

Gilbreath sequences and proof of conditions for Gilbreath conjecture

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Abstract

The conjecture attributed to Norman L. Gilbreath, but formulated by Francois Proth in the second half of the 1800s, concerns an interesting property of the ordered sequence of prime numbers P . Gilbreath conjecture stated that, if we compute the absolute values of differences of consecutive primes on ordered sequence of prime numbers, and if this calculation is repeated for the terms in the new sequence and so on, every sequence will start with 1. In this paper the concept of Gilbreath sequence, Gilbreath triangle and Gilbreath equation are defined and on the basis of the results obtained from their properties, an inductive proof is produced, which establishes the necessary condition to state that Gilbreath conjecture is true.

1 Introduction to Gilbreath conjecture

Let the ordered sequence $P = (2, 3, 5, 7, 11, 13, 17, \dots) = (p_1, \dots, p_n, \dots)$ formed by prime numbers, and set

$$p_a^b = \begin{cases} p_{a+1} - p_a, & \text{if } b = 1; \\ |p_{a+1}^{b-1} - p_a^{b-1}|, & \text{otherwise} \end{cases} \quad (1)$$

where $b \in [1; n - 1]$. Gilbreath conjectured that $p_1^b = 1$. It is likely that this conjecture is satisfied by many other sequences of integers, so it is necessary

to define the general properties of all sequences that satisfy this conjecture. There is an interesting computational proof of the Gilbreath conjecture for all prime numbers less than 10^{13} by A. M. Odlyzko [1].

2 Gilbreath sequence

Definition 1. Let $S = (s_1, s_2, s_3, \dots, s_n)$ be a finite increasing sequence of integers and

$$s_a^b = \begin{cases} s_{a+1} - s_a, & \text{if } b = 1; \\ |s_{a+1}^{b-1} - s_a^{b-1}|, & \text{otherwise} \end{cases} \quad (2)$$

where $b \in [1; n - 1]$. S is defined Gilbreath sequence if $s_1^b = 1 \forall b$.

For example, let $S = (2, 3, 5, 7, 11, 13, 17)$ be a sequence of length $n = 7$, Gilbreath triangle of S is defined by (2). Then

s_1	s_2	s_3	s_4	\dots	s_{n-3}	s_{n-2}	s_{n-1}	s_n
s_1^1	s_2^1	s_3^1	s_4^1	\dots	s_{n-3}^1	s_{n-2}^1	s_{n-1}^1	
\dots								
s_1^{n-2}	s_2^{n-2}							
s_1^{n-1}								

with values

2	3	5	7	11	13	17		
1	2	2	4	2	4			
1	0	2	2	2				
1	2	0	0					
1	2	0						
1	2							
1								

Let \mathbb{G}_n denote the set of all Gilbreath sequences of length n . In our example the first term of every sequence is equal to 1. Then $S \in \mathbb{G}_7$.

Lemma 1. Let $S = (s_1, \dots, s_n) \in \mathbb{G}_n$ and $S' = (s_1, \dots, s_{n-1})$ be finite sequences of integers. Then $S' \in \mathbb{G}_{n-1}$.

Proof. Consider the Gilbreath triangle of S

s_1	s_2	s_3	s_4	\dots	s_{n-3}	s_{n-2}	s_{n-1}	s_n
s_1^1	s_2^1	s_3^1	s_4^1	\dots	s_{n-3}^1	s_{n-2}^1	s_{n-1}^1	

$$\begin{matrix} \dots \\ s_1^{n-2} & s_2^{n-2} \\ s_1^{n-1} \end{matrix}$$

where $s_1^1 = \dots = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$. Removing the last element of each sequence gives

$$\begin{matrix} s_1 & s_2 & s_3 & s_4 & \dots & s_{n-3} & s_{n-2} & s_{n-1} \\ s_1^1 & s_2^1 & s_3^1 & s_4^1 & \dots & s_{n-3}^1 & s_{n-2}^1 & \\ \dots \\ s_1^{n-2} \end{matrix}$$

which is Gilbreath triangle of S' , $s_1^1 = \dots = s_1^{n-2} = 1$ as a consequence of $S \in \mathbb{G}_n$. Then $S' \in \mathbb{G}_{n-1}$. \square

Definition 2. Let $S = (s_1, \dots, s_n) \in \mathbb{G}_n$ and $S' = (s_1, \dots, s_n, k)$ be finite sequences of integers. We denote with \mathbb{K}_S the set of integers k such that $S' \in \mathbb{G}_{n+1}$.

Gilbreath triangle of S is

$$\begin{matrix} s_1 & s_2 & s_3 & s_4 & \dots & s_{n-3} & s_{n-2} & s_{n-1} & s_n \\ s_1^1 & s_2^1 & s_3^1 & s_4^1 & \dots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 & \\ \dots \\ s_1^{n-2} & s_2^{n-2} \\ s_1^{n-1} \end{matrix}$$

where $s_1^1 = \dots = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$. Gilbreath triangle of S' is

$$\begin{matrix} s_1 & s_2 & s_3 & s_4 & \dots & s_{n-3} & s_{n-2} & s_{n-1} & s_n & k \\ s_1^1 & s_2^1 & s_3^1 & s_4^1 & \dots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 & |s_n - k| & \\ \dots \\ s_1^{n-2} & s_2^{n-2} & |s_3^{n-3} - |s_4^{n-4} - | \dots - |s_{n-1}^1 - |s_n - k| | \dots ||| \\ s_1^{n-1} & |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - | \dots - |s_{n-1}^1 - |s_n - k| | \dots ||| \\ |s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - | \dots - |s_{n-1}^1 - |s_n - k| | \dots ||| \end{matrix}$$

where $s_1^1 = \dots = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$. If also $s_1^n = |s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - | \dots - |s_{n-1}^1 - |s_n - k| | \dots ||| = 1$, then $S' \in \mathbb{G}_{n+1}$.

$$|s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - | \dots - |s_{n-1}^1 - |s_n - k| | \dots ||| = 1 \quad (3)$$

The equation is defined as Gilbreath equation of S and $\mathbb{K}_S = (k_1, \dots, k_{2^n})$ is the set of all solutions for k .

Gilbreath equation is a 2^n degree equation. Then there are 2^n values of k that satisfy the equation (3)

$$k_{1,\dots,2^n} = \pm s_1^{n-1} \pm s_2^{n-2} \pm s_3^{n-3} \pm s_4^{n-4} \pm \dots \pm s_{n-1}^1 + s_n \pm 1 \quad (4)$$

With respect to \mathbb{K}_S , the largest value that solves (4) is $\max \mathbb{K}_S = s_1^{n-1} + s_2^{n-2} + s_3^{n-3} + s_4^{n-4} + \dots + s_{n-1}^1 + s_n + 1$ and the smallest value that solves (4) is $\min \mathbb{K}_S = -s_1^{n-1} - s_2^{n-2} - s_3^{n-3} - s_4^{n-4} - \dots - s_{n-1}^1 + s_n - 1 = 2s_n - \max \mathbb{K}_S$. A remarkable relation is

$$\max \mathbb{K}_S + \min \mathbb{K}_S = 2s_n \quad (5)$$

Lemma 2. Let $S = (s_1, \dots, s_n) \in \mathbb{G}_n$ and $S' = (s_1, \dots, s_n, s_n)$ finite sequences of integers, then $S' \in \mathbb{G}_{n+1}$.

Proof. Gilbreath equation of S' with $k = s_n$ is $|s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - | \dots - |s_{n-1}^1| \dots ||||| = 1$ which is true because $S \in \mathbb{G}_n$. Then $S' \in \mathbb{G}_{n+1}$. \square

It is useful, for the following proofs to introduce the definition of two Gilbreath sequences. Let $S = (s_1, \dots, s_n)$ be a finite sequence of integers, from (4), any solutions of the Gilbreath equation cannot be greater than $\max \mathbb{K}_S$, so the sequence $(s_1, \dots, s_n, \max \mathbb{K}_S)$ is the upper bound sequence for the sequence (s_1, \dots, s_n) . The new sequence $S' = (s_1, \dots, s_n, \max \mathbb{K}_S, k)$ will have the upper limit for $k = \max \mathbb{K}_{(s_1, \dots, s_n, \max \mathbb{K}_S)}$ and so on. Equally, let a finite sequence of integers $S = (s_1, \dots, s_n)$, from (4), any value of k cannot be smaller than $\min \mathbb{K}_S$ and the new sequence $S' = (s_1, \dots, s_n, \min \mathbb{K}_S, k)$ will have the lower limit for $k = \min \mathbb{K}_{(s_1, \dots, s_n, \min \mathbb{K}_S)}$ and so on. From this it is now possible to introduce the definition of upper bound sequence and lower bound sequence.

Definition 3. Let a finite sequence of integers $S = (s_1, \dots, s_n) \in \mathbb{G}_n$, let denote with U_S the upper bounds sequence for S and with L_S the lower bounds

sequence for S . Also, let denote U'_S and L'_S as below.

$$\begin{aligned}
U_S &= \left(s_1, \dots, s_n, \max \mathbb{K}_{(s_1, \dots, s_n)}, \max \mathbb{K}_{(s_1, \dots, s_n, \max \mathbb{K}_{(s_1, \dots, s_n)})}, \dots \right) = \\
&\quad (u_{S_1}, \dots, u_{S_m}, \dots) \\
L_S &= \left(s_1, \dots, s_n, \min \mathbb{K}_{(s_1, \dots, s_n)}, \min \mathbb{K}_{(s_1, \dots, s_n, \min \mathbb{K}_{(s_1, \dots, s_n)})}, \dots \right) = \\
&\quad (l_{S_1}, \dots, l_{S_m}, \dots) \\
U'_S &= \left(\max \mathbb{K}_{(s_1, \dots, s_n)}, \max \mathbb{K}_{(s_1, \dots, s_n, \max \mathbb{K}_{(s_1, \dots, s_n)})}, \dots \right) = \\
&\quad (u'_{S_1}, \dots, u'_{S_m}, \dots) \\
L'_S &= \left(\min \mathbb{K}_{(s_1, \dots, s_n)}, \min \mathbb{K}_{(s_1, \dots, s_n, \min \mathbb{K}_{(s_1, \dots, s_n)})}, \dots \right) = \\
&\quad (l'_{S_1}, \dots, l'_{S_m}, \dots)
\end{aligned} \tag{6}$$

From lemma 2 and definition 3

$$S = (s_1, \dots, s_n) \in \mathbb{G}_n \implies l_{(s_1, \dots, s_m)} \leq s_{m+1} \leq u_{(s_1, \dots, s_m)}, \text{ where } m = 1, \dots, n-1 \tag{7}$$

Lemma 3. Let a finite sequence of integers $S = (s_1, \dots, s_n) \in \mathbb{G}_n$ where $s_1 \in 2\mathbb{Z}$. Then $(s_2, \dots, s_n) \subset (2\mathbb{Z} + 1)^{n-1}$.

Proof. Let $S_1 = (s_1)$, where $s_1 \in 2\mathbb{Z}$. From definition 2, $S_2 = (s_1, k) \in \mathbb{G}_2$ if $k = s_1 \pm 1 \in 2\mathbb{Z} + 1$. Now let the sequence $S_3 = (s_1, s_1 \pm 1, k)$, from definition 2, $S_3 \in \mathbb{G}_3$ if $k = |\pm 1| + (s_1 \pm 1) \pm 1 = 1 + s_1 \pm 1 \pm 1$. From the previous step, $s_1 \pm 1 \in 2\mathbb{Z} + 1$. Then $1 + s_1 \pm 1 \in 2\mathbb{Z}$ and $1 + s_1 \pm 1 \pm 1 \in 2\mathbb{Z} + 1$. Iteratively, this can be proved for every element of S . If $S \in \mathbb{G}_n$ and the first element of S is an even number. Then all the other numbers of the sequence will be odd. \square

Lemma 4. Let a finite sequence of integers $S = (s_1, \dots, s_n) \in \mathbb{G}_n$ where $s_1 \in 2\mathbb{Z} + 1$. Then $(s_2, \dots, s_n) \subset (2\mathbb{Z})^{n-1}$.

Proof. Let $S_1 = (s_1)$, where $s_1 \in 2\mathbb{Z} + 1$. From definition 2, $S_2 = (s_1, k) \in \mathbb{G}_2$ if $k = s_1 \pm 1 \in 2\mathbb{Z}$. Now let the sequence $S_3 = (s_1, s_1 \pm 1, k)$, from definition 2, $S_3 \in \mathbb{G}_3$ if $k = |\pm 1| + (s_1 \pm 1) \pm 1 = 1 + s_1 \pm 1 \pm 1$. From the previous step, $s_1 \pm 1 \in 2\mathbb{Z}$. Then $1 + s_1 \pm 1 \in 2\mathbb{Z} + 1$ and $1 + s_1 \pm 1 \pm 1 \in 2\mathbb{Z}$. Iteratively, this can be proved for every element of S . Then if $S \in \mathbb{G}_n$ and the first element of S is an odd number. Then all the other numbers of the sequence will be even. \square

Lemma 5. Let $2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$ denote the sets $2\mathbb{Z}$ and $2\mathbb{Z} + 1$ and let a finite sequence of integers $S = (s_1, \dots, s_n) \in \mathbb{G}_n$ where $s_1 \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$. Then $(s_2, \dots, s_n) \subset \left[2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)\right]^{n-1}$.

Proof. From lemma 3 and lemma 4. □

Lemma 6. Let $S = (s_1, \dots, s_n) \in \mathbb{G}_n$ and $S' = (s_1, \dots, s_n, k) \in \mathbb{G}_{n+1}$ finite sequences of integers where $s_1 \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$. Then $\mathbb{K}_S = \{x \in [\min \mathbb{K}_S, \max \mathbb{K}_S] \wedge x \in 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)\}$.

Proof. From definition 2, there are 2^n values of k that satisfy $S' \in \mathbb{G}_{n+1}$ and $\min \mathbb{K}_S \leq k \leq \max \mathbb{K}_S$. \mathbb{K}_S is defined as the set of all solutions of k . Then it contains elements between $\min \mathbb{K}_S$ and $\max \mathbb{K}_S$. From lemma 5 it has already been shown that if $s_1 \in 2\mathbb{Z}$, then $s_n \in 2\mathbb{Z} + 1, \forall n > 1$ and if $s_1 \in 2\mathbb{Z} + 1$, then $s_n \in 2\mathbb{Z}, \forall n > 1$. □

An important result regarding (4) follows from lemma 5. (4) generates 2^n solutions for a finite sequence $(s_1, \dots, s_n, k) \in \mathbb{G}_{n+1}$ where $(s_1, \dots, s_n) \in \mathbb{G}_n$, but from lemma 5 it has been proved that these solutions are only even or only odd according s_1 . Therefore, the number of distinct solutions generated by (4) is 2^{n-1} since solutions are divided between even and odd: $\#\mathbb{K}_S = 2^{n-1}$ where $\#$ is the cardinality.

Theorem 1. Let $S = (s_1, \dots, s_n) \in \mathbb{G}_n$ and $S' = (s_1, \dots, s_n, k)$ finite sequences of integers, where $s_1 \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right)$. Then $k \in \mathbb{K}_S \Leftrightarrow S' \in \mathbb{G}_{n+1}$.

Proof. Prove the right implication first. From definition 2, $k \in [\min \mathbb{K}_S, \max \mathbb{K}_S]$ and from lemma 5, $k \in 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)$. Then $k \in \mathbb{K}_S \Rightarrow S' \in \mathbb{G}_{n+1}$. Prove the left implication by contradiction. Let suppose that $S' \in \mathbb{G}_{n+1}$ but $k \notin \mathbb{K}_S$. Then $k \in \{x \notin [\min \mathbb{K}_S, \max \mathbb{K}_S] \vee x \notin 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)\}$. From definition 2 and lemma 5 it is not possible to have $S' \in \mathbb{G}_{n+1}$ if $k > \max \mathbb{K}_S \vee k < \min \mathbb{K}_S \vee k \notin 2\mathbb{Z} + \left(\frac{1}{2} \mp \frac{1}{2}\right)$. Then it is also true $k \in \mathbb{K}_S \Leftarrow S' \in \mathbb{G}_{n+1}$. □

2.1 Notable upper and lower bound sequence

From definition 3, $U_S = (S, U'_S)$ and $L_S = (S, L'_S)$. Let $S = (s_1)$, $U'_S = (s_1 + 1, s_1 + 3, \dots, s_1 + 2^{n-1} - 1)$ and $L'_S = (s_1 - 1, s_1 - 3, \dots, s_1 - 2^{n-1} + 1)$.

The author did not find any remarkable expression to analytically define the trend of U'_S and L'_S for a generic sequence S but it was observed that

the exponential trend is preserved [2], [3], [4], [5]. However, this trend varies with the number of terms of U'_S and L'_S so it does not seem possible to establish what will be the $n + 1$ -th term of U'_S and L'_S given the previous n terms through an analytical formula. However, it is always possible use the recursive expression (4).

Let the sequence $S = (s_1, \dots, s_n) \in \mathbb{G}_n$, using definition 3, the (5) can be rewritten as:

$$u_{S_{n+m}} + l_{S_{n+m}} = u'_{S_m} + l'_{S_m} = 2s_n \quad (8)$$

equivalent to (7).

If it is true that exponential trend is preserved. Then elements of U'_S can be written in the form $u'_{S_n} = \alpha e^{\beta n}$ or $\log u'_{S_n} = \log \alpha + \beta n$.

The best fit for a dataset $D = (d_1, \dots, d_n)$ in a linear regression model is

$$\beta = \frac{n \sum_{i=1}^n i \log d_i - \sum_{i=1}^n i \sum_{i=1}^n \log d_i}{n \sum_{i=1}^n i^2 - (\sum_{i=1}^n i)^2} = \frac{12}{n(n^2 - 1)} \left(\sum_{i=1}^n i \log d_i - \frac{n+1}{2} \sum_{i=1}^n \log d_i \right) \quad (9)$$

$$\log \alpha = \frac{1}{n} \sum_{i=1}^n \log d_i - \frac{\beta(n+1)}{2} \quad (10)$$

hence

$$\alpha = e^{-\frac{\beta(n+1)}{2}} \left(\prod_{i=1}^n d_i \right)^{\frac{1}{n}} \quad (11)$$

and the coefficient of determination is

$$R^2 = 1 - \frac{\sum_{i=1}^n (d_i - \alpha e^{\beta i})^2}{\sum_{i=1}^n (d_i - \bar{d})^2} \quad (12)$$

Note that if $|d_a| < |d_{a+1}|$ and $d_a < 0$, it is not possible to calculate $\log d_b$, where $b > a$. To avoid this problem the transformation $d_i \rightarrow d_i + \frac{d_1}{2} \left(\frac{d_1}{|d_1|} - 1 \right)$ is performed. In this way if $d_1 > 0$, then $d_i \rightarrow d_i$. If $d_1 < 0$, then $d_i \rightarrow d_i - d_1$. After that, the fitting curve will be $d_n = \alpha e^{\beta n} - \frac{d_1}{2} \left(\frac{d_1}{|d_1|} - 1 \right)$.

Example 1. Let a finite sequence $S = (2, 3, 5, 7, 11, 13)$ of length 6, the first 18 terms of the upper bound sequence are

$$U'_S = (21, 47, 119, 297, 705, 1595, 3475, 7365, 15309, 31399, 63823, 128961, 259577, 521203, 1044907, 2092829, 4189253, 8382751)$$

The first 18 terms of the lower bound sequence are

$$L'_S = (5, -21, -93, -271, -679, -1569, -3449, -7339, -15283, -31373, -63797, -128935, -259551, -521177, -1044881, -2092803, -4189227, -8382725).$$

Note that $(S, U'_S) \in \mathbb{G}_{24}$ and $(S, L'_S) \in \mathbb{G}_{24}$. According to (5), $21 + 5 = 47 - 21 = 119 - 93 = 297 - 271 = \dots = 8382751 - 8382725 = 26 = 2s_6$. Let fitting U'_S to $\alpha_{U'_S} e^{\beta_{U'_S} n}$. From (9), $\beta_{U'_S} = \frac{6}{2907} (\sum_{i=1}^{18} i \log u'_{S_i} - \frac{19}{2} \sum_{i=1}^{18} \log u'_{S_i}) \approx 0.75$,

from (11) $\alpha_{U'_S} = e^{-\frac{\beta(19)}{2}} (\prod_{i=1}^{18} u'_{S_i})^{\frac{1}{18}} \approx 14.42$. The model fits the trend of u'_{S_n} with $R^2 \approx 0.92$ from (12). As regards L'_S , from (8), $l'_{S_n} = 2s_6 - \alpha_{U'_S} e^{\beta_{U'_S} n} \approx 26 - 14.42e^{0.75n} \approx -7.97e^{0.80n}$ with $R^2 \approx 0.99$.

As explained above, the addition of a term to U'_S leads to new values of α and β , therefore this analysis is carried out without pretending to evaluate the $n + 1$ -th element of a given U'_S of length n .

The numerical analysis of the values of the upper limit sequence was added only to show that no analytical formula has been found for the generations of the values of this sequence, with exception of (4).

3 Proof of theorem 1 for $P = (p_1, \dots, p_n)$

Let a finite sequence of integers $S = (s_1, \dots, s_n)$, from theorem 1 is true the following. The relationship $s_2 = s_1 \pm 1$ must be true, otherwise it would not be true that $s_1^1 = 1$. As a consequence of lemma 5, for all elements subsequent to s_1 , the absolute difference of two successive elements must be an integer multiple of 2 so as to maintain the absolute difference of two successive elements as an even value. So, if the first element in the sequence is even, the subsequent elements must be odd and if the first element is odd, the subsequent elements must be even.

As a consequence of definition 2, solution of Gilbreath equation (4) and definition 3, each n -th element of a sequence S must be within the range between the upper and the lower sequences calculated on all the elements prior to the n -th ones. From definition 2 and according to the solution of

Gilbreath equation (4), there cannot exist Gilbreath sequence in which the n -th is larger than $\max \mathbb{K}_{(s_1, \dots, s_{n-1})}$, since $\max \mathbb{K}_{(s_1, \dots, s_{n-1})}$ is the maximum value that the n -th value can take. The same goes for $\min \mathbb{K}_{(s_1, \dots, s_{n-1})}$, since it is the smallest value that the n -th value can take. Then

$$l_{(s_1, \dots, s_{n-1})_n} \leq s_n \leq u_{(s_1, \dots, s_{n-1})_n} \quad (13)$$

Following the results obtained in the previous paragraphs about Gilbreath sequence and Gilbreath equation, let proceed discussing Gilbreath conjecture. The results obtained so far will be used to establish if theorem 1 is true for the ordered sequence of prime numbers P .

Theorem 2. *For every n -th prime number, $n > 1$, it is true that $l_{(p_1, \dots, p_{n-1})_n} \leq p_n \leq u_{(p_1, \dots, p_{n-1})_n}$.*

Proof. By definition of L and U , $l_{(p_1, \dots, p_{n-1})_n} = \min \mathbb{K}_{(p_1, \dots, p_{n-1})}$ and $u_{(p_1, \dots, p_{n-1})_n} = \max \mathbb{K}_{(p_1, \dots, p_{n-1})}$. Then (13) becomes

$$\min \mathbb{K}_{(p_1, \dots, p_{n-1})} \leq p_n \leq \max \mathbb{K}_{(p_1, \dots, p_{n-1})} \quad (14)$$

Let $S = (p_1, p_2) = (2, 3) \in \mathbb{G}_2$ Gilbreath sequence formed by the first two prime numbers. As $S \in \mathbb{G}_2$, from (5), $\min \mathbb{K}_{(p_1, p_2)} \leq p_2 \leq \max \mathbb{K}_{(p_1, p_2)}$ and from lemma 2, $(p_1, p_2, p_2) \in \mathbb{G}_3$. By definition of P , $p_n > p_{n-1}$. Since $\min \mathbb{K}_{(p_1, p_2)} \leq p_2$, it is certainly true that $\min \mathbb{K}_{(p_1, p_2)} \leq p_3$. The left inequality of (14) is proved for $n = 3$. If $p_3 \leq \max \mathbb{K}_{(p_1, p_2)}$. Then, subtracting the quantity $2p_2$ from both sides,

$$p_3 - 2p_2 \leq \max \mathbb{K}_{(p_1, p_2)} - 2p_2 \quad (15)$$

Let Bertrand postulate [6]

$$p_n < 2p_{n-1} \quad (16)$$

Replacing (5) and (16) in (15) gets

$$\min \mathbb{K}_{(p_1, p_2)} \leq \alpha, \alpha \in \mathbb{N} \quad (17)$$

Then, exist a value $\alpha \in \mathbb{N}$ such that $\min \mathbb{K}_{(p_1, p_2)} \leq \alpha$. We have proved that $p_3 \leq \max \mathbb{K}_{(p_1, p_2)}$ implies (17). Replacing (16) and then (5) in (17) gets back $p_3 \leq \max \mathbb{K}_{(p_1, p_2)}$ hence we have that $p_3 \leq \max \mathbb{K}_{(p_1, p_2)} \Leftrightarrow \min \mathbb{K}_{(p_1, p_2)} \leq \alpha, \alpha \in \mathbb{N}$. $\min \mathbb{K}_{(p_1, p_2)}$ can be written using (4) as $\min \mathbb{K}_{(p_1, p_2)} = -p_1^1 + p_2 - 1$

where $-p_1^1 - 1 < 0$ and $p_2 > 0$. If $\alpha = p_2$, (17) is proven. Then the right inequality of (14) is proved for $n = 3$ and $(p_1, p_2, p_3) \in \mathbb{G}_3$.

At this point the proof can process showing that (14) is true for $n = 4$. Since, $p_4 > p_3$ and $\min \mathbb{K}_{(p_1, p_2, p_3)} \leq p_3 \leq \max \mathbb{K}_{(p_1, p_2, p_3)}$, it is true that $\min \mathbb{K}_{(p_1, p_2, p_3)} \leq p_4$. Again,

$$p_4 \leq \max \mathbb{K}_{(p_1, p_2, p_3)} \Leftrightarrow \min \mathbb{K}_{(p_1, p_2, p_3)} \leq \alpha, \alpha \in \mathbb{N} \quad (18)$$

From the equation (4), $\min \mathbb{K}_{(p_1, p_2, p_3)} = -p_1^2 - p_2^1 + p_3 - 1$ where $-p_1^2 - p_2^1 - 1 < 0$ and $p_3 > 0$. If $\alpha = p_3$ (18) is proved for $n = 4$ and $(p_1, p_2, p_3, p_4) \in \mathbb{G}_4$. Iteratively, this can be proved to verify (14) for every prime. \square

Lemma 3 is already proved for P since p_1 is even and all other elements are odd: by definition of prime number, there are no even prime numbers except for 2. Theorem 2 prove theorem 1 in the case of P .

4 Conclusions

Lemma 1, 2, 3, 4, 5 and 6, about fundamental properties of Gilbreath sequence, are used to prove theorem 1. This theorem summarizes previous results stating that if and only if the first element of a finite sequence of integers $(s_1, \dots, s_n) \in \mathbb{G}_n$ is even, any odd integer k between U_S and L_S makes the sequence $(s_1, \dots, s_n, k) \in \mathbb{G}_{n+1}$ and if and only if the first element of a finite sequence of integers $(s_1, \dots, s_n) \in \mathbb{G}_n$ is odd, any even integer k between U_S and L_S makes the sequence $(s_1, \dots, s_n, k) \in \mathbb{G}_{n+1}$. The most important result of this paper is contained in theorem 2 in which theorem 1 is proved for the particular case of the ordered sequence of prime numbers P .

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