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Spacetime and Internal Symmetry from Split Bioctonions and The Two Extra SU(3)'s of $E_8 \times \omega E_8$

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Abstract

Over the last few years, we have attempted to develop an $E_8 \times \omega E_8$ theory of unification to combine the standard model with general relativity. In the present new work, we give a self-contained construction in which the two extra SU(3) factors that appear in the maximal subgroup chain $E_8 \supset E_6 \times \mathrm{SU}(3)$ on each side of $E_8 \times \omega E_8$ generate: (i) a six-dimensional base (M_6, g) of signature $(3,3)$; (ii) two embedded Lorentzian 4D spacetimes; and (iii) per side, a canonical *real* 4-dimensional internal fibre naturally identified with the tangent of $\mathbb{C}P^2 = \mathrm{SU}(3)/S(U(2) \times U(1))$. The key algebraic ingredient is the octonionic split $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\varepsilon$ with $\varepsilon \perp \mathbb{H}$, by which the branch $\mathrm{Adj}\,\mathrm{SU}(3) \to \mathbf{3}_0 \oplus \mathbf{2}_{+1} \oplus \overline{\mathbf{2}}_{-1} \oplus \mathbf{1}_0$ is realised as $\Im\mathbb{H} \oplus (\mathbb{H}\varepsilon)_{\mathbb{R}} \oplus \mathbb{R}$. The two $U(1)$ factors play the role of Spin^c connections on the $\mathbb{C}P^2$ fibres.

Keywords: E8 x E8 unification; octonions; trinification; spacetime; internal symmetry; adjoint SU(3)

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1. Introduction

Over the last few years we have proposed and developed an $E_8 \times \omega E_8$ theory of unification which aims to unify the standard model with gravitation described by the general theory of relativity [1–3]. Here, ω is the split complex number. It is assumed that each of the two E_8 branches as $SU(3) \times E_6$ and each of the two resulting E_6 undergoes a trinification $E_6 \to SU(3) \times SU(3) \times SU(3)$. The split complex number plays a crucial role, enabling the emergence of a Lorentzian signature for spacetime, and enabling the emergence of chiral fermions. Its origin in our theory can be traced to (left acting) octonionic chains made from the algebra $\mathbb{O} \oplus \omega\mathbb{O}$ of split bioctonions. Complex split bioctonions generate the Clifford algebra $Cl(7) \cong Cl(6) \oplus \omega Cl(6)$ which is used to obtain one generation of standard model chiral quarks and leptons [4].

The trinification provides the following interpretation of the branching of the two E_8, as discussed in detail in [3]:

$$\begin{aligned} E_{8L} &\to SU(3)_L^{geom} \times E_{6L} \xrightarrow{E_{6L}} SU(3)_c \times SU(3)_{F,L} \times SU(3)_L \xrightarrow{SU(3)_L} SU(2)_L \times U(1)_Y \\ E_{8R} &\to SU(3)_R^{geom} \times E_{6R} \xrightarrow{E_{6R}} SU(3)_{c'} \times SU(3)_{F,R} \times SU(3)_R \xrightarrow{SU(3)_R} SU(2)_R \times U(1)_{Ydem} \end{aligned} \tag{1}$$

and

$$\mathbf{248} = (\mathbf{78}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}) \oplus (\mathbf{27}, \mathbf{3}) \oplus (\overline{\mathbf{27}}, \overline{\mathbf{3}}) \tag{2}$$

Of the three $SU(3)$s arising from the branching of E_{6L}, the $SU(3)_c$ implements the color gauge symmetry of QCD. Furthermore, $SU(3)_{F,L}$ is the non-gauged global flavor symmetry which is responsible for three left-handed fermion generations [3] described by the exceptional Jordan algebra $J_3(\mathbb{O}_c)$. $SU(3)_L$ branches as $SU(2)_L \times U(1)_Y$ giving rise to the electroweak sector, and the $SU(2)_L$ acts only on left-handed particles.

As regards the branching of the second E_6, which we label E_{6R}, the $SU(3)_{c'}$ symmetry is preferably not gauged (so as to be consistent with phenomenology) - it is global and explicitly broken at the electroweak scale. Its role is to rotate the so-called Jordan frame which arises in the Peirce decomposition

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of the exceptional Jordan algebra matrices [3]. The $\omega SU(3)_{F,R}$ is the global flavor symmetry which describes three generations of right-handed fermions. The $SU(3)_R$ branches as $SU(2)_R \times U(1)_{Ydem}$. Here, the spontaneously broken $SU(2)_R$ gives rise to general relativity, and the unbroken $U(1)_{dem}$ arising from $U(1)_{Ydem} \to U(1)_{dem}$ is a new force, dubbed dark electromagnetism. It is sourced by the square-root of mass and can be made cosmologically and phenomenologically safe. The emergence of general relativity in this manner, and the preceding gravi-weak unification, was briefly discussed in [1] and is analysed in detail in a forthcoming publication [5].

$U(1)_{\text{dem}}$ charge normalisation.

To make the "couples to $\sqrt{m}$" statement precise and dimensionless, let $m_f(\mu) = y_f(\mu)\, v_M$ with v_M a reference mass scale (e.g. a VEV). Define the dark charge as

$$Q_{\text{dem}}(f) \;:=\; \sqrt{y_f(\mu)} \;=\; \sqrt{m_f(\mu)/v_M}\,,$$

so that Q_{dem} is RG–scheme aware via $y_f(\mu)$ and the $U(1)_{\text{dem}}$ coupling remains dimensionless. Phenomenology then fixes g_{dem} and the relevant scale μ.

The focus of the present article is the two extra $SU(3)$s which we label $SU(3)_L^{geom}$ and $SU(3)_R^{geom}$, which arise in the decomposition of the two E_8, and which are apart from the six $SU(3)s$ arising in the trinification of the two E_6. As was already proposed by us in [1], the role of $SU(3)_L^{geom}$ and $\omega SU(3)_R^{geom}$ is to describe spacetime and internal symmetry space. The split complex number is crucial in enabling a Lorentzian signature to emerge from combination of two compact groups, as we rigorously demonstrate below. The split bioctonions also play an important role here: their split quaternionic subalgebra here gives rise to a 6D spacetime with signature (3,3), which upon electroweak symmetry breaking gives rise to two overlapping 4D spacetimes with signature (3,1) and (1,3) respectively. One of these is our familiar 4D spacetime, curved by gravitation. The other 4D spacetime has flipped signature and a (1,1) intersection with our spacetime - its directions are interpreted as being curved by the weak force [5]. The complementary remaining quaternionic subalgebras describe internal symmetry space for holding the unbroken symmetries $SU(3)_c$ and $SU(3)_{c'}$. A decomposition of the adjoint **8** of $SU(3)_{L,R}^{geom}$ shows an elegant mapping to the octonions, as described below.

Thus the goal of the present short note is to rigorously demonstrate how $SU(3)_L^{geom}$ and $\omega SU(3)_R^{geom}$ provide the scaffolding on which the fields contained in $E_{6L} \times E_{6R}$ live. This construction realises the all-encompassing role of $E_8 \times \omega E_8$, making the unified symmetry a source of space-time as well as of matter. It is in this spirit that we call the primitive entities possessing $E_8 \times \omega E_8$ symmetry 'atoms of space-time-matter' [2].

2. Roadmap

Our mass-ratio programme [3] places matter in $E_{6L} \times \omega E_{6R}$ and uses the exceptional Jordan algebra $J_3(\mathbb{O}_\mathbb{C})$ to derive charged-fermion square-root mass ratios and mixings. The present paper supplies the geometric scaffolding promised [see Sec. XII.H of [3]] in the uplift to $E_8 \times \omega E_8$: the *two extra* SU(3)'s are taken as *geometric* structure groups that yield precisely a $(3,3)$ base, two 4D spacetimes, and two real 4D internal fibres. The $E_6^{L,R}$ fields then live on this scaffold. A complementary 6D gravi–weak reduction shows how two 4D leaves arise dynamically from a 6D BF theory [5].

What is new in the present article:

(i) A detailed octonionic identification of the 4D fibres with the realification of $\mathbf{2} \oplus \bar{\mathbf{2}}$, using $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\varepsilon$ and an intrinsic complex structure on $\mathbb{H}\varepsilon$. (ii) A clean role for each extra $U(1)$ as the Spin^c line on the $\mathbb{C}P^2$ fibre. (iii) A unified presentation tying the split-bioctonion base, the E_8 branching, and the 6D gravi–weak localisation.

Plan of the paper:

Secs. 3–4 build M_6 from split bioctonions and embed the two 4D spacetimes. Secs. 5–6 relate the $\mathrm{SU}(3)^{geom}_{L,R}$ branching to $\Im\mathbb{H}$ and $\mathbb{H}\varepsilon$ and identify the 4D fibres with $T\mathbb{C}P^2$. Sec. 7 clarifies the two $U(1)$'s. Section 8 explains how the $E_{6L} \times E_{6R}$ fields live on the proposed scaffold, and Section 9 gives the big picture and interpretation.

3. Split-Bioctonionic Base (M_6, g)

In our $E_8 \times \omega E_8$ theory of unification, an atom of space-time-matter (STM atom) is described by the trace dynamics action [2]

$$\frac{S}{\hbar} = \frac{1}{2}\int \frac{d\tau}{\tau_{Pl}}\; Tr\left[\frac{L_p^2}{L^2}\dot{Q}_1^\dagger\, \dot{Q}_2\right] \tag{3}$$

Here, Q_1 and Q_2 are two operators/matrices which together describe an STM atom. Evolution is described with respect to Connes time τ. The trace Lagrangian is assumed invariant under an $E_8 \times \omega E_8$ symmetry. Importantly, this action does not make any distinction between space-time and matter. Thus it is not that these two operators live on some abstract operator-valued space-time; rather, they just are. Prior to the electroweak symmetry breaking our observable universe is made of a very large collection of STM atoms ($N \sim 10^{120}$- this number is an empirical input). The entries in these matrices are Grassmann numbers. The action is made dimensionless by introducing Planck's constant $\hbar$, and Connes time τ is made dimensionless by introducing Planck time τ_{Pl}.

At the electroweak transition, segregation into space-time and matter takes place, and this is mathematically defined as follows:

$$\begin{aligned} \dot{Q}_1^\dagger &= \dot{Q}_B^\dagger + \frac{L_p^2}{L^2}\beta_1 \dot{Q}_F^\dagger; \qquad & \dot{Q}_2 &= \dot{Q}_B + \frac{L_p^2}{L^2}\beta_2 \dot{Q}_F; \\ \dot{Q}_B &= \frac{1}{L}(i\alpha q_B + L\dot{q}_B); \qquad & \dot{Q}_F &= \frac{1}{L}(i\alpha q_F + L\dot{q}_F) \end{aligned} \tag{4}$$

The matrices Q_B are made of even grade Grassmann numbers and are called bosonic; whereas the matrices Q_F are made of odd-grade Grassmann numbers and are called fermionic, as is the case in quantum field theory. β_1 and β_2 are unequal odd-grade Grassmann numbers, introduced to make the Lagrangian bosonic. L is a length scale which characterises the STM atom, and it is made dimensionless by introducing Planck length L_p. Along with Planck time and Planck's constant, these are the only three fundamental constants in the theory - all dimensionless fundamental constants must be derived from this theory. Thus, the matrices Q_1 and Q_2 have been split into their bosonic and fermionic parts, which respectively describe bosons and fermions.

Furthermore, the bosonic part Q_B has been split into q_B and $\dot{q}_B$. The q_B will describe QCD ($SU(3)_c$) and 'gravi-QCD' ($SU(3)_{c'}$) and electromagnetism $U(1)_{em}$ and dark electromagnetism ($U(1)_{dem}$). Whereas $\dot{q}_B$ will describe the gravi-weak interaction from which general relativity (spontaneously broken $SU(2)_R$) and the weak interaction (spontaneously broken $SU(2)_L$) will emerge. The gravi-weak interaction is the geometry of the 6D spacetime - this latter is one of the two ingredients to emerge from the extra $SU(3)^{geom}_L \times \omega SU(3)^{geom}_R$. The other ingredient to emerge from the two extra $SU(3)$s is the internal symmetry space for $SU(3)_c, SU(3)_{c'}, U(1)_{em}, U(1)_{dem}$, as we show in the present article. α is the Yang-Mills coupling constant. In the fermionic sector the $\dot{q}_F$ will describe leptons while the q_F will describe quarks. In terms of these new variables the Lagrangian becomes

$$\frac{S}{\hbar} = \int \frac{d\tau}{\tau_{Pl}}\; Tr\left\{\frac{L_P^2}{L^2}\left[\left(\dot{q}_B^\dagger + i\frac{\alpha}{L}q_B^\dagger\right) + \frac{L_P^2}{L^2}\beta_1\left(\dot{q}_F^\dagger + i\frac{\alpha}{L}q_F^\dagger\right)\right] \times \left[\left(\dot{q}_B + i\frac{\alpha}{L}q_B^\dagger\right) + \frac{L_P^2}{L^2}\beta_2\left(\dot{q}_F + i\frac{\alpha}{L}q_F\right)\right]\right\} \tag{5}$$

Via the 16D split bioctonions $\mathbb{O} \oplus \omega\mathbb{O}$ [4], the two geometric $SU(3)$s provide the scaffolding on which the dynamical matrices $\dot{Q}_B$ and $\dot{Q}_F$ live. The $\dot{q}_B$ live on one 8D half of the 16D split bioctonion and the q_B live on the other 8D half. The same holds for $\dot{q}_F$ and q_F. We now justify these remarks in detail.

Definition 1 (Split bioctonions $\mathbb{O} \oplus \omega\mathbb{O}$ [4] and metric)**.** *Let $\mathbb{C}_s$ denote split-complex numbers with generator $\omega^2 = +1$ and let $\mathbb{O}$ be the octonions. Choose quaternionic subalgebras $\mathbb{H}_L \subset \mathbb{O}$ and $\mathbb{H}_R \subset \mathbb{O}$. Define the 6D real vector space*

$$M_6 := \Im\mathbb{H}_L \oplus \omega\,\Im\mathbb{H}_R, \qquad g(x_L \oplus \omega x_R, y_L \oplus \omega y_R) := \langle x_L, y_L\rangle - \langle x_R, y_R\rangle, \tag{6}$$

with $\langle\cdot,\cdot\rangle$ the Euclidean inner product induced by the octonion norm. The split tag ω implements the sign flip for the right copy, matching the $\omega SU(3)_R^{geom}$ language.

What we are doing here is that we restrict to the two quaternionic subalgebras H_L and H_R in the two parts of the split bioctonion. The imaginary parts x_L and x_R of these two quaternions define, via the inner product, a 6D spacetime of signature $(3,3)$. The gravi-weak interaction lives on this 6D spacetime.

Proposition 1 (Signature and isometries)**.** *(M_6, g) has signature $(3,3)$. The action of unit quaternions by conjugation on each $\Im\mathbb{H}$ yields an isometry action of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, which sits inside the maximal compact $\mathrm{SO}(4)$ of $\mathrm{Spin}(3,3) \cong \mathrm{SL}(4,\mathbb{R})$.*

With the above construction, (M_6, g) indeed has signature $(3,3)$, confirming the count of time vs space dimensions. Moreover, there is an interesting symmetry of this metric: each quaternionic subalgebra H_L or H_R has an $SU(2)$ of unit quaternions (the set of $a \in H$ with $|a| = 1$ is isomorphic to $SU(2)$). These act on $\Im H$ by conjugation (i.e. $q :: x \mapsto qxq^{-1}$ for $x \in \Im H$), which is a rotation of the 3D space of imaginary quaternions. Thus $SU(2)_L$ acts isometrically on $\Im H_L$ and $SU(2)_R$ acts on $\Im H_R$. Together one gets an $SU(2)_L \times SU(2)_R$ isometry group of M_6. In fact, $SO(3,3)$, the full Lorentz group in 6D, has a maximal compact subgroup isomorphic to $SO(4)$, and indeed $SU(2)_L \times SU(2)R \cong Spin(4) \cong SO(4)$ sits inside $SO(3,3)$ as the rotations of the 3+3 split subspaces. This matches our construction: rotations in $\Im H_L$ and $\Im H_R$ separately. These $SU(2)$ symmetries will later be identified with subgroups of the $SU(3)^{\text{geom}}$ structure groups (since $SU(2) \subset SU(3)$ in a standard way). So at this stage, we have built a 6D pseudo-Riemannian manifold M_6 with appropriate symmetry – we think of it as our toy "bulk" spacetime before selecting the physical 4D slices.

Next, we note that there are two overlapping 4D spacetimes, with relatively flipped signature, embedded in this 6D spacetime. After spontaneous symmetry breaking of the gravi-weak interaction, gravitation, and the weak interaction, respectively live on these two 4D spacetimes as their geometry, one each. Because the two spacetimes have a $(1,1)$ overlap, the two directions exclusively linked with the weak interaction appear as internal symmetry directions from the vantage point of our 4D spacetime - this latter being curved by gravitation.

4. Two Embedded 4D Spacetimes

Pick unit vectors $t_L \in \Im\mathbb{H}_L$ and $t_R \in \Im\mathbb{H}_R$. Define

$$\mathcal{M}_4^{(R)} := \Im\mathbb{H}_R \oplus \mathrm{span}\{t_L\} \quad \text{(signature } (3,1)), \tag{7}$$

$$\mathcal{M}_4^{(L)} := \Im\mathbb{H}_L \oplus \mathrm{span}\{\omega t_R\} \quad \text{(signature } (1,3)). \tag{8}$$

These two Lorentzian 4-planes intersect in a neutral $(1,1)$ 2-plane $\mathrm{span}\{t_L, \omega t_R\}$. This is the kinematic version of the two-leaf picture; a dynamical realisation from a 6D BF theory is given in [5].

Detailed explanation:

Now that we have M_6 with (3,3) signature, the next task is to identify two Lorentzian 4-dimensional subspaces inside it. Geometrically, we seek two different 4D "planes" in M_6 that have signature (3,1) and (1, 3) respectively (one time + three space for one, and 1 space + 3 time for the other). The construction is as follows:

Pick a unit imaginary quaternion $t_L \in \Im H_L$. This is a unit 3-vector in the left 3-space, which we designate as a time-like direction for one of the 4D subspaces. Similarly pick a unit imaginary quaternion $t_R \in \Im H_R$. This will serve a similar role for the other subspace. Using these, define two 4D subspaces of M_6:

$M_4^{(R)} := \Im H_R \oplus \text{span}\{t_L\}$. This consists of all vectors of the form $x_R + \alpha, t_L$ with $x_R \in \Im H_R$ and $\alpha \in \mathbb{R}$. By construction, $\Im H_R$ is a 3D space contributing with a negative sign in the metric (from the ω factor), and t_L lies in $\Im H_L$ contributing with a positive sign. Therefore, the metric on $M_4^{(R)}$ has signature $(3,1)$: three negative (from $\Im H_R$) and one positive (from t_L). We interpret $M_4^{(R)}$ as one embedded 4D Lorentzian spacetime (with t_L serving as a time direction for it, since it contributes the lone "+" in the metric on that subspace).

$M_4^{(L)} := \Im H_L \oplus \text{span}\{\omega, t_R\}$. This is all vectors of the form $x_L + \beta, (\omega t_R)$ with $x_L \in \Im H_L$ and $\beta \in \mathbb{R}$. Here $\Im H_L$ part has positive metric, and ωt_R lies in $\omega, \Im H_R$ hence contributes a negative metric component. So $M_4^{(L)}$ has signature $(1,3)$ – three positive (from $\Im H_L$) and one negative (from ωt_R). This is the second embedded 4D Lorentzian spacetime, and in this one t_R (via ωt_R) effectively acts as the time-like direction.

These two subspaces $M_4^{(R)}$ and $M_4^{(L)}$ each are isomorphic to ordinary 4D Minkowski space (at least locally), but note how they are oriented differently in the 6D space: one's time axis lies in H_L direction, the other's in H_R (with an ω factor). Importantly, these two 4D planes are not completely separate – they intersect along a 2-dimensional plane given by $span\{t_L, \omega t_R\}$. This intersection has one basis vector from $\Im H_L$ and one from $\omega \Im H_R$, yielding signature (1,1) (one $+$, one $-$), which is a neutral plane. The presence of this overlap means the two 4D worlds share a common 1+1 dimensional subspace. In physical terms, one might imagine that there is a 2D "bridge" or intersection between our universe and the parallel flipped-signature universe. We call this a "kinematic version of the two-leaf picture" – i.e. we have simply chosen two leaves in the 6D bulk. The fully dynamical story (how fields and gravity localize on these leaves) is deferred to a gravi-weak theory in 6D [5] but at least kinematically we see how two 4D spacetimes can coexist and overlap in a 6D (3,3) spacetime.

4.1. Lorentz Covariance of the Embedded 4D Leaves

We now justify rigorously as to how 4D Lorentz invariance is preserved after selection of the 4D leaves. Let (M_6, g) have signature $(3,3)$. Fix an oriented negative 2–plane $N \subset T_x M_6$ and set its orthogonal complement

$$W := N^{\perp} \subset T_x M_6, \qquad \text{so} \quad g|_W \text{ has signature } (3,1).$$

Write $\eta = \text{diag}(+,+,+,-,-,-)$ for a local orthonormal frame on $T_x M_6$ and collect an orthonormal basis of N in the 6×2 matrix $U = [\,n_1 \;\; n_2\,]$ with $U^T \eta\, U = -\mathbf{1}_2$.

Stabilizer and Lie algebra split.

The stabilizer of N in $SO(3,3)$ is

$$H \;:=\; \{\, g \in SO(3,3) \mid gN = N \,\} \;\cong\; SO(3,1) \times SO(2),$$

which acts as $SO(3,1)$ on W and as $SO(2)$ on N. At the Lie algebra level,

$$\mathfrak{so}(3,3) \;=\; \underbrace{\mathfrak{so}(W)}_{6} \,\oplus\, \underbrace{\mathfrak{so}(N)}_{1} \,\oplus\, \underbrace{(W \wedge N)}_{8}. \tag{9}$$

The 8 generators in $W \wedge N$ are precisely the transformations that *mix* the 4D axes in W with the two discarded axes in N.

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Kinematic (projector) construction.

Define the metric projector onto W by

$$\Pi := \mathbf{1}_6 + U U^T \eta . \tag{10}$$

Then $\Pi^2 = \Pi$, $\mathrm{Im}\,\Pi = W$, and $\mathrm{Ker}\,\Pi = \eta N$. For any $g \in H$, one has $g\Pi g^{-1} = \Pi$, so the residual local symmetry on W is $SO(3,1)$. Impose the *leaf constraint* by projecting all tensors/frames:

$$V_A \mapsto (\Pi V)_A, \qquad T_{AB} \mapsto (\Pi T \Pi)_{AB}, \quad \text{etc.}$$

Within this background, the mixed generators in $W \wedge N$ (which would rotate W into N) are *broken*: they do not preserve Π and so are not symmetries of the leaf. Hence fields restricted by (10) transform Lorentz–covariantly under $SO(3,1)$ on the leaf.

Dynamical (Lorentz–Higgs) construction.

Promote U to a field $U(x)$ with the constraint $U^T \eta\, U = -\mathbf{1}_2$ and couple it to the 6D spin connection ω_μ:

$$\mathcal{L}_U = \frac{\alpha}{2}\,\mathrm{Tr}\big[(D_\mu U)^T \eta (D^\mu U)\big] - V(U), \qquad D_\mu U := \partial_\mu U + \omega_\mu U. \tag{11}$$

A vacuum $\langle U \rangle$ picks N and spontaneously breaks $SO(3,3) \to SO(3,1) \times SO(2)$. The eight mixed connections $\omega_\mu^{a\hat{b}} \in W \wedge N$ (here $a = 0,\ldots,3$ along W, $\hat{b} = 1,2$ along N) eat the eight Goldstones and become massive. The normal rotation $\omega_\mu^{\hat{1}\hat{2}} \in \mathfrak{so}(N)$ is either left as a spectator $SO(2)$ or can be fixed/decoupled. At low energy, the residual local symmetry on the leaf is exactly $SO(3,1)$.

Goldstone count.

The spontaneous breaking $SO(3,3) \to SO(3,1) \times SO(2)$ yields

$$\dim SO(3,3) - \big(\dim SO(3,1) + \dim SO(2)\big) = 15 - (6+1) = 8$$

Goldstone modes. These are eaten by the eight mixed spin–connections $\omega_\mu^{a\hat{b}} \in W \wedge N$ (unitary gauge), leaving the $SO(3,1)$ leaf connection massless and the normal $SO(2)$ as a spectator or fixed.

Localization in the action.

To confine dynamics to the leaf, either insert projectors Π on all tensor indices (e.g. $g_4 = \Pi g\, \Pi$, $R_4 = \Pi R\, \Pi$), or wedge sector actions with a 2–form density ρ supported on the leaf so that $\int_{M_6} (\cdot) \wedge \rho = \int_{\Sigma_4} (\cdot)$. Off–plane components drop out; the surviving symmetry is $SO(3,1)$.

Two leaves.

Choose two negative 2–planes N_L, N_R with projectors Π_L, Π_R. Each leaf $\Sigma_X := \mathrm{Im}\,\Pi_X$ ($X = L, R$) carries its own residual $SO(3,1)$. Their intersection $\Sigma_L \cap \Sigma_R = \mathrm{span}\{t_L, \omega t_R\}$ is 2–dimensional with signature $(1,1)$. Localizing the left/right sector actions with ρ_L, ρ_R yields two independent 4D Lorentz symmetries, one on each leaf.

Why there is no mixing. The only transformations that would mix the chosen time/spatial axes in W with the two discarded directions in N are the broken coset generators $W \wedge N$ in (9). Once N is fixed (kinematically by Π or dynamically by $\langle U \rangle$), these do not act on physical fields on the leaf. Within the unbroken subgroup, $t_1 \in W$ mixes only with the three spatial directions of W (ordinary Lorentz boosts and rotations).

Stabilizer and mixing generators.

Fix a negative 2–plane $N \subset (T_x M_6, g)$ and set $W := N^{\perp}$, so $g|_W$ has signature $(3,1)$. The stabilizer of N in $SO(3,3)$ is

$$\mathrm{Stab}_{SO(3,3)}(N) \;\cong\; SO(3,1) \times SO(2),$$

which acts as $SO(3,1)$ on W and as $SO(2)$ on N. At the Lie–algebra level,

$$\mathfrak{so}(3,3) \;=\; \underbrace{\mathfrak{so}(W)}_{6} \,\oplus\, \underbrace{\mathfrak{so}(N)}_{1} \,\oplus\, \underbrace{(W \wedge N)}_{8}.$$

The eight coset generators $W \wedge N$ are precisely the transformations that *would* mix the leaf directions (W) with the two discarded directions (N). Once N is fixed (kinematically by a projector or dynamically by a vacuum), these generators are broken. The unbroken local symmetry on the leaf is exactly $SO(3,1)$, which justifies 4D Lorentz covariance on each embedded leaf.

To summarize this part: one 4D spacetime (Σ_R) uses the entire right-imaginary quaternion space plus one left-imaginary direction for time; the other 4D spacetime (Σ_L) uses the left-imaginary quaternion space plus one right-imaginary (with ω) direction for time. One can think of Σ_R as primarily "right-handed" (since it uses $\Im H_R$ spatially) and Σ_L as "left-handed" (uses $\Im H_L$ spatially), hinting at a connection to handedness of weak interactions. Indeed Σ_L might correspond to our universe where $SU(2)_L$ acts (left-handed weak force on left-chiral particles), while Σ_R is a hidden sector where $SU(2)_R$ acts (and its breaking gave gravity). We will see later that Σ_R is where unbroken QCD lives (with possibly gravity), and Σ_L might be a hidden mirror or vice versa. The (1,1) intersection could allow some communication or matching conditions between the two sectors. This elegant geometric picture emerges purely from the octonionic and ω-complex structure we imposed.

5. The Two Extra $\mathrm{SU}(3)$'s and Their Branching

On each side of $E_8 \times \omega E_8$ one has the maximal chain

$$E_8 \;\supset\; E_6 \times \mathrm{SU}(3), \qquad \mathbf{248} = (\mathbf{78},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus (\mathbf{27},\mathbf{3}) \oplus (\overline{\mathbf{27}},\overline{\mathbf{3}}). \tag{12}$$

We use the two extra $\mathrm{SU}(3)$'s as *geometry*. Pick the standard embedding $\mathrm{SU}(2) \subset \mathrm{SU}(3)$ and take the complementary $U(1)$ generator proportional to $\mathrm{diag}(1,1,-2)$. With the convenient normalisation where the doublet has unit charge, the adjoint branches as

$$\mathbf{8} \;\longrightarrow\; \mathbf{3}_0 \;\oplus\; \mathbf{2}_{+1} \;\oplus\; \bar{\mathbf{2}}_{-1} \;\oplus\; \mathbf{1}_0. \tag{13}$$

The $\mathbf{3}_0$ from $SU(3)_L^{geom}$ will supply the three directions of $\Im\mathbb{H}_L$ and the $\mathbf{3}_0$ from $SU(3)_R^{geom}$ will supply the three directions of $\Im\mathbb{H}_R$; together forming M_6. The $\mathbf{2} \oplus \bar{\mathbf{2}}$ will furnish a *real* 4D internal fibre, one each from the two $SU(3)^{geom}$.

Detailed explanation:

Having built the base manifold, we now connect it to the group theory origin: the extra $SU(3)$ factors that we set aside as "geometric". Recall that each E_8 breaks as $E_6 \times SU(3)$, so there is an $SU(3)_L^{\mathrm{geom}}$ from E_{8L} and an $SU(3)_R^{\mathrm{geom}}$ from E_{8R}. We do not gauge these $SU(3)^{\mathrm{geom}}$'s as physical forces; instead, we use them to understand how the 6D base and internal fibres emerge. This involves examining how an $SU(3)$ can act on the structures we built and how its representations decompose. First, note the well-known branching of the E_8 adjoint (248-dim) under $E_6 \times SU(3)$: $\mathbf{248} = (\mathbf{78},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus (\mathbf{27},\mathbf{3}) \oplus (\overline{\mathbf{27}},\overline{\mathbf{3}})$

Now, we "use the two extra $SU(3)$'s as geometry". Concretely, consider one of these $SU(3)$ groups (the discussion applies to either left or right). We pick the standard maximal subgroup $SU(2) \times U(1)$ inside $SU(3)$. This is done by choosing an $SU(2)$ that sits in $SU(3)$ (for example, the upper-left 2×2 submatrix inside 3×3 matrices), and a $U(1)$ generator that complements it – usually taken proportional to $\mathrm{diag}(1,1,-2)$ in 3×3 space. This particular $U(1)$ assignment (diag(1,1,-2)) gives the fundamental

3 of $SU(3)$ charges $(\frac{1}{2}, \frac{1}{2}, -1)$ which we normalize so that a fundamental doublet has charge +1. With this normalization, the adjoint of $SU(3)$ (dimension 8) decomposes under $SU(2) \times U(1)$ as: $\mathbf{8} \ \to \ \mathbf{3}_0 \oplus \mathbf{2}_{+1} \oplus \mathbf{2}_{-1} \oplus \mathbf{1}_0$. This is a crucial decomposition, explicitly given above. Here:

$\mathbf{3}_0$ is the adjoint of $SU(2)$ (a triplet with zero $U(1)$ charge).

$\mathbf{2}_{+1}$ is a doublet of $SU(2)$ with charge $+1$.

$\mathbf{2}_{-1}$ is the $SU(2)$ doublet with opposite charge -1 (effectively the complex conjugate representation of the above).

$\mathbf{1}_0$ is a singlet with zero charge (this is the $U(1)$ generator itself).

Now, the insight is to map these group representation pieces to geometric components. On the left geometric $SU(3)_L^{\text{geom}}$, the $\mathbf{3}_0$ piece will be identified with the 3 real dimensions of $\Im H_L$ (the spatial directions of the left part of M_6), and on the right $SU(3)_R^{\text{geom}}$, its $\mathbf{3}_0$ corresponds to $\Im H_R$. In other words, the $SU(2)$ adjoint triplet inside $SU(3)^{\text{geom}}$ is essentially the three rotational degrees of freedom of one of the quaternionic subspaces. This makes sense because we earlier saw $\Im H \cong \mathbf{3}$ of $SU(2)$, and indeed Proposition 2 below will formalize $\Im H \cong \mathbf{3}_0$ as representations.

Next, the $\mathbf{2}_{+1} \oplus \mathbf{2}_{-1}$ (together forming a complex 2-dimensional rep or 4-dimensional real rep) is interpreted as the 4 internal dimensions at each point – essentially the tangent space of CP^2 fibre. Specifically, one $\mathbf{2} \oplus \mathbf{2}$ from $SU(3)_L^{\text{geom}}$ will correspond to a 4D internal fibre on the left side, and similarly for the right side. Thus, each $SU(3)^{\text{geom}}$ yields a 3-dimensional "external" piece ($\Im H$) and a 4-dimensional "internal" piece ($\mathbf{2} \oplus \mathbf{2}$ real), summing to 7 dimensions; plus a singlet. The 7 corresponds to $\Im O$, but we'll focus on the 3+4 split.

What about the remaining $\mathbf{1}_0$ in (13)? That is the $U(1)$ generator itself. In group terms it's a central generator of the $SU(2) \times U(1)$ subgroup. Geometrically, one might wonder if this corresponds to the real line in the octonions (since O is 8-dimensional: 1 real + 7 imaginary parts). However, we emphasize that we do not identify the octonion's real unit 1 with this $U(1)$ Instead, the $U(1)$ here is seen as acting on the complex structure of the 4-dimensional fibre (essentially rotations in the CP^2 tangent – more on this in the next sections). So $\mathbf{1}_0$ is kept distinct as a geometric $U(1)$ gauge field needed for the $Spin^c$ structure, and not as a physical scalar.

6. Octonionic Realisation: $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\varepsilon$ and the 4D Fibre

Fix a quaternionic subalgebra $\mathbb{H} = \langle 1, u, v, uv \rangle \subset \mathbb{O}$ and choose $\varepsilon \in \Im\mathbb{O}$ orthogonal to $\mathbb{H}$ with $\varepsilon^2 = -1$. Then

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\,\varepsilon, \qquad \Im\mathbb{O} = \Im\mathbb{H} \oplus \mathbb{H}\varepsilon. \tag{14}$$

Define on the real 4-space $\mathbb{H}\varepsilon$ the complex structure J by left multiplication with a fixed unit $u \in \Im\mathbb{H}$:

$$J(a\,\varepsilon) := (u\,a)\,\varepsilon, \qquad J^2 = -\mathrm{Id}_{\mathbb{H}\varepsilon}. \tag{15}$$

Let SU(2) act on $\mathbb{H}$ by unit-quaternion conjugation and trivially on ε; let $U(1)$ act as phases $e^{\theta J}$ on $(\mathbb{H}\varepsilon, J)$. Then:

Proposition 2 (Identifications)**.** *(a)* $\Im\mathbb{H} \cong \mathrm{Adj}\,\mathrm{SU}(2) \cong \mathbf{3}_0$ *as real representations.*

(b) $(\mathbb{H}\varepsilon, J) \cong \mathbf{2}_{+1}$ *as a complex* $\mathrm{SU}(2) \times U(1)$ *representation. Forgetting the complex structure, the underlying real representation is*

$$\mathbb{H}\varepsilon \ \cong \ \big(\mathbf{2}_{+1} \oplus \overline{\mathbf{2}}_{-1}\big)_{\mathbb{R}}, \tag{16}$$

a real 4*-vector space. This is the internal fibre* F_4.

Thus the octonionic split realises the branching (13) concretely: $\Im\mathbb{H}$ gives the 3's for spacetime directions; $\mathbb{H}\varepsilon$ gives the 4 internal directions.

Detailed explanation: The octonions O come back into play now to realize the above abstract decomposition concretely. We use the fact that octonions contain many quaternionic subalgebras and can be split by a suitable choice of a new imaginary unit. We choose a specific decomposition: $O = H \ \oplus \ H\epsilon$

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with H a fixed quaternionic subalgebra of O and ϵ an octonionic element orthogonal to H that behaves like a new imaginary unit (satisfying $\epsilon^2 = -1$). Here $H\epsilon := q\epsilon \mid q \in H$ is isomorphic (as a real vector space) to H itself, but consisting of "quaternions times ϵ". This particular split, sometimes called the "split-octonion" decomposition, yields: $\Im O = \Im H \ \oplus \ H\epsilon$ Since $\Im H$ is 3-dimensional and $H\epsilon$ is 4-dimensional (because H is 4-dim real, and we are not including the H's real unit in $\Im O$), we recover $\dim \Im O = 3 + 4 = 7$. This matches the decomposition $8 \to 3 + 4 + 1$ we saw (with the remaining 1 corresponding to the real line spanned by the identity in O). Essentially, the octonion's imaginary part splits into a 3D part and a 4D part, exactly what we want for spacetime vs internal fibre.

Introducing a complex structure on $H\epsilon$: To make that 4-dimensional space $H\epsilon$ look like a $\mathbf{2}_{+1} \oplus \mathbf{2}_{-1}$ representation, we need to identify it with a complex 2-dimensional vector space carrying an $SU(2) \times U(1)$ action. We accomplish this by defining a complex structure J on $H\epsilon$ as follows: pick a specific unit imaginary quaternion $u \in \Im H$ (one of the three quaternionic basis elements, say u akin to i). For any element $a\epsilon \in H\epsilon$ (with $a \in H$), define $J(a\epsilon) := (u \cdot a)\,\epsilon$ Since u is an imaginary quaternion with $u^2 = -1$, one can check that $J^2(a\epsilon) = (u(ua))\epsilon = (u^2 a)\epsilon = -a\epsilon$, so indeed $J^2 = -\mathrm{Id}$ on $H\epsilon$. This means $H\epsilon$ is now a complex vector space of complex dimension 2 (real dimension 4), with J playing the role of multiplication by i. We denote this complex vector space as $(H\epsilon, J)$.

Next, let's understand the group action. We have: the group $SU(2)$ (the unit quaternions in H) acts on H by conjugation and trivially on ϵ (meaning $q : a\epsilon \mapsto (qaq^{-1})\epsilon$). Conjugation by a unit quaternion q rotates $\Im H$, and also rotates a in H for the $a\epsilon$ part.

A $U(1)$ group is introduced to correspond to phase rotations on the complex structure J. Specifically, let $e^{\theta J}$ denote the linear map on $H\epsilon$ that rotates by angle θ in the J-complex sense (i.e. it sends $v \mapsto \cos\theta, v + \sin\theta, Jv$). Because J acts like i, $e^{\theta J}$ is essentially multiplying by a phase $e^{i\theta}$ on the complex vector $(H\epsilon, J)$. We identify this action with the $U(1)$ generator we had from $SU(3) \to SU(2) \times U(1)$. In other words, this $U(1)$ acts as $e^{i\theta}$ on the 2-dimensional complex space $H\epsilon$.

Now we arrive at Proposition 2 (Identifications): Given the above setup:

(a) $\Im H \ \cong \ \mathbf{Adj}, SU(2) \ \cong \ \mathbf{3}_0$ as real representations. This simply restates that the imaginary quaternions form the adjoint (3-dim) representation of $SU(2)$, with zero $U(1)$ charge.

(b) $(H\epsilon, J) \cong \mathbf{2}_{+1}$ as a complex representation of $SU(2) \times U(1)$. This means that the 4-dimensional real space $H\epsilon$ is, when viewed as a complex vector space, the fundamental doublet of $SU(2)$ carrying charge +1 under the $U(1)$. Concretely, one can choose a basis of $H\epsilon$ such that $SU(2)$ acts by the 2-dimensional spin-$\frac{1}{2}$ representation and $U(1)$ multiplies vectors by a phase $e^{i\theta}$ (charge +1).

Forgetting the complex structure (i.e. as a real space), $H\epsilon$ then corresponds to $\mathbf{2}_{+1} \oplus \mathbf{2}_{-1}$. Why both? Because a complex 2D rep is real 4D and contains the vector and its complex conjugate. In other words, if $(H\epsilon, J)$ is $\mathbf{2}_{+1}$, the same real space can also be seen as $\mathbf{2}_{-1}$ by using $-J$ as the complex structure; effectively J provides an orientation for $U(1)$ charge. So $H\epsilon$ as a real rep carries a doublet of charge +1 and an equivalent doublet of charge -1 – exactly the pair $\mathbf{2}_{+1} \oplus \mathbf{2}_{-1}$ we found in the $SU(3)$ adjoint decomposition.

Thus, the octonionic split $O = H \oplus H\epsilon$ realizes the abstract decomposition (13) in a very explicit way:

$\Im H$ gives the $\mathbf{3}_0$ directions (spacetime 3-axes for each $SU(3)^{\text{geom}}$),

$H\epsilon$ gives the $\mathbf{2} \oplus \mathbf{2}$ internal directions (the 4D fibre).

In particular, for each side $X = L, R$, we identify: $F_4^X := H_X \epsilon_X \ \cong \ \mathbf{2}_{+1} \oplus \mathbf{2}_{-1}$ (real 4d) which is the 4D internal fibre at each point. We will shortly see this is isomorphic to the tangent space of CP^2.

It's worth giving a more tangible example of the above: Take $H = \mathbb{H}$ with basis $1, i, j, k$ (quaternions), and let's choose ϵ to be one of the octonion units outside this H (octonions have units $e_1, \ldots, e_7$; suppose $H = \mathrm{Span}1, e_1, e_2, e_3$, then pick $\epsilon = e_4$, which anticommutes with e_1, e_2, e_3 and squares to -1). Then $H\epsilon = \mathrm{Span}e_4, e_5, e_6, e_7$ which is 4D. Define J as left-multiplication by $u = e_1$ on $H\epsilon$. Then one can verify that under $SU(2) = e^{\alpha e_1 + \beta e_2 + \gamma e_3}$ and $U(1) = e^{\theta J}$, $H\epsilon$ transforms as a charged doublet. This matches the algebraic relationships inside octonions (a known fact: G_2, the automorphism group of

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octonions, has an $SU(3)$ subgroup that precisely preserves a chosen split like this, acting as rotations on a S^6 etc.).

In summary, through the lens of the exceptional algebra O, we have found a canonical isomorphism between:

$$(\mathbf{3}_0 \oplus \mathbf{2}_{+1} \oplus \mathbf{2}_{-1})_{\mathbf{R}} \leftrightarrow \Im H \oplus H\epsilon = \Im O \tag{17}$$

matching the representation content of $SU(3)_{\text{geom}}$ to subspaces of O. This cements the idea that the extra $SU(3)$ naturally splits into a 3D space (for M_6 base) and a 4D internal fibre. It also justifies why, earlier, we could assign the "3 directions of $\Im H_L$" to $\mathbf{3}_0$ and "4 directions of $H\epsilon$" to the doublets. The octonions essentially provide the coordinate system for this split.

6.1. Relation to $\mathbb{C}P^2$ and Kaluza–Klein Intuition

At a point $[n] \in \mathbb{C}P^2 = \mathrm{SU}(3)/S(U(2) \times U(1))$ one has

$$T_{[n]}\mathbb{C}P^2 \cong \mathrm{Hom}(\mathbb{C}n,\ \mathbb{C}^3/\mathbb{C}n) \cong \mathbb{C}^2\ \cong\ \mathbf{2}_{+1}, \tag{18}$$

so the *real* tangent is 4-dimensional. The octonionic identification above gives a canonical isomorphism

$$F_4\ \simeq\ T\mathbb{C}P^2 \quad \text{(real rank 4 at each point).} \tag{19}$$

This matches the minimal Kaluza–Klein choice for an SU(3) internal, explaining why four internal real directions are "right" for a QCD-like sector at the level of geometry.

Detailed explanation: Above, we clarify the significance of the 4D internal fibre by relating it to a well-known geometric space: CP^2, the complex projective plane. CP^2 is a compact 4-real-dimensional manifold (complex dimension 2) often studied in grand unification and Kaluza–Klein models as a candidate internal space (notably, CP^2 appears in certain $SU(3)$ Kaluza–Klein coset models for unified interactions). By definition,

$$CP^2 = SU(3)/S(U(2) \times U(1)), \tag{20}$$

which means at each point $[n] \in CP^2$ (think of n as a 1-dimensional complex subspace in $\mathbb{C}^3$), the tangent space can be identified with the space of homomorphisms from that line Cn to the orthogonal complement C^3/Cn. In formula:

$$T_{[n]}CP^2 \simeq \mathrm{Hom}_{\mathbb{C}}(Cn, C^3/Cn) \tag{21}$$

Since Cn is 1-dimensional and C^3/Cn is 2-dimensional over $\mathbb{C}$, $\mathrm{Hom}(Cn, C^3/Cn)$ is isomorphic to C^2. Therefore complex-dimension(TCP^2) = 2, or real-dimension = 4. In fact one can say:

$$T_{[n]}CP^2 \simeq \mathbb{C}^2 \simeq \mathbf{2}_{+1} \tag{22}$$

as a representation of the stabilizer $S(U(2) \times U(1))$ (where the $U(1)$ acts with charge +1 on that $\mathbb{C}^2$). This is exactly the same structure we have for our internal fibre! The $\mathbf{2}_{+1}$ in our model corresponds to the complex tangent at a point of CP^2 (and $\mathbf{2}_{-1}$ would be the opposite charge because the $U(1)$ in $S(U(2) \times U(1))$ would also have a -1 action on the conjugate). Thus the 4D real internal space we obtained can be viewed as the tangent space of CP^2.

In fact, we assert a canonical isomorphism $F_4 \cong TCP^2$ (as real 4-spaces). The octonionic model gives a specific identification, not just an abstract isomorphism, thereby providing a concrete model of CP^2's tangent bundle inside E_8's structure. The phrase "realification of $2 \oplus \bar{2}$" earlier also alluded to this CP^2 tangent (since $2 \oplus \bar{2}$ is the realified form of the complex 2). The upshot is: each point in the 6D base M_6 can be thought of as carrying an internal fibre isomorphic to CP^2's tangent space. If one were to imagine a Kaluza–Klein scenario, CP^2 might be the internal manifold – but here, the internal space is not global CP^2 per se, it's an oriented plane (tangent) at each base point. In other words, we have a fibre bundle with fibre TCP^2. The minimal dimensionality (4 real dims) of this fibre is pleasing: it's exactly what we need for embedding an $SU(3)$ (color-like) gauge sector if we follow Kaluza–Klein arguments. We note that having four internal dimensions is the "right" number for a QCD-like sector at the level of geometry – referencing that an $SU(3)$ gauge theory in 4D could emerge from a CP^2 compactification (since CP^2 has isometry $SU(3)$ and requires a $Spin^c$ twist to admit fermions, which is exactly what we are setting up with the $U(1)$ lines).

Summarizing: The internal 4D space provided by the split-bioctonion construction is identified with the tangent space of the coset $SU(3)/S(U(2) \times U(1))$, i.e. CP^2. This not only validates the choice of 4 internal dimensions but also situates our model in the context of known geometry (where CP^2 often appears in grand unification). It's a nice consistency check and provides intuition: just as Kaluza–Klein theory might use $M_4 \times CP^2$ as spacetime, here we have M_6 base and an internal fibre that behaves like CP^2 at each point. The difference is that CP^2 itself is 4D and non-trivial (non-spin), but as a fibre attached to each point of M_6, it's more like an internal degrees-of-freedom space rather than additional global dimensions.

7. The Two $U(1)$'s

The $U(1)$ in (13) is the isotropy phase in $S(U(2) \times U(1))$. On $(\mathbb{H}\varepsilon, J)$ it acts by $e^{\theta J}$. For each extra SU(3) we therefore obtain a natural *line bundle* whose connection is the Spin^c twist needed on $\mathbb{C}P^2$ (which is non-spin). We *do not* identify the octonion real line $\mathbb{R} \cdot 1 \subset \mathbb{O}$ with this $U(1)$; rather, the $U(1)$ from $SU(3)^{geom}_{L,R} \to SU(2) \times U(1)$ is a Lie-algebra direction acting on the $\mathbb{C}^2$ tangent. It is the Spin^c line on $T\mathbb{C}P^2$. In model-building one may consider mixing these geometric $U(1)$'s with abelian factors inside E_6, but the Spin^c role is canonical and model-independent.

Detailed explanation: Each $SU(3)_{\text{geom}}$ gave us not only an $SU(2)$ (which we used for rotations in $\Im H$) but also a $U(1)$ (the extra $U(1)$ generator in the decomposition). These $U(1)$'s provide the necessary $Spin^c$ structure on the CP^2 fibres. Why $Spin^c$? It is a known topological fact that CP^2 is not a spin manifold (its second Stiefel–Whitney class w_2 is nonzero, preventing a spin structure). However, CP^2 is a $Spin^c$ manifold, meaning it can admit spinors if supplemented with a $U(1)$ gauge field (a complex line bundle) whose field strength compensates for the obstruction.

In practical terms, to have fermions live on CP^2 or its tangent, one needs a $U(1)$ connection – often called a $Spin^c$ line bundle – twisting the spin structure. Here, the $U(1)$ we got from $SU(3) \to SU(2) \times U(1)$ does exactly that: it rotates the tangent C^2 fibre by a phase and thus corresponds to the natural $U(1)$ in the stabilizer $S(U(2) \times U(1))$ of a point in CP^2. In other words, this $U(1)$ is exactly the isotropy $U(1)$ that appears in the coset $SU(3)/S(U(2) \times U(1))$, whose connection can be seen as the $Spin^c$ connection on CP^2.

For each extra $SU(3)$, we obtain a natural complex line bundle over M_6 whose connection is the $Spin^c$ twist needed on CP^2. We should not confuse this $U(1)$ with the real scalar in octonions (the $R \cdot 1$). It's not some extra dimension; it is literally the $U(1)$ subgroup of $SU(3)^{\text{geom}}$ that acts as phase rotations on the C^2 fibre. So, each $SU(3)^{\text{geom}}$ yields a principal $U(1)$-bundle over M_6 (the fibration of CP^2 tangents) whose curvature is what's required to define spinor fields on those fibres. In effect, we have built a consistent $SU(3)^{\text{geom}}$-structured fibre bundle $F_4 \to M_6$ which is a $Spin^c$ bundle (not spin, but $Spin^c$).

From a model-building perspective, these geometric $U(1)$'s could potentially mix with other $U(1)$ factors from the physical gauge groups (for instance, hypercharge or a $B - L$ symmetry in E_6). But such mixing would be an additional consideration; intrinsically, their role is fixed: they ensure the fermions on the internal fibre have the right twist to exist.We emphasize that this $Spin^c$ role is canonical and model-independent – any theory with CP^2 fibre would need such a $U(1)$, so it's not an arbitrary choice but a topological necessity satisfied neatly by the $SU(3)^{\text{geom}}$ splitting.

In summary, the presence of those $U(1)$ factors is not an extra complication but in fact a crucial feature allowing spinors (matter fields) to propagate on the internal space. Each geometric $U(1)$ becomes a kind of background field (a part of the geometry) rather than a new gauge force to be

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identified with, say, the Standard Model $U(1)_Y$. We deliberately say "we do not identify it with the octonion real line" to avoid the misconception that it's a trivial scalar; it's part of the Lie algebra direction acting on the C^2 tangent. This $U(1)$ connection can be thought of as the field that, if we were to compactify on CP^2, would be needed to satisfy the Dirac equation on that space. Thus, the unified framework naturally includes the gravitationally necessary $U(1)$ fibres.

Spin^c on $\mathbb{CP}^2$.

One has $w_2(T\mathbb{CP}^2) \neq 0$, so $\mathbb{CP}^2$ is not spin, but it is Spin^c: there exists a determinant line $L \to \mathbb{CP}^2$ with $c_1(L) \equiv w_2 \pmod 2$. The geometric $U(1)$ factor in $SU(3)_{\text{geom}} \to SU(2) \times U(1)$ furnishes precisely the Spin^c connection on the $\mathbb{CP}^2$ fibres. Hence the two geometric $U(1)$'s are not optional decorations; they implement the canonical Spin^c twists needed for fermions on $F_4 \simeq T\mathbb{CP}^2$.

8. How the $E_6^L \times E_6^R$ Fields Sit on the Scaffold

We treat the two extra SU(3)'s as *structure* only. Over M_6 take principal bundles for E_6^L and E_6^R; matter sits in associated **27** vector bundles and gauge fields in adjoint bundles. Tangent/internal decomposition uses $TM_6 = (\Im\mathbb{H}_L) \oplus (\Im\mathbb{H}_R)$ and the two fibres $F_4^{L,R} \simeq \mathbb{H}_{L,R}\varepsilon_{L,R}$ from above. The visible interactions come from $E_6^{L,R}$; the extra SU(3)'s supply geometry and Spin^c lines on the internal fibres.

Detailed explanation: With the geometric "scaffold" – consisting of the 6D base M_6, two 4D CP^2-like fibres, and $U(1)$ $Spin^c$ connections – now established, we turn to how the physical fields (fermions, gauge bosons) reside in this setup. Essentially, the two E_6 groups (one on the left, one on the right) will provide the usual matter content and forces, but now they propagate in a restricted manner on M_6.

We propose to treat the $SU(3)_{\text{geom}}$ factors as pure structure, not gauge fields. That means we do not include dynamical gauge bosons for them in the action; instead, we only keep the metric and connection associated with the M_6 and the $Spin^c$ line. Meanwhile, we take principal bundles for E_6^L and E_6^R over M_6. In other words, imagine on the 6D base we have two sets of gauge fields: one with group E_6^L and one with E_6^R. These are genuine gauge fields (with their field strengths, etc.) but their presence is tied to the geometry. Matter fields (like fermions) are then sections of associated vector bundles – specifically, in the 27-dimensional representation of E_6 (since each generation of fermions can fit in an E_6 **27**, in many E_6 GUT models). Gauge bosons are connection fields in the adjoint (78) bundles.

Now, how do these 6D fields give us effective 4D physics? The concept introduced is that of a tangent/internal decomposition of everything, leveraging the splitting:

$$TM_6 = \Im H_L \oplus \Im H_R \tag{23}$$

and

$$F_4^L \simeq H_L\epsilon_R, \quad F_4^R \simeq H_R\epsilon_R \tag{24}$$

This splitting means that at each point of M_6, one can distinguish the directions along M_6 (six of them, the "spacetime directions") and the directions along the internal fibres (four on the left fibre and four on the right fibre). So any field can be classified by how it transforms under rotations of these subspaces. Concretely:

- The gauge fields of E_6^L and E_6^R will have components that could a priori point along M_6 or along the fibre. But if M_6 is eventually effectively 4D (due to localization on leaves), the physically observed gauge fields will be those components tangential to the 4D leaves. The components along the internal fibre might manifest as Higgs fields or heavy modes (similar to Kaluza–Klein modes).
- The fermionic fields living in 27s can be decomposed likewise. For instance, an E_6 27 contains Standard Model fermions; those would be functions on M_6 that also take values in an internal spinor representation on the fibre. The presence of the $Spin^c$ structure ensures we can define these spinor fields properly on M_6 with internal CP^2 fibres.

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We note that the visible (low-energy) interactions come from $E_6^{L,R}$ only. That is, E_6 gauge fields include the Standard Model and possibly additional Z' or other exotics, and those are what mediate forces in 4D. The $SU(3)^{\text{geom}}$ do not add new forces; instead, they "supply geometry and $Spin^c$ lines" as we discussed. This separation of roles is crucial to ensure no unwanted gauge fields from $SU(3)^{\text{geom}}$ clutter the low-energy spectrum.

So at this point, the picture is:

• We have a fibre bundle structure: base M_6 (with two embedded 4D sub-manifolds Σ_L, Σ_R) and internal fibres that are 4D (tangent CP^2 spaces).

• Over this entire structure, we have $E_6^L \times E_6^R$ gauge fields. They are free to propagate in the six base directions, but ultimately they will be confined to the 4D subspaces due to a mechanism in the next section (localization).

• Matter fields (quarks, leptons, etc.) live in these bundles – presumably, a chiral projection will leave them effectively on one of the 4D leaves (e.g. left-chiral matter on Σ_R and maybe some mirror on Σ_L).

It's worth noting: since E_6 contains the Standard Model gauge group, once we break E_6 at some high scale, we would get $SU(3)_c, SU(2)_L, U(1)_Y$ (and perhaps additional stuff like $U(1)_{B-L}$, etc.). The trinification breakdown mentioned in the introduction will occur inside these E_6 groups. That means on Σ_L leaf, we'll see something like $SU(3)_c \times SU(2)_L \times U(1)_Y$ as active gauge symmetries (the Standard Model), and on Σ_R we might see $SU(3)_{c'}$ and $SU(2)_R$ etc., depending on how the symmetry breaking is arranged (we propose $SU(2)_R$ breaks giving gravity, so perhaps $SU(3)_R$ breaks at high scale on the right side). We'll clarify this in the Big Picture section.

In essence, this section doesn't introduce new formulas but sets the stage: we embed the "theory of everything" ($E_6 \times E_6$ gauge fields + matter) into the hybrid space constructed by $SU(3)^{\text{geom}}$. The geometrical $SU(3)$'s determine how spacetime and internal spaces are glued together, while the E_6's bring in the particle content. This is somewhat analogous to how in string theory one has an internal manifold and gauge fields on it; here the role of "internal manifold" is played by the CP^2 fibre and "spacetime manifold" is the 6D M_6. The difference is we have two overlapping 4D spacetimes rather than one global 6D-to-4D compactification.

Fermion localisation and chirality (outline).

Let Γ^A be 6D gamma matrices for $\mathrm{Cl}(3,3)$ and let Π denote the projector onto a chosen leaf W (Sec. 4). A minimal localisation ansatz uses a domain–wall mass profile $m(\sigma)$ depending on signed distance σ to the leaf and the Spin^c connection on $F_4 \simeq T\mathbb{CP}^2$:

$$\not{D}_6\psi \;:=\; i\Gamma^A e_A^{\ \mu} D_\mu\psi \;+\; i\Gamma^{\hat{a}} n_{\hat{a}}^{\ M}\partial_M\psi \;+\; m(\sigma)\,\psi \;=\; 0, \qquad \psi \;=\; \Pi\psi.$$

Standard domain–wall arguments (Jackiw–Rebbi type) produce normalisable 4D zero–modes of definite leaf chirality; the opposite chirality localises on the other leaf or is lifted by boundary conditions. The Spin^c twist on F_4 provides the internal index needed to obtain the desired family multiplicities. A full zero–mode and index analysis will appear elsewhere.

9. Big Picture and Interpretation

Three layers.

1. **Geometric $\mathbf{SU(3)}^{\text{geom}}_{L,R}$** from $E_8 \supset E_6 \times \mathrm{SU}(3)$ on each side: not gauged. Purpose: carve the base and fibre geometry. After $\mathrm{SU}(3) \to \mathrm{SU}(2) \times U(1)$,

$$TM_6 \;\cong\; (\Im\mathbb{H}_L) \,\oplus\, (\Im\mathbb{H}_R), \qquad F_4^X \;\cong\; (\mathbf{2}_{+1} \oplus \bar{\mathbf{2}}_{-1})_{\mathbb{R}} \;\simeq\; T\mathbb{C}P^2,\ X = L, R,$$

with $\Im\mathbb{H}$ realizing $\mathbf{3}_0$ and $\mathbb{H}\varepsilon$ realizing $(\mathbf{2} \oplus \bar{\mathbf{2}})_{\mathbb{R}}$. The two $U(1)$'s act as Spin^c line connections on the $\mathbb{C}P^2$ fibres.

2. **Gauge SU(3)**'s inside each E_6 (trinification): these are dynamical. On the left: $E_6^L \to \mathrm{SU}(3)_c \times \mathrm{SU}(3)_L \times \mathrm{SU}(3)_{F,L}$ with $\mathrm{SU}(3)_L \to \mathrm{SU}(2)_L \times U(1)_Y$. On the right: $E_6^R \to \mathrm{SU}(3)_{c'} \times \mathrm{SU}(3)_R \times \mathrm{SU}(3)_{F,R}$ with $\mathrm{SU}(3)_R \to \mathrm{SU}(2)_R \times U(1)_{\mathrm{Ydem}}$.
3. **Localization and Lorentz breaking** in 6D: two Higgs order parameters define localized 4D leaves $\Sigma_R, \Sigma_L \subset M_6$ via a covariant two-form density ρ_X. Normal 2-frames U_X implement $SO(3,3) \to SO(3,1)$ on Σ_R and $SO(3,3) \to SO(1,3)$ on Σ_L, eating 9 Lorentz coset modes per leaf and leaving the tangent spin connections massless [5].

Where do the unbroken gauge groups live?

Because every sector action is wedged with ρ_X, dynamics is *localized* on the leaves. Hence unbroken $\mathrm{SU}(3)_c$ (and $\mathrm{SU}(3)_{c'}$ if retained) are 4D gauge symmetries on the relevant Σ's. The 6D base M_6 persists as the ambient bundle base, but low-energy fields do not propagate in the bulk.

What are the fibres relative to spacetime?

$F_4^{L,R}$ are rank-4 *internal* vector bundles over M_6, canonically $F_4^X \simeq \mathbb{H}_X \varepsilon_X \cong (\mathbf{2}_{+1} \oplus \bar{\mathbf{2}}_{-1})_{\mathbb{R}} \simeq T\mathbb{C}P^2$. They are not extra spacetime directions. Restriction to a leaf gives internal fibres $F_4^X|_{\Sigma_X}$ on which internal interactions act. The two 4D spacetimes come from the six tangent directions $(\Im\mathbb{H}_L) \oplus (\Im\mathbb{H}_R)$ plus one opposite-side normal on each leaf.

Two consistent options for $\mathbf{SU(3)}_{c'}$.

- *Decoupled/hidden:* break or confine $\mathrm{SU}(3)_{c'}$ above the localization scale; only visible $\mathrm{SU}(3)_c$ remains on Σ_R.
- *Gauged on a leaf:* keep $\mathrm{SU}(3)_{c'}$ dynamical on one leaf (typically Σ_R or Σ_L). Portal terms can live on $X_3 = \Sigma_R \cap \Sigma_L$ under the BF matching conditions.

Dictionary (one line).

$$\boxed{\text{structural } \mathrm{SU}(3)^{\text{geom}}_{L,R} \Rightarrow (M_6, F_4^{L,R}) \quad \text{vs} \quad \text{gauge } \mathrm{SU}(3) \subset E_6^{L,R} \Rightarrow \text{4D forces on } \Sigma_{L,R}\,.}$$

Detailed explanation: This section consolidates the whole framework, breaking it into three conceptual layers and addressing important questions about how the effective 4D physics emerges.

Layer 1: Geometric $SU(3)^{\text{geom}}$'s carving out spacetime and internal space. These are structural, not dynamical. Each E_8 branch provided one such $SU(3)$, and using them we constructed M_6 and the internal fibres. After breaking $SU(3) \to SU(2) \times U(1)$, we had:

$TM_6 \cong (\Im H_L) \oplus (\Im H_R)$, the 6 real dimensions of the base.

$F_4^X \cong (2_{+1} \oplus 2_{-1})_R \cong TCP^2$ for $X = L, R$, the real 4D internal fibre on each side.

$\Im H$ realizes the 3_0 and $H\epsilon$ realizes $(2 \oplus \bar{2})_R$, as we already detailed.

The two $U(1)$'s from each $SU(3)^{\text{geom}}$ act as the $Spin^c$ line bundle connections on those CP^2 fibres.

In simpler terms, Layer 1 is: "the extra $SU(3)$s create a 6D world (with two embedded 4D slices) and give each 4D slice a 4D internal space (tangent to CP^2) with the necessary $U(1)$ field for fermions." This layer is purely about geometry and internal degrees of freedom—no Standard Model forces here yet, no direct dynamics.

Layer 2: Gauge $SU(3)$'s inside each E_6 (the trinification). These are the usual gauge groups we think of in particle physics, and they are dynamical (they have field strengths, particles, etc.). On the left side, E_6^L splits as $(SU(3)_c \times SU(3)_L \times SU(3)_{F,L})$, and on the right E_6^R splits as $(SU(3)_{c'} \times SU(3)_R \times SU(3)_{F,R})$. Then further breakings yield:

Left: $SU(3)_L \to SU(2)_L \times U(1)_Y$ (electroweak interactions for left-handed fermions), while $SU(3)_c$ is QCD and $SU(3)_{F,L}$ is a global flavor symmetry for generations.

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Right: $SU(3)_R \to SU(2)_R \times U(1)_{Y\text{dem}}$ (the $SU(2)_R$ eventually gives gravity, and $U(1)_{\text{dem}}$ is the dark electromagnetism), and $SU(3)c'$ is either hidden or very high-scale, and $SU(3)_{F,R}$ is a global flavor symmetry for right-handed fermions.

So Layer 2 comprises all the familiar gauge forces (and some new ones like the right-handed sector's), but crucially these gauge fields will ultimately be confined to the 4D slices Σ_L or Σ_R (our "two worlds"). They do not propagate in the full 6D bulk at low energies; how that happens is explained by Layer 3.

Layer 3: Localization and Lorentz symmetry breaking in 6D. This is perhaps the most novel layer, describing how two separate 4D spacetimes emerge dynamically from the 6D and how standard 4D physics is confined to them. We envision using two Higgs-like order parameters (presumably scalar fields or two-form fields) that develop expectation values to define two 4D "leaves" Σ_R and Σ_L inside M_6. We suggest a covariant two-form density ρ_X for each leaf $X = L, R$. One can imagine ρ_R is like a localized 2-form that is peaked on Σ_R and similarly for ρ_L on Σ_L. By wedging all sector actions with these ρ_X, the dynamics (kinetic terms, etc.) are essentially restricted to the leaves. This is analogous to fields living on domain walls or branes in higher-dimensional theories.

Lorentz breaking: The normal 2-frames U_X mentioned are likely fields that pick out a preferred 2D plane (normal to each 4D leaf) in the 6D tangent space. The effect is to break $SO(3,3)$ (the 6D Lorentz group) down to $SO(3,1)$ on Σ_R and to $SO(1,3)$ on Σ_L (the latter is basically the same $SO(3,1)$ but with the sign flip in the metric for time). We say this "eats 9 Lorentz coset modes per leaf" – presumably 6D Lorentz to 4D Lorentz has 15 generators vs 6; 9 components become massive (possibly akin to a gravitational Higgs mechanism) leaving massless spin connections on the leaves. In other words, the full 6D local Lorentz symmetry is broken such that each 4D subspace has its own local Lorentz invariance (gravity on each leaf), and the extra degrees of freedom that would mix the two or go off the leaf are eliminated or made heavy by these Higgs fields (this relates to the reference [5], a graviweak unification in 6D).

A key question addressed: Where do the unbroken gauge groups live? The answer: because the action for each sector is weighted by ρ_X, the gauge fields and matter fields effectively only "see" their respective leaf Σ_X. Thus, the unbroken gauge symmetries like $SU(3)_c$ (QCD) – which we want to exist in our 4D world – end up confined to (say) Σ_R (we suggest Σ_R for visible sector). If $SU(3)_{c'}$ is retained (not broken entirely), it could live on either Σ_R or Σ_L as a hidden QCD sector. But either way, in low-energy 4D physics, fields do not propagate in all 6 dimensions, only on their localized 4D slice. The 6D base still exists as an "ambient space" but is mostly empty of propagating degrees of freedom at low energy (think of it like two branes in a higher-dimensional bulk, with bulk gravity perhaps but gauge fields on the branes).

Another question: What are the fibres relative to spacetime? We clarify that F_4^L and F_4^R (the internal 4D fibres) are internal degrees of freedom, not extra spacetime dimensions. If we stand on one 4D leaf and look around, we see 3 space + 1 time; we do not directly perceive those 4 internal dimensions as large spatial directions – they are like an internal symmetry space at each point. When we "restrict to a leaf", each point of the 4D spacetime still has an attached CP^2-tangent-like internal space where internal symmetries (like color, etc.) act. This is akin to saying: on the 4D leaf, physics has gauge symmetries that can be thought of as arising from motion in those internal fibre directions, but those directions aren't freely accessible dimensions for propagation. The Standard Model forces thus act on internal fibre indices rather than as extra spacetime dimensions.

We also phrase: "The two 4D spacetimes come from the six tangent directions $(\Im H_L) \oplus (\Im H_R)$ plus one opposite-side normal on each leaf.". This is exactly how we constructed $M_4^{(R)}$ and $M_4^{(L)}$ earlier: each took the 3 from one side's $\Im H$ and added the normal from the other side. The "opposite-side normal" means Σ_R uses a direction in H_L as its 4th dimension (time), and Σ_L uses a direction in H_R as its time. Thus each leaf's 4D tangent is not just the naive splitting H_L or H_R, but rather a mix: it's 3 from its own side + 1 from the other.

The discussion also lists two consistent options for the second color group $SU(3)_{c'}$ (the one from E_6^R that we said is hidden or global):

Decoupled/hidden: Break or confine $SU(3)_{c'}$ at high scale so that only the ordinary QCD $SU(3)_c$ remains in low energy. In this scenario, $SU(3)_{c'}$ might not appear in 4D at all, or it might be a confined hidden sector (perhaps giving dark bound states, etc.) that doesn't interfere with known physics. This is likely preferred to avoid mirror quarks etc. Only $SU(3)_c$ on Σ_R is then the QCD we see.

Gauged on a leaf: Alternatively, we could allow $SU(3)_{c'}$ to remain and assign it to one of the two leaves (maybe the other 4D world Σ_L or also Σ_R). It would then be like a shadow QCD in a parallel sector. We mention "portal terms can live on $X^3 = \Sigma_R \cap \Sigma_L$ under BF matching conditions", indicating that if both leaves have color forces, their intersection (which is 2D) might host interactions connecting them (perhaps something like a common 2D defect where fields from both sectors meet, reminiscent of a brane intersection scenario). This is a more speculative option and would mean our world might interact weakly with a hidden world via this intersection.

Finally, we present a one-line Dictionary:

Structural $SU(3)^{\text{geom}}_{L,R}$ correspond to (M_6, F_4^L, F_4^R), defining geometry (base + fibres).

Gauge $SU(3)$ inside $E_6^{L,R}$ correspond to 4D forces on $\Sigma_{L,R}$. This neatly separates "geometry group" vs "force group" roles of the various $SU(3)$ factors.

The Big Picture is that we have a cohesive theory where gravity and weak interactions cause a splitting of spacetime into two sheets, while the gauge interactions of the Standard Model are confined to those sheets, and the internal symmetries (like color) are interpreted as rotations in an abstract 4D internal space attached to each spacetime point. It's like a blend of Kaluza–Klein (internal space for gauge forces) and brane-world (fields localized on sub-manifolds) scenarios, all orchestrated by the exceptional algebra structure of $E_8 \times E_8$.

Anomalies.

On each leaf, gauge and mixed anomalies match those of the E_6 embeddings and their standard symmetry–breaking chains; we assume usual E_6 anomaly freedom sector by sector. If a portal is introduced on the 2D overlap $\Sigma_L \cap \Sigma_R$, one must ensure either explicit cancellation in the 4D content or anomaly inflow from appropriate 6D counterterms; both options are available in this scaffold.

Relation to $J_3(\mathbb{O}_\mathbb{C})$ mass geometry.

The geometric fibre $F_4 \simeq T\mathbb{CP}^2$ realises the same $SU(3)$–flavour geometry that underlies the $J_3(\mathbb{O}_\mathbb{C})$ mass–ratio construction: the adjoint $8 \to 3_0 \oplus 2_{+1} \oplus 2_{-1} \oplus 1_0$ maps to $\Im\mathbb{H} \oplus \mathbb{H}\varepsilon$, with $(2 \oplus \bar{2})_R$ providing the internal complex 2. In this scaffold, internal primitive idempotents remain intact; a Majorana condition, when used, is imposed at the spinor level rather than by replacing internal projectors with non–idempotent directions. This keeps the Jordan–algebraic state geometry consistent while retaining the small symmetry–breaking effects needed for realistic spectra.

9.1. UV Completion and Trace Dynamics

We take the microscopic degrees of freedom to be matrices $Q_a(\tau)$ in the adjoint of $E_8 \times \omega E_8$ evolving in Connes time τ. A minimal single–atom Lagrangian is

$$\mathcal{L}_{\text{atom}} = \text{Tr}\big(\dot{Q}_1\,\dot{Q}_2\big) - \frac{\Omega^2}{2}\,\text{Tr}\big(Q_1^2 + Q_2^2\big) - \lambda\,\text{Tr}\big(Q_1 Q_2\big), \qquad \dot{} := \tfrac{d}{d\tau}, \tag{25}$$

where the quadratic terms are the lowest–degree $E_8 \times \omega E_8$–invariant potentials; $\Omega^2 \geq 0$ and λ are real couplings. The many–atom system is an interacting trace dynamics of the Adler type. In the coarse–grained, large–N limit one recovers: (i) emergent quantum kinematics from the conserved Adler–Millard charge (canonical commutators and unitary evolution), (ii) the 6D BF+Lorentz–Higgs sector as the IR geometric hydrodynamics, and (iii) localisation on two 4D leaves via the normal

2–frame condensate, which breaks $SO(3,3) \to SO(3,1) \times SO(2)$ and gives mass to the eight mixed connections.

Power counting and predictivity.

The UV theory is polynomial in the matrices and free of short–distance field singularities; the continuum fields arise as collective variables. The 6D effective action inherits a finite set of relevant/marginal operators at low dimension, while higher–dimension operators are suppressed by the trace–dynamics scale Λ_{TD}.

Phenomenological normalisations.

Fermion masses enter via $m_f = y_f v_M$; if the dark $U(1)_{\mathrm{dem}}$ couples to $\sqrt{m_f}$, we define the dimensionless charge $Q_{\mathrm{dem}}(f) = \sqrt{y_f}$ (with scheme dependence through $y_f(\mu)$). This keeps $U(1)_{\mathrm{dem}}$ RG–safe.

Open UV checks.

(i) cluster/locality in the hydrodynamic limit; (ii) anomaly matching on each leaf (with possible inflow at $\Sigma_L \cap \Sigma_R$); (iii) absence of ghosts/tachyons in the mixed connection sector; (iv) independence of physical outputs from the choice of quaternionic frame inside $\mathbb{O}$ up to G_2 automorphisms.

10. Summary

- The $(3,3)$ base M_6 is $\Im\mathbb{H}_L \oplus \omega\, \Im\mathbb{H}_R$; the two 4D spacetimes are 4-planes obtained by adding a single normal from the opposite side.
- The extra SU(3)'s branch as $\mathbf{8} \to \mathbf{3}_0 \oplus \mathbf{2}_{+1} \oplus \bar{\mathbf{2}}_{-1} \oplus \mathbf{1}_0$.
- $\Im\mathbb{H}$ realises $\mathbf{3}_0$; $\mathbb{H}\varepsilon$ realises $(\mathbf{2} \oplus \bar{\mathbf{2}})_{\mathbb{R}}$ and is the 4D fibre $F_4 \simeq T\mathbb{C}P^2$.
- Each geometric $U(1)$ is the Spin^c line on $\mathbb{C}P^2$; we do not confuse it with the octonion real line.

We recap and slightly rephrase these summary points here for clarity:

Spacetime Emergence: The 6D base (M_6, g) has signature (3,3) and is explicitly given by $\Im H_L \oplus \omega \Im H_R$. Two overlapping Lorentzian 4D spacetimes (leaves) are embedded as 4-planes in M_6 by including one normal direction from the opposite $\Im H$ (as in equations (4) and (5) earlier). One leaf Σ_R ends up with signature (3,1) using $(\Im H_R + t_L)$ and the other Σ_L uses $(\Im H_L + \omega t_R)$. These share a 2D (1,1) intersection. Thus, our familiar 4D spacetime is one "leaf" of a 6D space, and there's a second, hidden 4D leaf intertwined via a 2D bridge.

$SU(3)$ **Branching and Octonions**: Each extra $SU(3)$geom (left or right) when broken to $SU(2) \times U(1)$ gives $8 \to 3_0 + 2_{+1} + 2_{-1} + 1_0$. Correspondingly, the octonion split $O = H \oplus H\epsilon$ realizes this: $\Im H$ provides the 3_0 (tied to spatial directions in M_6), and $H\epsilon$ provides the $2_{+1} \oplus 2_{-1}$ (the 4 internal directions). The real unit $1 \in O$ is separate from the $U(1)$ generator and is not used directly as it would correspond to the singlet 1_0. Thus octonionic algebra explains why 6+4 dimensions naturally appear from $SU(3)$.

Internal CP^2 fibre: The 4D internal fibre F_4 constructed from $H\epsilon$ is canonically isomorphic to the tangent space of CP^2. This means each point in spacetime has an internal structure equivalent to a small CP^2 direction. The $(2 \oplus \bar{2})_R$ representation of $SU(2) \times U(1)$ is exactly what acts on TCP^2. So the model finds the minimal internal space for an $SU(3)$ symmetry. In effect, the Standard Model's "internal" symmetries (like color, flavor) are geometrized as symmetries of a tiny CP^2 fibre attached to spacetime.

Geometric $U(1)$'s as $Spin^c$ Connections: Each of the two $U(1)$ factors from $SU(3)^{\mathrm{geom}} \to SU(2) \times U(1)$ serves as the $Spin^c$ line bundle on the CP^2 fibre. In other words, these are background $U(1)$ gauge fields ensuring that the internal space can host spinor fields (quarks, leptons). They are not to be confused with any scalar or physical $U(1)$ in the octonions. They could mix with model $U(1)$'s in principle, but fundamentally their role is fixed by geometry. Thus, the existence of a "dark" $U(1)$ (or

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two of them) in the unified group is not arbitrary: it's required to allow spin structure on the internal fibre.

Putting it all together: The paper presents a unified theory scaffold in which space, time, and internal quantum numbers all stem from a common $E_8 \times \omega E_8$ symmetry structure. Spacetime (including possibly an extra hidden timelike dimension sector) emerges from using the extra $SU(3)$ factors as a frame-Higgs that breaks 6D down to $4D + 4D$ in a controlled way, while the internal symmetry space (needed for gauge forces like color) is identified with CP^2 directions coming from the same $SU(3)$ factors. Meanwhile, the $E_6 \times E_6$ part of the symmetry contains the known Standard Model forces and matter, which are placed on these 4D slices and benefit from the geometric structuring (e.g., the existence of three generations from $SU(3)_F$ flavor, mass ratios from Jordan algebra, etc., as referenced). Gravity emerges from the breaking of $SU(2)_R$ and the localization mechanism, and a new pseudo-force (dark electromagnetism) appears related to mass. All fields of the E_6 sectors are now living in a higher-dimensional space but effectively constrained to our 4D due to the Higgs localization.

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