second variable, and which verifies the growth condition:

$$|f(x, s)| \leq c_1 |s|^{q-1} + \beta(x) \quad \forall s \in \mathbf{R}, \forall x \in \Omega \setminus A, \mu(A) = 0, \quad (3.12)$$

where $c_1 \geq 0, \beta \in L^{q'}(\Omega)$.

Proposition 6. Under the above conditions, for any $\lambda \neq 0$ and for any h in $W^{-1,p'}(\Omega)$, there exists u_0 in $W_0^{1,p}(\Omega)$ such that

$$\lambda(-\Delta_p)u_0 = (i' \circ N_i \circ i)u_0 + h \quad [35]. \quad (3.13)$$

Remark. All the new notions can be found in [35].

Solution for the problem (3.1) & (3.2). One can also formulate the problem in such a manner to superpose it on one of the situations provided by Propositions 5 and 6 and then, in case the asked conditions are fulfilled, one may prove the existence *via* these two results.

3.2. Other Types of Characterization

3.2.1. Characterization of Weak Solutions Starting from Ekeland Variational Principle

Let Ω be an open bounded nonempty set in $\mathbf{R}^N$, $N > 1$, $f: \Omega \times \mathbf{R} \rightarrow \mathbf{R}$, and $u_0 \in W_0^{1,p}(\Omega)$. Consider the problem:

$$(**) \quad \begin{cases} -\Delta_p u = f(x, u), & x \in \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.14)$$

$$(3.15)$$

and $f: \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ a Carathéodory function with the growth condition:

$$|f(x, s)| \leq c|s|^{p-1} + b(x), \quad (3.16)$$

where $c > 0$, $2 \leq p \leq \frac{2N}{N-2}$ when $N \geq 3$ and $2 \leq p < +\infty$ when $N = 1, 2$, and where $b \in L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 7. Let Ω be an open bounded of C^1 class set in $\mathbf{R}^N$, $N \geq 3$, $f: \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ a Carathéodory function and u_1, u_2 from $W_0^{1,p}(\Omega)$ bounded weak subsolution and weak supersolution of (**), respectively, with $u_1(x) \leq u_2(x)$ a.e. on Ω . Suppose that f verifies (3.16) and there is $\rho > 0$ such that the function $g: g(x, s) = f(x, s) + \rho s$ is strictly increasing in s on $[\inf u_1(\Omega), \sup u_2(\Omega)]$. Then there is a weak solution $\bar{u}$ of (*) in $W_0^{1,p}(\Omega)$ with the property

$$u_1(x) \leq \bar{u}(x) \leq u_2(x) \text{ a.e. on } \Omega \quad [35]. \quad (3.17)$$

Clarification. For the definitions and other elements necessary in the last result, [35] can be consulted.

Solution for the problem (3.1) & (3.2). In this case, the place of f from (3.14) is taken by

$$(x, u) \rightarrow \frac{1}{h} \left(-B(u(x)) + f(x) \right). \quad (3.18)$$

When the conditions of Proposition 7 are fulfilled, then one obtains this characterization of the solution of the studied problem is obtained.

3.2.2. Solutions involving Critical Points for Nondifferentiable Functionals

To introduce the next results, let Ω be a bounded domain of $\mathbf{R}^N$ with the smooth boundary $\partial\Omega$ (topological boundary). Consider the nonlinear boundary value problems (**) from (3.14)+(3.15), where $f: \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ is a measurable function with *subcritical growth*, i.e.

$$(I) \quad |f(x, s)| \leq a + b|s|^\sigma \quad \forall s \in \mathbf{R}, x \in \Omega \text{ a.e.}, \quad (3.19)$$

where $a, b > 0$, $0 \leq \sigma < \frac{N+2}{N-2}$ for $N > 2$ and $\sigma \in [0, +\infty)$ for $N = 1$ or $N = 2$.

Set [37]:

$$\underline{f}(x, t) = \lim_{s \rightarrow t} f(x, s), \quad \overline{f}(x, t) = \overline{\lim}_{s \rightarrow t} f(x, s). \quad (3.20)$$

Suppose

(II) $\underline{f}, \overline{f}: \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ are measurable with respect to x .

Emphasize that (II) is verified in the following two cases:

1⁰ f is independent of x ;

2⁰ f is Baire measurable and $s \rightarrow f(x, s)$ is decreasing $\forall x \in \Omega$, in which case we have:

$$\overline{f}(x, t) = \max\{f(x, t+), f(x, t-)\}, \quad \underline{f}(x, t) = \min\{f(x, t+), f(x, t-)\}. \quad (3.21)$$

For the announced result, the following definition is necessary: u from $W_0^{1,p}(\Omega)$, $p > 1$ is solution of (**) if $u = 0$ on $\partial \Omega$ in the sense of trace (see details in [35]) and

$$-\Delta_p u(x) \in [\underline{f}(x, u(x)), \overline{f}(x, u(x))] \text{ in } \Omega \text{ a.e.} \quad (3.22)$$

Associate to the problem (**) the locally Lipschitz functional $\Phi: W_0^{1,p}(\Omega) \rightarrow \mathbf{R}$,

$$\Phi(u) = \frac{1}{p} \|u\|_{1,p}^p - \int_{\Omega} F(x, u) dx, \quad u \in W_0^{1,p}(\Omega), \quad (3.23)$$

Proposition 8. *If (I) and (II) are verified, every critical point of Φ is solution for (**).*

Solution for the problem (3.1) & (3.2). The conditions (I) and (II) are fulfilled for a largeset of functions having the form (3.18), hence one can propose in this manner another characterization of the solution with Proposition 8.

3.3. Weak Solutions Using a Perturbed Variational Principle

For the following statement, some clarifications are necessary. Let Ω be an open bounded set of C^1 class in $\mathbf{R}^N$, $N \geq 3$. Consider the above problem (**), where $f: \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ is a Carathéodory function with the growth condition

$$|f(x, s)| \leq c |s|^{p-1} + b(x), \quad c > 0, \quad 2 \leq p \leq \frac{2N}{N-2}, \quad b \in L^{p'}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (3.24)$$

The functional $\varphi: W_0^{1,p}(\Omega) \rightarrow \mathbf{R}$,

$$\varphi(u) = \int_{\Omega} \left[\frac{1}{p} \left(|u|^p + \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^p \right) - F(x, u(x)) \right] dx \quad (3.25)$$

with $F(x, s) := \int_0^s f(x, t) dt$, are of Fréchet C^1 -class and their critical points are the weak solutions of the problem (**).

Proposition 9. *Under the above assumptions and in addition the growth condition*

$$F(x, s) \leq c_1 \frac{s^p}{p} + \alpha(x)s, \quad (3.26)$$

with $0 < c_1 < \lambda_1$, $\alpha \in L^{q'}(\Omega)$ for some $2 \leq q \leq \frac{2N}{N-2}$ and $f(x, -s) = -f(x, s)$, $\forall x$ from Ω , the following assertions hold:

(i) *The set of functions h from $W^{-1,p'}(\Omega)$, having the property that the functional $\varphi_h: W_0^{1,p}(\Omega) \rightarrow \mathbf{R}$,*

$$\varphi_h(u) = \frac{1}{p} \|u\|_p^p - \int_{\Omega} (F(x, u(x)) + h(u(x))) dx \quad (3.27)$$

has in only one point an attained minimum, includes a G_δ set everywhere dense;

(ii) The set of functions h from $W^{-1,p'}(\Omega)$, having the property the problem
$$\begin{cases} -\Delta_p u = f(x, u) + h(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$
 has

solutions, includes a G_δ set everywhere dense;

(iii) Moreover, if $s \rightarrow f(x, s)$ is increasing, then the set of functions h from $W^{-1,p'}(\Omega)$, having the property the problem
$$\begin{cases} -\Delta_p u = f(x, u) + h(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$
 has a unique solution, includes a G_δ set everywhere dense [35].

Clarification. For any notion, see details and explanations in paper [35].

Solution for the problem (3.1) & (3.2). In this last case, the role of

$$(x, u) \rightarrow f(x, u(x)) + h(x) \quad (3.28)$$

is played by (3.18). So, when the condition of Proposition 9 are fulfilled, then one finds this kind of description of the solution of the considered problem.

3.4. A Theorem based on a Result of Mountain Pass Type

To enounce the last result applied in this work, we need the function $f: \overline{\Omega} \times \mathbf{R} \rightarrow \mathbf{R}$, which is of Carathéodory type satisfying the growth condition:

$$|f(x, s)| \leq c|s|^{p-1} + \alpha(x), \quad \forall x \in \Omega \setminus A \text{ with } \mu(A) = 0 \quad \forall s \in \mathbf{R}, \quad (3.29)$$

where $c \geq 0$, $\alpha \in L^{p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$ and $1 \leq p \leq \frac{2N}{N-2}$ if $N \geq 3$ and $1 \leq p \leq +\infty$ if $N = 2$ and let $F(x, t) = \int_0^t f(x, s) ds$.

Proposition 10. Assume that f satisfies:

(f1) $f(x, s) = o(|s|)$, $s \rightarrow 0$, uniformly for $x \in \overline{\Omega}$;

(f2) There exist constants $\mu > p$ and $r > 0$ e.g., for $|s| \leq r$,

$$0 < \mu F(x, s) \leq s f(x, s).$$

Then the above problem (**) possesses a nontrivial solution. [36]

Solution for the problem (3.1) & (3.2). One can remark that the conditions (f1) and (f2) are fulfilled for a great class of functions and it is possible to be fulfilled for the function f of the form (3.18) and thus another solving method for our problem in study is established.

4. Discussion

This paper discussed novel mathematical aspects of the filtration through porous membrane. We studied these problems when we discovered them in removal of pharmaceutical micropollutants. The issue of these contaminants was quite recently highlighted, and their enormous amount together with the inability of the usual cleaning plants to remove them from water sources, opened an extensive field of research. In another work, we proposed as a viable solution (which is not the unique possible!) the rehabilitation of the existing treatment plants by introducing some special kinds of membranes that retain the PhP contaminants. Hence specific dimensions and geometry of the pores are needed together with the construction of a mathematical model for a better control of this process. For this reason, we firstly adapted a given model from [5] for the flow through porous membrane in order to comply with the requirements imposed by the PhP particles obtained from analysis bulletins. The detailed description of pores will be the subject of a future work.

We developed the problem involving p -Laplacian and, by implicit time discretization, the parabolic problem was changed in one elliptic, arriving to a Dirichlet problem for p -Laplacian. The author applied ten existence results obtained in papers [35,36]. These approaching methods actually suggests the continuation in ten types of numerical methods proposed for future researches. A comparison and ranking of these ways should be realized in relation to the pore conditions that will

be introduced in the base model. A last verification will be performed by introducing the model into CFD program to visualize the flow through the proposed pores.

5. Conclusions

In this work, a model of flow through porous membranes has been adapted to the requirements imposed by retention of PhP micropollutants.

The next step was to formulate the flow model as a mathematical physics problem involving a parabolic partial differential equation with p -Laplacian which was transformed, *via* the method of implicit time discretization, in a Dirichlet problem for an elliptic equation with p -Laplacian.

A series of ten results previously found by the author has been applied to solve and characterize the solution of the Dirichlet problem obtained. This is an intermediate step on the way to visualize a concrete solution to the studied problem.

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