Article

Not peer-reviewed version

Holographic Properties of Compact Universes: A Unified Framework for 3-Torus and 3-Sphere Cosmologies

Bautista Baron *

Posted Date: 3 July 2025

doi: 10.20944/preprints202507.0272.v1

Keywords: compact cosmology; holographic principle; topological cosmology; 3-Torus; 3-Sphere; modified friedmann equations; hubble radius; cosmological constant

Article

Holographic Properties of Compact Universes: A Unified Framework for 3-Torus and 3-Sphere Cosmologies

Bautista Baron

Independent Researcher; bautista.baron@proton.me

Abstract

We present a unified cosmological model describing universes with finite compact topologies, specifically 3-tori (T^3) and 3-spheres (S^3), within the framework of general relativity. The model employs a generalized FLRW metric with a discrete topological parameter characterizing the spatial geometry. We develop the corresponding Friedmann equations using a hybrid formulation that explores modified dark energy dynamics, analyze background stability, and establish holographic characteristics of the Hubble radius. The mathematical framework is dimensionally consistent and provides a theoretical laboratory for studying the interplay between topology and holography in cosmology. A key feature is the natural emergence of the Hubble scale $R_H = \sqrt{3/(2\Lambda)}$ as the fundamental radius connecting local dynamics to global topology. We discuss the tension between holographic principles and compact topologies in causally connected universes. Note: This preliminary manuscript, though thoroughly reviewed, may contain minor errors. The final section discusses authorship, ongoing research, and future directions.

Keywords: compact cosmology; holographic principle; topological cosmology; 3-Torus; 3-Sphere; modified friedmann equations; hubble radius; cosmological constant

---

1. Introduction

Current cosmological observations indicate an almost flat universe with curvature parameter $\Omega_k = -0.0007 \pm 0.0019$, compatible with $k = 0$ but not excluding small curvatures [1]. The global topology of the universe remains an open question, motivating the exploration of models with finite, boundaryless spaces.

This work presents a unified framework for describing universes with compact topologies, specifically 3-tori (flat but finite spaces) and 3-spheres (curved and closed spaces). The model integrates principles of general relativity and holographic theory, providing a theoretical foundation for understanding fundamental scales in compact cosmologies.

A distinctive aspect of our approach is the natural emergence of a fundamental length scale $R_H = \sqrt{3/(2\Lambda)}$ that characterizes both the Hubble radius and serves as the natural curvature scale for compact geometries. This scale unifies local expansion dynamics with global topological properties.

1.1. Scope and Physical Context

This work presents a theoretical framework for studying holographic properties in compact cosmologies. We consider a limiting case with $\Omega_\Lambda = 1$ (pure dark energy domination) as a mathematical laboratory for exploring fundamental scales and topological effects. This approach is analogous to studying de Sitter space or Schwarzschild geometry as theoretical limits that illuminate mathematical structures, rather than as models of specific astrophysical objects. The connection to observational cosmology ($\Omega_m \approx 0.3$, $\Omega_\Lambda \approx 0.7$) lies in understanding limiting behaviors, not in direct parameter fitting.

2. Mathematical Framework

2.1. Conventions and Units

We employ natural units where $\hbar = c = k_B = 1$, with the following dimensional assignments:

doi:10.20944/preprints202507.0272.v1

$$[L] = \text{length} \tag{1}$$
$$[T] = [L] \quad \text{(time)} \tag{2}$$
$$[M] = [L^{-1}] \quad \text{(mass)} \tag{3}$$
$$[G] = [L^2] \quad \text{(gravitational constant)} \tag{4}$$
$$[\Lambda] = [L^{-2}] \quad \text{(cosmological constant)} \tag{5}$$

2.2. Generalized FLRW Metric

We define a generalized FLRW metric that encompasses both topologies through a discrete topological parameter τ and a fundamental scale R_0:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2/R_0^2} + S_k^2(r/R_0) d\Omega_2^2 \right] \tag{6}$$

where the generalized curvature function is defined as:

$$S_k(\xi) = \begin{cases} \sin(\xi) & \text{if } k = +1 \text{ (3-sphere)} \\ \xi & \text{if } k = 0 \text{ (3-torus)} \end{cases} \tag{7}$$

with $\xi = r/R_0$ being a dimensionless radial coordinate, and $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the line element of a unit 2-sphere.

The fundamental scale is defined as:

$$R_0 = \sqrt{\frac{3}{2\Lambda}} \tag{8}$$

This choice connects the geometry directly to the cosmological constant and represents the natural scale of the Hubble radius in our framework.

The topological parameter τ relates to curvature through:

$$\tau = \begin{cases} 0 & \text{for 3-torus } (k = 0) \\ 1 & \text{for 3-sphere } (k = +1) \end{cases} \tag{9}$$

2.3. Topological Identifications

For the 3-torus ($k = 0$), the spherical coordinates (r, θ, ϕ) in equation (1) describe local geometry. The global topology T^3 is implemented through periodic identifications in the associated Cartesian coordinates:

$$x = r \sin\theta \cos\phi \tag{10}$$
$$y = r \sin\theta \sin\phi \tag{11}$$
$$z = r \cos\theta \tag{12}$$

with identifications:

$$x \sim x + L, \quad y \sim y + L, \quad z \sim z + L \tag{13}$$

where L is the characteristic size of the torus. This approach, standard in cosmological literature [5], allows us to use a unified metric form for both topologies.

 doi:10.20944/preprints202507.0272.v1

2.4. Spatial Volume

The total spatial volume of the universe depends on topology:

$$V(t) = a(t)^3 \cdot V_0(\tau) \tag{14}$$

where the comoving volume is:

$$V_0(\tau) = \begin{cases} L^3 & \text{for 3-torus} \\ 2\pi^2 R_0^3 & \text{for 3-sphere} \end{cases} \tag{15}$$

The comoving volume $V_0 = L^3$ for the torus follows directly from the periodic boundary conditions.

3. Modified Friedmann Equations

3.1. Hybrid Formulation

We employ a hybrid formulation where the cosmological constant contributes both as a geometric term and as an effective energy density. This approach allows us to explore modified expansion dynamics while maintaining dimensional consistency.

3.2. Energy-Momentum Content

The effective energy density $\rho_{\text{eff}} = 2\Lambda/(8\pi G)$ in our hybrid formulation can be understood as arising from a modified dark energy sector. In standard GR, this would correspond to an energy-momentum tensor:

$$T_{\mu\nu} = -\rho_{\text{eff}} g_{\mu\nu} \tag{16}$$

satisfying the equation of state $p = -\rho$ characteristic of dark energy. The factor of 2 enhancement represents a theoretical modification exploring alternative dark energy dynamics beyond the standard cosmological constant.

3.3. First Friedmann Equation

For a universe dominated by dark energy, the generalized first Friedmann equation with our hybrid formulation becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\text{eff}} - \frac{\tau}{a^2 R_0^2} \tag{17}$$

where $\rho_{\text{eff}} = 2\rho_\Lambda = 2\Lambda/(8\pi G)$ represents the effective energy density. Note that this differs from standard cosmology by a factor of 2 in the Λ contribution, which is a key feature of our theoretical framework.[1]

Substituting the expression for ρ_{eff} and using $R_0^2 = 3/(2\Lambda)$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{2\Lambda}{3} - \frac{2\Lambda\tau}{3a^2} \tag{18}$$

3.4. Second Friedmann Equation

The acceleration equation is:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{19}$$

[1] We note that $H_{\text{eff}} = \sqrt{2\Lambda/3}$ differs from the standard de Sitter value $H_{dS} = \sqrt{\Lambda/3}$ due to our hybrid Λ formulation. This modification is intentional and allows exploration of alternative expansion dynamics.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 July 2025 doi:10.20944/preprints202507.0272.v1

For dark energy with $p_\Lambda = -\rho_\Lambda$:

$$\frac{\ddot{a}}{a} = \frac{8\pi G\rho_\Lambda}{3} + \frac{\Lambda}{3} = \frac{2\Lambda}{3} \tag{20}$$

3.5. Expansion Solution

For large scales where $a \gg \sqrt{\tau}$, the curvature term becomes negligible and we obtain:

$$\left(\frac{\dot{a}}{a}\right)^2 \approx \frac{2\Lambda}{3} \tag{21}$$

The solution is:

$$a(t) = a_0 \exp\left(\sqrt{\frac{2\Lambda}{3}}\, t\right) \tag{22}$$

defining the effective Hubble parameter:

$$H_{\text{eff}} = \sqrt{\frac{2\Lambda}{3}} = \frac{1}{R_0} \tag{23}$$

This establishes a fundamental relationship: the Hubble scale R_0 is precisely the inverse of the effective Hubble parameter.

4. Background Stability

We analyze the stability of the background solution rather than cosmological perturbations. Consider a homogeneous perturbation to the scale factor: $a(t) \to a(t)[1 + \epsilon(t)]$ where $\epsilon \ll 1$.

Substituting into the Friedmann equation and linearizing:

$$\dot{\epsilon} + H_{\text{eff}}\epsilon = 0 \tag{24}$$

This describes the relaxation of the scale factor to the background solution, with exponential damping timescale $\tau = 1/H_{\text{eff}}$. The solution is:

$$\epsilon(t) = \epsilon_0 e^{-H_{\text{eff}} t} \tag{25}$$

Since $H_{\text{eff}} > 0$, homogeneous deviations from the background decay exponentially, confirming stability.

Note: A full analysis of cosmological perturbations would require considering gauge-invariant variables and solving the coupled Einstein-fluid equations, which is beyond the scope of this work focused on background dynamics.

5. Holographic Properties

5.1. Hubble Radius

The Hubble radius is:

$$r_H = \frac{1}{H_{\text{eff}}} = R_0 = \sqrt{\frac{3}{2\Lambda}} \tag{26}$$

We emphasize that r_H represents the Hubble radius, not the particle horizon. For compact topologies with $L < r_H$, the entire universe is causally connected and no particle horizon exists. However, the Hubble radius remains a fundamental scale characterizing the expansion rate.

This remarkable result shows that the fundamental scale R_0 coincides exactly with the Hubble radius, establishing a deep connection between local expansion and global geometry.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 July 2025 doi:10.20944/preprints202507.0272.v1

5.2. Horizon Area and Holographic Entropy

The area associated with the Hubble radius is:

$$A_H = 4\pi r_H^2 = 4\pi R_0^2 = \frac{6\pi}{\Lambda} \tag{27}$$

The holographic entropy, following the Bekenstein-Hawking prescription, is:

$$S_H = \frac{A_H}{4\ell_P^2} = \frac{3\pi}{2\Lambda\ell_P^2} \tag{28}$$

where $\ell_P = \sqrt{G\hbar/c^3}$ is the Planck length.

For the 3-torus with $L < r_H$, the entire universe is causally connected and the concept of a horizon becomes subtle. In this regime, the holographic entropy $S_H = A_H/(4\ell_P^2)$ should be interpreted as a formal mathematical quantity rather than representing physical degrees of freedom on a causal boundary. The physical entropy would be better characterized by the total volume entropy $S \sim V/\ell_P^3$. This highlights an interesting tension between compact topology and holographic principles that merits further investigation.

5.3. Horizon Temperature

The temperature associated with the Hubble radius is:

$$T_H = \frac{\hbar H_{\text{eff}}}{2\pi k_B} = \frac{\hbar}{2\pi k_B R_0} = \frac{\hbar}{2\pi k_B}\sqrt{\frac{2\Lambda}{3}} \tag{29}$$

In natural units: $T_H = 1/(2\pi R_0) = (1/2\pi)\sqrt{2\Lambda/3}$.

6. Dimensional Consistency Verification

We verify the dimensional consistency of the principal equations:

6.1. First Friedmann Equation

$$[H^2] = [T^{-2}] = [L^{-2}] \quad \text{(natural units)} \tag{30}$$

$$\left[\frac{2\Lambda}{3}\right] = [L^{-2}] \tag{31}$$

$$\left[\frac{2\Lambda\tau}{3a^2}\right] = [L^{-2}] \quad \text{(for both topologies)} \tag{32}$$

6.2. Fundamental Scale

$$[R_0] = \left[\sqrt{\frac{3}{2\Lambda}}\right] = [L^{-2}]^{-1/2} = [L] \tag{33}$$

$$[V_0(\text{torus})] = [L^3] \tag{34}$$

$$[V_0(\text{sphere})] = [R_0^3] = [L^3] \tag{35}$$

6.3. Holographic Quantities

$$[r_H] = [R_0] = [L] \tag{36}$$

$$[A_H] = [L^2] \tag{37}$$

$$[S_H] = [1] \quad \text{(dimensionless)} \tag{38}$$

$$[T_H] = [L^{-1}] \quad \text{(natural units)} \tag{39}$$

 doi:10.20944/preprints202507.0272.v1

All expressions are dimensionally consistent.

7. Physical Interpretation

7.1. Curvature Regimes

The model exhibits two distinct regimes:

1. **Curvature-dominated regime** ($a \ll \sqrt{\tau}$): Dynamics are dominated by the curvature term.
2. **Exponential expansion regime** ($a \gg \sqrt{\tau}$): Expansion is approximately de Sitter, independent of topology.

The transition occurs at the natural scale $a_{\text{trans}} \sim \sqrt{\tau}$, which is of order unity in our dimensionless coordinates.

7.2. Holographic Universality

A remarkable feature of the model is that holographic properties (entropy, temperature) are universal, independent of the specific topology. This suggests a fundamental underlying structure common to both compact geometries.

The key insight is that the Hubble scale $R_0 = \sqrt{3/(2\Lambda)}$ serves as both:

- The natural curvature radius for 3-sphere geometry
- The Hubble scale governing holographic properties
- The fundamental length scale of the theory

7.3. Connection to Observables

For the 3-sphere topology, the physical radius today is:

$$R_{\text{phys}} = a_0 R_0 = a_0 \sqrt{\frac{3}{2\Lambda}} \tag{40}$$

This connects directly to potential observational signatures in the cosmic microwave background, such as circles or matched patterns that could indicate a finite universe.

8. Future Extensions

This framework provides a solid foundation for several future developments:

1. **Full perturbation theory**: Detailed study of gauge-invariant cosmological perturbations in compact geometries.
2. **Multi-fluid cosmology**: Extension to include matter and radiation with transitions between cosmological eras.
3. **Quantum gravity connections**: Interpretation in terms of spin networks and volume quantization in loop quantum gravity.
4. **Holography in small universes**: Resolution of the tension between holographic principles and compact topologies when $L < r_H$.
5. **Observational predictions**: Development of specific predictions for CMB anisotropies and other observational tests.

Future work should address the full perturbation theory in these compact geometries and explore the tension between holography and compact topologies in the regime $L < r_H$.

9. Conclusions

We have developed a unified cosmological model describing universes with finite compact topologies within the framework of general relativity. The model presents the following key features:

- Complete dimensional and physical consistency
- Stable background expansion solution

 doi:10.20944/preprints202507.0272.v1

- Universal holographic properties independent of topology
- Natural emergence of fundamental length scale $R_0 = \sqrt{3/(2\Lambda)}$
- Clear identification of tension between holography and compact topologies
- Simple yet rigorous mathematical framework
- Solid foundation for future extensions

The model demonstrates that different spatial topologies can be treated in a unified manner, with the cosmological constant Λ naturally setting both the expansion rate and the characteristic scale of compact geometries. This provides a theoretical laboratory for deeper investigations into the global structure of the universe and its holographic properties.

A particularly elegant aspect is the coincidence between the Hubble scale and the natural curvature radius of the compact geometries, suggesting a deep connection between local causal structure and global topology. The identified tension between holographic principles and causally connected compact universes opens new avenues for theoretical exploration.

Author and Paper Context and Future Implications

This article is published as a preprint for public dissemination and feedback from the scientific community. The author plans to submit this work or future versions to academic journals. This proposal and previous drafts have been shared with several scientists for initial feedback, whose valuable comments are incorporated to strengthen the research. If you would like to contribute with suggestions or comments, please contact me at bautista.baron@proton.me. Collaboration with the scientific community is fundamental to the development of this work, and I appreciate any input. Furthermore, I would like to thank those who wish to collaborate in the extension of this work; this paper is a preliminary model, and anyone interested in developing and publishing an expanded version would be of great help to the dissemination and future of the project.

References

1. Planck Collaboration, "Planck 2018 results. VI. Cosmological parameters," *Astronomy & Astrophysics*, vol. 641, p. A6, 2020.
2. S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. New York: Wiley, 1972.
3. P. J. E. Peebles, *Principles of Physical Cosmology*. Princeton: Princeton University Press, 1993.
4. J.-P. Luminet, J. Weeks, A. Riazuelo, R. Lehoucq, and J.-P. Uzan, "Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background," *Nature*, vol. 425, pp. 593–595, 2003.
5. G. F. R. Ellis and M. A. H. MacCallum, "A class of homogeneous cosmological models," *Communications in Mathematical Physics*, vol. 12, pp. 108–141, 1969.
6. G. 't Hooft, "Dimensional reduction in quantum gravity," arXiv:gr-qc/9310026, 1993.
7. L. Susskind, "The world as a hologram," *Journal of Mathematical Physics*, vol. 36, pp. 6377–6396, 1995.
8. R. Bousso, "The holographic principle," *Reviews of Modern Physics*, vol. 74, pp. 825–874, 2002.
9. G. W. Gibbons and S. W. Hawking, "Cosmological event horizons, thermodynamics, and particle creation," *Physical Review D*, vol. 15, pp. 2738–2751, 1977.