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Article

Noncommutative Equiangular Lines: Van Lint-Seidel Relative and Gerzon Universal Bounds

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Abstract: We introduce the notion of noncommutative equiangular lines and derive noncommutative versions of fundamental van Lint-Seidel relative and Gerzon universal bounds.

Keywords: equiangular lines; C*-algebra; Hilbert C*-module

MSC: 46L05; 46L08

1. Introduction

Let $d \in \mathbb{N}$ and $\gamma \in [0, 1]$. Recall that a collection $\{\tau_j\}_{j=1}^n$ of unit vectors in $\mathbb{R}^d$ or $\mathbb{C}^d$ is said to be γ -**equiangular lines** [1,2] if

$$|\langle \tau_j, \tau_k \rangle| = \gamma, \quad \forall 1 \leq j, k \leq n, j \neq k.$$

Two fundamental problems associated with equiangular lines are the following.

Problem 1.1. Given $d \in \mathbb{N}$ and $\gamma \in [0, 1]$, what is the upper bound on n such that there exists a collection $\{\tau_j\}_{j=1}^n$ of γ -equiangular lines in $\mathbb{R}^d$?

Problem 1.2. For which parameters $d, n \in \mathbb{N}$ and $\gamma \in [0, 1]$, there exists a collection $\{\tau_j\}_{j=1}^n$ of γ -equiangular lines in $\mathbb{R}^d$? If exists, how to construct them?

Two answers to Problem (1.1) which are fundamental driving force in the study of equiangular lines are the following results of van Lint and Seidel [1,3] and Gerzon [4].

Theorem 1.3 ([1,3,5,6]). (*van Lint-Seidel Relative Bound*) Let $\{\tau_j\}_{j=1}^n$ be γ -equiangular lines in $\mathbb{R}^d$ or $\mathbb{C}^d$. Then

$$n(1 - d\gamma^2) \leq d(1 - \gamma^2).$$

In particular, if

$$\gamma < \frac{1}{\sqrt{d}},$$

then

$$n \leq \frac{d(1 - \gamma^2)}{1 - d\gamma^2}. \tag{1}$$

Theorem 1.4 ([4]). (*Gerzon Universal Bound*)

(i) Let $\{\tau_j\}_{j=1}^n$ be γ -equiangular lines in $\mathbb{R}^d$. Then

$$n \leq \frac{d(d+1)}{2}. \quad (2)$$

(ii) Let $\{\tau_j\}_{j=1}^n$ be γ -equiangular lines in $\mathbb{C}^d$. Then

$$n \leq d^2. \quad (3)$$

Bounds in Inequalities (1), (2) and (3) are improved for various special values of γ , n and d (including asymptotic). Regarding Problem (1.2), various special cases are solved but full generality remains open. We refer [7–30] for a look on these achievements in the real case and [31–37] in the complex case. In the real case, it is known that equality is not always attained for every d in Inequality (2). Celebrated Zauner conjecture says that there exists a $d^2 \frac{1}{\sqrt{d+1}}$ -equiangular lines in $\mathbb{C}^d$ for every d satisfying equality in Inequality (3) [38–44].

We wish to record that there is a vector space version of equiangular lines [45,46] and there is also a notion of equiangular subspaces of Euclidean spaces [47,48]. Recently, p-adic version of equiangular lines have been introduced [49].

As noncommutative geometry plays major role in modern Mathematics, our fundamental objective is to initiate a study of noncommutative equiangular lines. As most important results in the theory of equiangular lines are van Lint-Seidel relative and Gerzon universal bounds, we derive noncommutative versions of them.

2. Noncommutative Equiangular Lines

We want the notion of Hilbert C^* -modules. They are first introduced by Kaplansky [50] for modules over commutative C^* -algebras and later developed for modules over arbitrary C^* -algebras by Paschke [51] and Rieffel [52]. A friendly introduction is given in the book [53]. We want only standard Hilbert C^* -module. Let $\mathcal{A}$ be a unital C^* -algebra with identity 1. For $d \in \mathbb{N}$, let $\mathcal{A}^d$ be the standard left Hilbert C^* -module over $\mathcal{A}$ with inner product

$$\langle (a_j)_{j=1}^d, (b_j)_{j=1}^d \rangle := \sum_{j=1}^d a_j b_j^*, \quad \forall (a_j)_{j=1}^d, (b_j)_{j=1}^d \in \mathcal{A}^d.$$

Hence norm on $\mathcal{A}^d$ is

$$\|(a_j)_{j=1}^d\|_2 := \left\| \sum_{j=1}^d a_j a_j^* \right\|^{1/2}, \quad \forall (a_j)_{j=1}^d \in \mathcal{A}^d.$$

We introduce two notions of equiangular lines for modules.

Definition 2.1. Let $a \in \mathcal{A}$ be a positive element. A collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{A}^d$ is said to be **modular a -equiangular lines** if following conditions hold.

- (i) $\langle \tau_j, \tau_j \rangle = 1, \forall 1 \leq j \leq n$.
- (ii) $\langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle = a, \forall 1 \leq j, k \leq n, j \neq k$.

Definition 2.2. Let $\gamma \in [0, 1]$. A collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{A}^d$ is said to be **modular γ -norm-equiangular lines** if following conditions hold.

- (i) $\langle \tau_j, \tau_j \rangle = 1, \forall 1 \leq j \leq n$.
- (ii) $\|\langle \tau_j, \tau_k \rangle\| = \gamma, \forall 1 \leq j, k \leq n, j \neq k$.

It is clear that if $\{\tau_j\}_{j=1}^n$ is modular a -equiangular lines, then $\{\tau_j\}_{j=1}^n$ is modular $\sqrt{\|a\|}$ -norm-equiangular lines. We now derive two modular versions of Theorem 1.3 for modules over commutative C^* -algebras.

Theorem 2.3. (Modular van Lint-Seidel Relative Bound) Let $\mathcal{A}$ be a commutative unital C^* -algebra. Let $\{\tau_j\}_{j=1}^n$ be modular a -equiangular lines in $\mathcal{A}^d$. Then

$$n(1 - da) \leq d(1 - a).$$

In particular, if $1 - da$ is invertible, then

$$n \leq \frac{d(1 - a)}{1 - da} \leq d \left\| \frac{1 - a}{1 - da} \right\|.$$

Proof. Let $\tau_j := (a_1^{(j)}, a_2^{(j)}, \dots, a_d^{(j)})$ for all $1 \leq j \leq n$. Define

$$B := \sum_{j=1}^n \sum_{k=1}^n \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle = n + (n^2 - n)a.$$

Then using Cauchy-Schwarz inequality in Hilbert C^* -modules,

$$\begin{aligned} B &= \sum_{j=1}^n \sum_{k=1}^n \left(\sum_{r=1}^d a_r^{(j)} (a_r^{(k)})^* \right) \left(\sum_{s=1}^d a_s^{(k)} (a_s^{(j)})^* \right) \\ &= \sum_{r,s=1}^d \left(\sum_{j=1}^n a_r^{(j)} (a_s^{(j)})^* \right) \left(\sum_{k=1}^n a_r^{(k)} (a_s^{(k)})^* \right)^* \\ &= \sum_{r=1}^d \left(\sum_{j=1}^n a_r^{(j)} (a_r^{(j)})^* \right) \left(\sum_{k=1}^n a_r^{(k)} (a_r^{(k)})^* \right)^* + \sum_{r,s=1, r \neq s}^d \left(\sum_{j=1}^n a_r^{(j)} (a_s^{(j)})^* \right) \left(\sum_{k=1}^n a_r^{(k)} (a_s^{(k)})^* \right)^* \\ &\geq \sum_{r=1}^d \left(\sum_{j=1}^n a_r^{(j)} (a_r^{(j)})^* \right) \left(\sum_{k=1}^n a_r^{(k)} (a_r^{(k)})^* \right)^* \\ &\geq \frac{1}{d} \left(\sum_{r=1}^d \left(\sum_{j=1}^n a_r^{(j)} (a_r^{(j)})^* \right) \cdot 1 \right) \left(\sum_{s=1}^d 1 \cdot \left(\sum_{k=1}^n a_s^{(k)} (a_s^{(k)})^* \right)^* \right) \\ &= \frac{1}{d} \left(\sum_{j=1}^n \sum_{r=1}^d a_r^{(j)} (a_r^{(j)})^* \right) \left(\sum_{k=1}^n \sum_{s=1}^d a_s^{(k)} (a_s^{(k)})^* \right) \\ &= \frac{1}{d} \left(\sum_{j=1}^n 1 \right) \left(\sum_{k=1}^n 1 \right) = \frac{n^2}{d}. \end{aligned}$$

Hence

$$(n - 1)a + 1 \geq \frac{n}{d} \implies d(n - 1)a + d \geq n \implies d(1 - a) \geq n(1 - da).$$

□

Theorem 2.4. Let $\mathcal{A}$ be a commutative unital C^* -algebra. Let $\{\tau_j\}_{j=1}^n$ be modular γ -norm-equiangular lines in $\mathcal{A}^d$. Then

$$n(1 - d\gamma^2) \leq d(1 - \gamma^2).$$

In particular, if

$$\gamma < \frac{1}{\sqrt{d}},$$

then

$$n \leq \frac{d(1 - \gamma^2)}{1 - d\gamma^2}.$$

Proof. By using the proof of Theorem 2.3,

$$\begin{aligned} \frac{n^2}{d} &\leq \sum_{j=1}^n \sum_{k=1}^n \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle = \sum_{j,k=1, j \neq k}^n \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle + \sum_{j=1}^n \langle \tau_j, \tau_j \rangle^2 \\ &= \sum_{j,k=1, j \neq k}^n \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle + n \leq \sum_{j,k=1, j \neq k}^n \|\langle \tau_j, \tau_k \rangle\|^2 + n = (n^2 - n)\gamma^2 + n. \end{aligned}$$

□

Proof of previous theorems used commutativity of C^* -algebra. We are unable to derive noncommutative versions of them. However, we introduce a subclass of norm-equiangular lines for which we derive modular relative bound.

Definition 2.5. Let $\gamma \in [0, 1]$. A collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{A}^d$ is said to be **special modular γ -norm-equiangular lines** if following conditions hold.

- (i) $\langle \tau_j, \tau_j \rangle = 1, \forall 1 \leq j \leq n$.
- (ii) $\|\langle \tau_j, \tau_k \rangle\| = \gamma, \forall 1 \leq j, k \leq n, j \neq k$.
- (iii)

$$n^2 \leq d \sum_{j=1}^n \sum_{k=1}^n \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle.$$

Theorem 2.6. (Noncommutative van Lint-Seidel Relative Bound) Let $\mathcal{A}$ be a unital C^* -algebra. Let $\{\tau_j\}_{j=1}^n$ be special modular γ -norm-equiangular lines in $\mathcal{A}^d$. Then

$$n(1 - d\gamma^2) \leq d(1 - \gamma^2).$$

In particular, if $\gamma < \sqrt{d^{-1}}$, then

$$n \leq \frac{d(1 - \gamma^2)}{1 - d\gamma^2}.$$

Proof.

$$n^2 \leq d \sum_{j=1}^n \sum_{k=1}^n \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle \leq (n^2 - n)\gamma^2 + n.$$

□

Next we derive modular Gerzon bound for modules over C^* -algebras with invariant basis number (IBN) property. These C^* -algebras are studied in [54].

Theorem 2.7. (Modular Gerzon Universal Bound) Let $\mathcal{A}$ be a unital C^* -algebra with IBN. Let $\{\tau_j\}_{j=1}^n$ be modular a -equiangular lines in $\mathcal{A}^d$. If $1 - a$ is invertible, then

$$n \leq d^2. \quad (4)$$

Proof. For $1 \leq j \leq n$, define

$$\tau_j \otimes \tau_j : \mathcal{A}^d \ni x \mapsto (\tau_j \otimes \tau_j)x := \langle x, \tau_j \rangle \tau_j \in \mathcal{A}^d.$$

We show that the collection $\{\tau_j \otimes \tau_j\}_{j=1}^n$ is linearly independent over $\mathcal{A}$. Let $c_1, \dots, c_n \in \mathcal{A}$ be such that

$$\sum_{j=1}^n c_j (\tau_j \otimes \tau_j) = 0. \quad (5)$$

Let $1 \leq k \leq n$ be fixed. Then previous equation gives

$$0 = \sum_{j=1}^n c_j (\tau_j \otimes \tau_j) \tau_k = \sum_{j=1}^n c_j \langle \tau_k, \tau_j \rangle \tau_j.$$

Taking inner product with τ_k gives

$$\begin{aligned} 0 &= \sum_{j=1}^n c_j \langle \tau_k, \tau_j \rangle \langle \tau_j, \tau_k \rangle = \sum_{j=1, j \neq k}^n c_j \langle \tau_k, \tau_j \rangle \langle \tau_j, \tau_k \rangle + c_k \langle \tau_k, \tau_k \rangle^2 \\ &= \sum_{j=1, j \neq k}^n c_j a + c_k = \sum_{j=1}^n c_j a - c_k a + c_k = \sum_{j=1}^n c_j a + c_k (1 - a). \end{aligned}$$

Now define

$$c := \frac{-1}{1-a} \sum_{j=1}^n c_j a.$$

Then $c_k = c$ for all $1 \leq k \leq n$. We wish to show that $c = 0$. Now Equation (5) gives

$$0 = \sum_{j=1}^n c (\tau_j \otimes \tau_j).$$

Let $1 \leq k \leq n$ be fixed. Then previous equation gives

$$0 = \sum_{j=1}^n c (\tau_j \otimes \tau_j) \tau_k = \sum_{j=1}^n c \langle \tau_k, \tau_j \rangle \tau_j.$$

Taking inner product with τ_k gives

$$\begin{aligned} 0 &= \sum_{j=1}^n c \langle \tau_k, \tau_j \rangle \langle \tau_j, \tau_k \rangle = \sum_{j=1, j \neq k}^n c \langle \tau_k, \tau_j \rangle \langle \tau_j, \tau_k \rangle + c \langle \tau_k, \tau_k \rangle^2 \\ &= \sum_{j=1, j \neq k}^n ca + c = (n-1)ca + c = c((n-1)a + 1). \end{aligned}$$

Since $a \geq 0$, $((n-1)a + 1)$ is invertible, hence $c = 0$. Note that the matrix of $\tau_j \otimes \tau_j$ is $\tau_j \tau_j^*$ (viewing τ_j as column vector). The rank of d by d matrices over $\mathcal{A}$ is d^2 . Since $\{\tau_j \otimes \tau_j\}_{j=1}^n$ is linearly independent and $\mathcal{A}$ has IBN property, $n \leq d^2$. $\square$

We can slightly generalize Definition 2.1. In the vector space case, such a generalization was done by Greaves, Iverson, Jasper and Mixon [46].

Definition 2.8. Let $a, b \in \mathcal{A}$ be positive elements. A collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{A}^d$ is said to be **modular** (a, b) -**equiangular lines** if following conditions hold.

- (i) $\langle \tau_j, \tau_j \rangle = b, \forall 1 \leq j \leq n.$
- (ii) $\langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle = a, \forall 1 \leq j, k \leq n, j \neq k.$

Note that b is necessarily positive but need not be invertible. Thus we may not be able to reduce Definition 2.8 to Definition 2.1. By modifying earlier proofs, we easily get following theorems.

Theorem 2.9. Let $\mathcal{A}$ be a commutative unital C^* -algebra. Let $\{\tau_j\}_{j=1}^n$ be modular (a, b) -equiangular lines in $\mathcal{A}^d$. Then

$$n(b^2 - da) \leq d(b - a).$$

In particular, if $b^2 - da$ is invertible, then

$$n \leq \frac{d(b - a)}{b^2 - da} \leq d \left\| \frac{b - a}{b^2 - da} \right\|.$$

Theorem 2.10. Let $\mathcal{A}$ be a unital C^* -algebra with IBN. Let $\{\tau_j\}_{j=1}^n$ be modular (a, b) -equiangular lines in $\mathcal{A}^d$. If $b^2 - a$ and $(n - 1)a + b^2$ are invertible, then

$$n \leq d^2.$$

Similar to scalar equiangular lines and Zauner conjecture, we formulate following problem.

Problem 2.11. Let $\mathcal{A}$ be a unital C^* -algebra. For which parameters $d, n \in \mathbb{N}$ and $a \in \mathcal{A}$, there exists a collection $\{\tau_j\}_{j=1}^n$ of modular a -equiangular lines in $\mathcal{A}^d$? In particular, for which d for all d , there is d^2 modular $\frac{1}{d+1}$ -equiangular lines in $\mathcal{A}^d$ (making equality in Inequality (4))?

Note that Problem 2.11 contains Zauner conjecture (whenever $\mathcal{A} = \mathbb{C}$).

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