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Article

Uniform Approximation by Rational Functions with Prescribed Poles: Operator-Theoretic Perspective and Symmetries

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Abstract

In this paper the uniform approximation of continuous functions on $[0, 1]$ by rational functions with prescribed poles and bounded multiplicities is studied. A classical theorem of Fichera characterizes density in $C([0, 1])$ through the divergence of a conformally invariant series involving the pole distribution. A modern reformulation of this result is developed and it is given an operator-theoretic interpretation in which the approximation property is equivalent to cyclicity and to the absence of nontrivial invariant subspaces in an associated Hardy-space model. In this framework, the classical Blaschke condition emerges as the fundamental obstruction to density, linking rational approximation to the structure of model spaces and non-selfadjoint operator algebras. The density criterion is interpreted in terms of symmetry: divergence corresponds to a balanced distribution of poles compatible with the conformal geometry of the slit domain, while convergence induces symmetry breaking and the emergence of invariant structures. Numerical models illustrate the sharpness of the criterion and provide a concrete manifestation of the Blaschke obstruction and cyclicity mechanism. This new approach places Fichera's theorem within a broader operator-theoretic and spectral framework, connecting classical approximation theory with Hardy spaces, invariant subspace theory, and modern rational approximation methods.

Keywords: rational approximation; prescribed poles; hardy spaces; invariant subspaces; cyclic vectors; Blaschke products; conformal mapping; operator theory; non-selfadjoint operators; spectral completeness; symmetry breaking

1. Introduction

In this paper the uniform approximation of rational functions with prescribed poles is studied from the modern operator-theoretic perspective. The proposed operator-theoretic reinterpretation aims to develop a novel framework in which rational approximation is understood as a problem of cyclicity, invariant subspaces, and spectral completeness for non-selfadjoint operators. This shift of perspective reveals that uniform rational approximation is not merely an approximation-theoretic phenomenon, but an instance of a more general principle: the generation of dense subspaces by functional calculi of operators.

The problem of approximating continuous functions by analytic families has a classical origin in the theorem of Weierstrass [1]. Rational approximation with constrained pole sets was subsequently investigated by Szegő, Szász, and Porcelli [2–5], culminating in the fundamental result of Fichera [6], who provided a necessary and sufficient condition for density in $C([0, 1])$.

Fichera's theorem characterizes approximation in terms of the divergence of a conformally invariant series depending on the location and multiplicity of the poles. This invariance reflects a deeper geometric structure: the approximation problem is governed not only by the distribution of poles, but by also on how this distribution interacts with the conformal symmetries of the slit domain $\mathbb{C} \setminus [0, 1]$.

The purpose of this paper is to revisit Fichera's result from a modern perspective that combines complex analysis with operator theory and symmetry considerations. Our main contribution is an operator-theoretic reinterpretation in which the density of rational functions is equivalent to cyclicity and to the absence of nontrivial invariant subspaces in an associated Hardy-space model. A probabilistic generalization of the Fichera's deterministic theorem is also proposed in the application to random pole distributions.

Within this framework, the classical Blaschke condition emerges as a symmetry-breaking mechanism: when the pole distribution respects a global balance condition encoded by a divergent conformal series, the system remains cyclic; when this balance fails, a nontrivial invariant subspace appears, obstructing approximation.

This interpretation places Fichera's theorem within a broader principle: approximation can be viewed as a manifestation of spectral completeness under symmetry constraints, while failure of approximation corresponds to the emergence of hidden structures (invariant subspaces) induced by asymmetries in the pole configuration.

Recent developments in rational approximation have emphasized both computational and functional-analytic aspects of the problem. The AAA algorithm introduced by Nakatsukasa, Sète, and Trefethen [7] provides an adaptive framework for rational approximation, highlighting the role of pole selection and its impact on stability and convergence. From a theoretical perspective, the work of Gonchar and Rakhmanov [8] develops potential-theoretic methods to describe asymptotic distributions governing rational approximation, revealing deep connections between pole configurations and equilibrium measures. Hardy-space techniques have been further developed by Baratchart and Olivi [9], who analyze rational approximation problems using analytic function spaces and operator-theoretic tools closely related to those employed in the present work. The structure of invariant subspaces associated with rational systems is closely linked to model space theory, as discussed in recent work by Bessonov [10], where Toeplitz kernels and shift-invariant subspaces play a central role. Extremal problems in Hardy spaces, studied by Bénéteau, Khavinson, and Rakhmanov [11], further illustrate the interplay between analytic constraints and rational approximation, particularly in connection with boundary behavior and zero distributions. From a numerical viewpoint, stability issues in rational representations have been investigated by Huybrechs and Vandewalle [12], emphasizing the sensitivity of approximation schemes to pole placement and parametrization. Finally, recent work by Pushnitski and Yafaev [13] connects rational approximation with spectral theory of operators, reinforcing the perspective that approximation properties can be interpreted in terms of spectral completeness and functional calculus.

The paper is organized as follows. Novelty and main results are given in Sect. 2. Section 3 deals with preliminary remarks. Sections 4 develop the Hardy-space and operator-theoretic reformulation. Section 5 introduces a symmetry-based interpretation of the result. Section 6 provides illustrative concrete models. Technical details are collected in the two appendices.

2. Novelty and Main Results

While Fichera's theorem provides a complete characterization of density in terms of a conformally invariant series, its structural implications in operator-theoretic and symmetry terms have not been fully developed. The present work contributes to this direction by establishing explicit equivalences between approximation, cyclicity, and invariant subspace structure, and by formalizing these correspondences in a functional-analytic framework.

The main new contributions of this paper are summarized as follows:

- an operator-theoretic reformulation of rational approximation as a cyclicity problem,
- a symmetry-based interpretation of the density condition,
- explicit connections between Blaschke products and invariant subspaces,
- and quantitative toy-model realizations of the approximation mechanism.

We now state the main results in a unified form and explained in the following sections.

Theorem 1 (Operator-theoretic characterization of density). *Let $\{z_k\} \subset \mathbb{C} \setminus [0, 1]$ with multiplicities $v_k \in \mathbb{N}$, and let $\mathcal{R}$ be the associated rational system. Then the following are equivalent:*

- i. $\overline{\mathcal{R}}^{\|\cdot\|_\infty} = C([0, 1])$,
- ii. the constant function 1 is cyclic for the operator algebra $\{r(M_z) : r \in \mathcal{R}\}$ on $H^2(\mathbb{D})$,
- iii. the associated model space $\mathcal{K}_B = H^2 \ominus BH^2$ is trivial,
- iv. the conformal series

$$\sum_{k=1}^{\infty} v_k \Phi(z_k)$$

diverges.

Proof. The equivalence between (i) and (iv) is the classical theorem of Fichera. The equivalence between (i) and (ii) follows from the Hardy-space reformulation via functional calculus. The equivalence with (iii) follows from the standard correspondence between cyclicity and invariant subspaces generated by Blaschke products. $\square$

Theorem 2 (Symmetry—breaking criterion). *Let the pole system $\{z_k\}$ be given. Then:*

- If $\sum_k v_k \Phi(z_k) = +\infty$, the system is asymptotically symmetry-balanced and generates no nontrivial invariant subspaces.
- If $\sum_k v_k \Phi(z_k) < \infty$, the symmetry is broken and there exists a nontrivial invariant subspace $\mathcal{K}_B \neq \{0\}$ obstructing approximation.

In particular, symmetry breaking is equivalent to the existence of a nontrivial Blaschke product associated with the pole sequence.

Proof. This follows from the equivalence between the Blaschke condition and invariant subspace generation in the Nevanlinna class, together with the geometric comparability between $\Phi(z_k)$ and $(1 - |w_k|)$. $\square$

Theorem 3 (Spectral completeness via rational functional calculus). *Let T be the multiplication operator on $L^2([0, 1])$. Then*

$$\overline{\{r(T) : r \in \mathcal{R}\}} = C^*(T)$$

if and only if

$$\sum_k v_k \Phi(z_k) = +\infty.$$

Thus, rational approximation is equivalent to spectral completeness of the rational functional calculus associated with T .

Proof. This is a direct consequence of the continuous functional calculus for normal operators combined with Theorem 1. $\square$

Theorem 4 (Quantitative obstruction estimate). *Assume*

$$\sum_k v_k \Phi(z_k) < \infty.$$

Then there exists a nonzero $f \in C([0, 1])$ and a constant $c > 0$ such that for all $r \in \mathcal{R}$,

$$\|f - r\|_\infty \geq c.$$

Thus, the failure of density produces a uniform approximation gap.

Proof. The existence of a nontrivial annihilating measure implies the existence of a function orthogonal to $\mathcal{R}$ in the dual space, yielding a uniform lower bound via Hahn–Banach. $\square$

At last a novel probabilistic generalization of the Fichera’s deterministic theorem is proposed in section 6 in the application to random pole distributions.

3. Preliminary Remarks

3.1. The Density Criterion

Let $\{z_k\} \subset \mathbb{C} \setminus [0, 1]$ and let $v_k \in \mathbb{N}$. Denote by $\mathcal{R}$ the set of rational functions whose poles are among the z_k with pole orders bounded by v_k .

The approximation problem is to characterize when

$$\overline{\mathcal{R}}^{\|\cdot\|_\infty} = C([0, 1]). \quad (1)$$

Define

$$\Phi(z) := \frac{d(z, [0, 1])}{|z(z-1)|^{1/2}}, \quad (2)$$

where $d(z, [0, 1])$ is the Euclidean distance.

Theorem 5 (Fichera [6]). *The space $\mathcal{R}$ is dense in $C([0, 1])$ if and only if*

$$\sum_{k=1}^{\infty} v_k \Phi(z_k) = +\infty. \quad (3)$$

where $\Phi(z)$, defined in Eq. (2), is a conformally invariant weight measuring the analytic influence of the pole z on the interval $[0, 1]$.

Proof. For the classical proof given by Fichera see [6]. $\square$

The quantity $\Phi(z)$ is invariant under conformal transformations of $\mathbb{C} \setminus [0, 1]$ and reflects the boundary distortion of the associated slit domain [14,15]. The divergence condition is formally analogous to the Blaschke condition for zero sets of Nevanlinna-class functions [16,17].

This condition has a clear structural meaning: approximation succeeds precisely when the total analytic influence of the pole system diverges.

3.2. Hardy-Space Reformulation

Let $H^2(\mathbb{D})$ denote the Hardy space on the unit disk and let M_z be the unilateral shift operator (multiplication by z). For rational functions analytic on $\mathbb{D}$, the functional calculus $r \mapsto r(M_z)$ is well defined.

Define the rational generated subspace

$$\mathcal{H}_{\mathcal{R}} := \overline{\{r(M_z)\mathbf{1} : r \in \mathcal{R}\}} \subset H^2(\mathbb{D}),$$

where $\mathbf{1}$ is the constant function.

The uniform density of $\mathcal{R}$ is equivalent to the cyclicity of $\mathbf{1}$ for the operator algebra generated by $\{r(M_z)\}$. This places the approximation problem within the classical theory of cyclic vectors and invariant subspaces [18,19].

3.3. Blaschke Products and Invariant Subspaces

Let w_k be the images of z_k under a conformal map from $\mathbb{D}$ onto $\mathbb{C} \setminus [0, 1]$. Define the Blaschke product

$$B(w) = \prod_k \left(\frac{|w_k|}{w_k} \frac{w_k - w}{1 - \overline{w_k}w} \right)^{v_k}.$$

The associated model space

$$\mathcal{K}_B := H^2(\mathbb{D}) \ominus BH^2(\mathbb{D}) \quad (4)$$

is invariant under the backward shift. The failure of rational approximation is equivalent to the nontriviality of $\mathcal{K}_B$, reflecting the presence of a Blaschke obstruction [17,20].

There follows that rational approximation fails iff the pole system generates a nontrivial model space $\mathcal{K}_B \neq 0$. So that, density is equivalent to triviality of the associated invariant subspace.

4. Operator-Theoretic Interpretation

Let T be the multiplication operator by x on $L^2([0,1])$. By the spectral theorem and continuous functional calculus,

$$C([0,1]) \cong C^*(T),$$

see [21,22]. Consequently, uniform density of $\mathcal{R}$ in $C([0,1])$ is equivalent to norm density of $\{r(T) : r \in \mathcal{R}\}$ in $C^*(T)$. Thus, Fichera's criterion may be interpreted as a statement about cyclicity and invariant subspaces for a non-selfadjoint operator algebra generated by rational functions.

The Spectral Interpretation is as follows: Let T be the multiplication operator by x on $L^2([0,1])$. The rational functions $r(T)$ define a non-selfadjoint operator algebra.

Define:

$$\mathcal{A}_{\mathcal{R}} := \overline{r(T) : r \in \mathcal{R}}.$$

Then density is equivalent to: $\mathcal{A}_{\mathcal{R}}$ is strongly dense in $\mathcal{B}(C([0,1]))$ acting on constants. This means that poles define a non-normal spectral deformation and the approximation corresponds to spectral completeness.

There follows the structural Operator-Theoretic Principle: based on the following conceptual equivalence: a) Rational Approximation Principle (Operator Form) b) Uniform density of rational functions with prescribed poles is equivalent to the absence of nontrivial closed invariant subspaces for the operator algebra generated by their functional calculus.

Symbolically:

$$\text{Density} \iff \text{Cyclicity} \iff \text{Trivial invariant subspace} \iff \text{No Blaschke obstruction}.$$

This reframes approximation theory as operator generation theory.

4.1. Operator Algebra Reduction

Theorem 6 (Functional calculus reduction). *Let A be a normal operator with $\sigma(A) = [0,1]$. Then the continuous functional calculus gives an isometric-isomorphism*

$$\Gamma : C([0,1]) \rightarrow C^*(A), \quad \Gamma(f) = f(A),$$

and

$$|f(A)| = \sup_{x \in [0,1]} |f(x)|.$$

Hence for any set $S \subset C([0,1])$,

$$\overline{\Gamma(s) : s \in S}^{\|\cdot\|} = C^*(A) \iff \overline{S}^{\|\cdot\|_{\infty}} = C([0,1]).$$

Proof. Standard continuous functional calculus for normal operators. $\square$

So the quantum/operator version for normal A is not a different theorem: it is exactly the same density question transported into $C^*(A)$.

The classical theorem 5 can be tightened within the operator's theory perspectives as follows.

Let $z_k \subset \mathbb{C} \setminus [0, 1]$ and multiplicities $v_k \in \mathbb{N}$. Let $\mathcal{R}$ be the rational functions whose poles are among the z_k with order $\leq v_k$.

Define the conformal weight according to (2) where $d(z, [0, 1])$ is Euclidean distance.

Theorem 7 (Fichera's theorem, from the operator's perspective).

$$\overline{\mathcal{R}}^{|\cdot|^\infty} = C([0, 1]) \iff \sum_k v_k = 1^\infty, \Phi(z_k) = +\infty. \quad (5)$$

Proof. It will be given a proof chain that is based on the following steps

- (i) duality,
- (ii) the disk map,
- (iii) the Nevanlinna/Blaschke step, and
- (iv) the necessity via construction.

□

(i) Duality

Lemma 1 (Hahn—Banach annihilator criterion). $\overline{\mathcal{R}}^{|\cdot|^\infty} = C([0, 1])$ iff the only finite complex Borel measure μ on $[0, 1]$ such that

$$\int_0^1 r(t) d\mu(t) = 0 \quad \forall r \in \mathcal{R} \quad (6)$$

is $\mu \equiv 0$.

Proof. Use $C([0, 1])^* \cong M([0, 1])$. A closed subspace is all of $C([0, 1])$ iff its annihilator is 0. □

There follows that density is equivalent to no nonzero annihilating measure.

(ii) The disk map

Let us build the analytic function carrying pole data.

Given a finite complex measure μ on $[0, 1]$, define its Cauchy transform

$$F(z) := \int_0^1 \frac{d\mu(t)}{z - t}, \quad z \in \mathbb{C} \setminus [0, 1], \quad (7)$$

we can show that

Lemma 2 ((Annihilation implies zeros at the prescribed poles)). If μ annihilates $\mathcal{R}$, then for every k ,

$$F^{(j)}(z_k) = 0, \quad j = 0, 1, \dots, v_k - 1.$$

Equivalently, F has a zero at z_k of order $\geq v_k$.

Proof. For each k and $j \leq v_k - 1$, the function $r_{k,j}(t) = (t - z_k)^{-(j+1)}$ belongs to $\mathcal{R}$. Annihilation gives

$$0 = \int_0^1 (t - z_k)^{-(j+1)} d\mu(t) = (-1)^{j+1} \frac{1}{j!} F^{(j)}(z_k).$$

□

So any nonzero annihilating measure forces an analytic function with many zeros.

(iii) The Nevanlinna/Blaschke step

Transport to the disk and invoke the Blaschke obstruction. Let $A = \mathbb{C} \setminus [0, 1]$. Use the standard conformal map $\psi : \mathbb{D} \rightarrow A$ (Joukowski-type; Fichera uses this explicitly), and set

$$w_k := \psi^{-1}(z_k) \in \mathbb{D}, \quad G(w) := F(\psi(w)).$$

Then G is analytic on $\mathbb{D}$ and has zeros at w_k with multiplicity $\geq v_k$.

Now we need the growth class:

Lemma 3 (Nevanlinna class membership). *For any finite complex measure μ , the function G belongs to the Nevanlinna class $\mathcal{N}(\mathbb{D})$.*

Proof. Cauchy transforms of finite measures admit harmonic majorants after conformal transport; equivalently

$$\sup_{0 < r < 1} \int_0^{2\pi} \log^+ |G(re^{i\theta})|, d\theta < \infty.$$

This is classical for Poisson–Stieltjes / Cauchy integrals of finite measures composed with a conformal map from a slit domain to $\mathbb{D}$. $\square$

Now the zero-set theorem:

Lemma 4 (Blaschke condition for $\mathcal{N}$). *If $G \in \mathcal{N}(\mathbb{D})$ is not identically zero and has zeros a_n with multiplicities m_n , then*

$$\sum_n m_n (1 - |a_n|) < \infty. \quad (8)$$

Proof. This is the classical Blaschke condition for zero sets of Nevanlinna-class functions that are Blaschke sequences; see e.g. [16,17]. $\square$

Apply to our zeros w_k (with multiplicities $\geq (v_k)$):

$$G \not\equiv 0 \implies \sum_k v_k (1 - |w_k|) < \infty.$$

So if we can relate $(1 - |w_k|)$ to $\Phi(z_k)$, we have done.

Lemma 5 (Geometric comparability). *There exist two constants $c, C > 0$ such that for all k large enough,*

$$c, \Phi(z_k) \leq 1 - |w_k| \leq C, \Phi(z_k).$$

Proof. Quantitative distortion control for ψ near the boundary slit $[0, 1]$: radial approach to $\partial\mathbb{D}$ corresponds to normal approach to $[0, 1]$, with Jacobian factor governed by $|z(z-1)|^{-1/2}$. This is exactly the estimate Fichera proves in his Lemma 1. $\square$

Putting it together we have: If $\mu \neq 0$, then $F \not\equiv 0$, hence $G \not\equiv 0$. Then $\sum_k v_k (1 - |w_k|) < \infty$ and hence $\sum_k v_k \Phi(z_k) < \infty$.

So that we have proved the contrapositive: If Eq. (3) holds true, then no nonzero annihilating measure exists, hence $\mathcal{R}$ is dense. That proves the sufficiency.

(iv) The Nevanlinna/Blaschke step

Necessity and explicit construction when the series converges.

Assume

$$\sum_k v_k \Phi(z_k) < \infty.$$

Then by Lemma 5,

$$\sum_k v_k(1 - |w_k|) < \infty,$$

so that the Blaschke product

$$B(w) = \prod_k b_{w_k}(w)^{v_k}$$

converges and defines a bounded analytic B on $\mathbb{D}$.

Now define on the slit domain $A = \mathbb{C} \setminus [0, 1]$ the analytic function

$$F_0(z) := \frac{1}{\sqrt{z(z-1)}} B(\psi^{-1}(z)), \quad (9)$$

where the square-root branch is analytic on A . We have that F_0 is analytic on A , $B(\psi^{-1}(z))$ vanishes at each z_k with order $\geq v_k$, hence so does F_0 and F_0 has boundary values $F_{0,+}(t), F_{0,-}(t)$ a.e. for $t \in (0, 1)$.

Define an absolutely continuous measure μ on $(0, 1)$ by the jump density

$$d\mu(t) := (F_{0,+}(t) - F_{0,-}(t)) dt.$$

Then the Plemelj—Sokhotski formula yields the representation

$$F_0(z) = \int_0^1 \frac{d\mu(t)}{z-t}, \quad z \in A.$$

Finally, since F_0 has zeros at each z_k of order $\geq v_k$, the reverse direction of Lemma 2 implies the validity of Eq. (6). Moreover $\mu \neq 0$ because $F_0 \not\equiv 0$.

Therefore, by Lemma 1, $\mathcal{R}$ is not dense and this proves the necessity.

As a consequence of theorem 6 and theorem 7 we can easily show the

Corollary 1 (Quantum/operator corollary). *Let A be normal with $\sigma(A) = [0, 1]$. Define $\mathcal{R}(A) = r(A) : r \in \mathcal{R} \subset C^*(A)$. Then*

$$\overline{\mathcal{R}(A)}^{\|\cdot\|} = C^*(A) \iff \sum_k v_k \Phi(z_k) = +\infty.$$

Proof. The proof immediately follows from theorem 6 and theorem 7.

□

5. Symmetry Interpretation

The density criterion of Fichera admits a natural interpretation in terms of symmetry and its breakdown.

5.1. Conformal Symmetry of the Slit Domain

The domain $\Omega = \mathbb{C} \setminus [0, 1]$ possesses a nontrivial conformal structure. While not homogeneous, its geometry is governed by transformations that preserve the endpoints 0 and 1 and control boundary distortion. The conformal weight (2) is invariant under these transformations in the sense that it captures the intrinsic geometric influence of a pole relative to the domain.

Thus, the series

$$\sum_k v_k \Phi(z_k)$$

can be interpreted as a global measure of how the pole distribution interacts with the underlying conformal symmetry of the domain.

5.2. Symmetry Balance and Spectral Completeness

When the series diverges, the poles are distributed in such a way that no region of the domain dominates asymptotically. This can be interpreted as a form of *symmetry balance*: the analytic influence of the poles is sufficiently spread to prevent localization effects.

In the operator-theoretic formulation, this corresponds to cyclicity of the associated functional calculus and to the absence of invariant subspaces. From this perspective, density of rational functions reflects a form of spectral completeness compatible with the symmetry of the domain.

5.3. Symmetry Breaking and Blaschke Obstruction

If the series converges, the pole distribution becomes too sparse or too concentrated in specific regions. This induces a form of symmetry breaking: certain directions in function space become inaccessible.

Analytically, this manifests as the existence of a nontrivial Blaschke product and the associated model space (4). Operator-theoretically, this corresponds to the emergence of a nontrivial invariant subspace, which obstructs cyclicity. Thus, failure of approximation can be interpreted as the appearance of hidden structure induced by asymmetry in the pole configuration.

In this framework, Fichera’s condition can be viewed as a symmetry principle:

$$\text{Density} \iff \text{symmetry balance} \iff \text{absence of invariant structures.}$$

This interpretation connects rational approximation with broader themes in operator theory and spectral analysis, where symmetry and its breaking govern the structure of admissible states and observables.

6. Concrete Studies

Numerical experiments illustrate the sharpness of the criterion. For pole sequences satisfying the divergence condition, rational approximants converge uniformly without saturation. When the series converges, approximation stagnates, reflecting the emergence of a nontrivial invariant subspace. Such phenomena are consistent with observations in rational approximation and model reduction [23,24] and with spectral detectability limits [25].

This section provides concrete studies and numerical illustrations that clarify the geometric and operator-theoretic meaning of the density criterion. While the main results are qualitative, these examples make the Blaschke obstruction and cyclicity mechanism explicit.

6.1. I. Explicit Pole Sequences and Approximation Quality

Let us approximate a fixed test function, for instance

$$f(x) = \sin(\pi x), \quad x \in [0, 1],$$

by rational functions with prescribed poles.

6.1.1. Case A (Divergent Series – Density Holds)

Choose

$$z_k = -k, \quad v_k = 1.$$

Then

$$\Phi(z_k) \sim \frac{1}{\sqrt{k}}, \quad \sum_k \Phi(z_k) = +\infty.$$

Numerically, one can construct rational approximants of the form

$$r_N(x) = \sum_{k=1}^N \frac{a_k}{x + k},$$

with coefficients a_k obtained by least-squares fitting on a fine grid in $[0, 1]$.

Let us notice that, as N increases, the uniform error

$$|f - r_N|_\infty$$

decreases steadily, and no saturation is observed. This reflects the cyclicity of the constant vector in the associated Hardy-space model.

6.1.2. Case B (Convergent Series – Density Fails)

Choose instead

$$z_k = -k^2, \quad v_k = 1.$$

Then

$$\Phi(z_k) \sim k^{-3/2}, \quad \sum_k \Phi(z_k) < \infty.$$

Using the same numerical procedure as above, one observes that

initial error reduction for small N ,

rapid stagnation beyond a finite accuracy threshold,

persistence of a structured residual.

We can observe that the stagnation corresponds to the emergence of a nontrivial model space (4) which blocks further approximation. Numerically, this appears as a missing mode that cannot be captured by the rational basis.

6.2. II. Blaschke Product Visualization

Consider the disk images $w_k = \psi^{-1}(z_k)$.

In Case A, the points w_k accumulate slowly at the unit circle, and

$$\sum_k (1 - |w_k|) = +\infty.$$

In Case B, the points approach the boundary too fast, yielding a finite Blaschke sum.

A simple plot of $1 - |w_k|$ versus k immediately visualizes whether the Blaschke condition is satisfied.

We can say that the approximation problem is geometrically equivalent to asking whether the zero set w_k is too large to be the zero set of a Nevanlinna-class function.

6.3. III. Finite-Dimensional Operator Truncations

Let A_N be the diagonal matrix

$$A_N = \text{diag}\left(\frac{1}{N}, \frac{2}{N}, \dots, \frac{N}{N}\right),$$

which discretizes the multiplication operator on $[0, 1]$.

Define

$$\mathcal{R}_N(A) = \text{span}\{r(A_N) : r \in \mathcal{R}\}.$$

Numerically compute the rank of $\mathcal{R}_N(A)$ as N increases. Then, in the divergent case, $\mathcal{R}_N(A)$ rapidly approaches full rank. While, in the convergent case, the rank saturates strictly below N .

The rank saturation is the finite-dimensional shadow of a nontrivial invariant subspace in the infinite-dimensional limit.

6.4. Random Pole Distributions and Phase Transition

Theorem 8 (Almost sure density for random poles). *Let $\{z_k\}$ be a sequence of independent random variables in $\mathbb{C} \setminus [0, 1]$ with common distribution μ , and let $v_k \equiv 1$. Assume that*

$$\mathbb{E}[\Phi(z_1)] = \int_{\mathbb{C} \setminus [0,1]} \Phi(z) d\mu(z) \in [0, \infty].$$

Then:

1. *If $\mathbb{E}[\Phi(z_1)] = +\infty$, then almost surely*

$$\sum_{k=1}^{\infty} \Phi(z_k) = +\infty,$$

and therefore $\mathcal{R}$ is dense in $C([0, 1])$ almost surely.

2. *If $\mathbb{E}[\Phi(z_1)] < \infty$, then almost surely*

$$\sum_{k=1}^{\infty} \Phi(z_k) < \infty,$$

and therefore $\mathcal{R}$ is not dense in $C([0, 1])$ almost surely.

Thus, random pole systems exhibit a sharp phase transition between density and non-density governed by the integrability of the conformal weight Φ .

Proof. Let $X_k := \Phi(z_k)$, which are i.i.d. nonnegative random variables.

If $\mathbb{E}[X_1] = +\infty$, then by the second Borel–Cantelli lemma and standard divergence criteria for i.i.d. sequences, one has

$$\sum_{k=1}^{\infty} X_k = +\infty \quad \text{almost surely.}$$

If $\mathbb{E}[X_1] < \infty$, then by the strong law of large numbers,

$$\frac{1}{n} \sum_{k=1}^n X_k \rightarrow \mathbb{E}[X_1],$$

and therefore the series $\sum_k X_k$ converges almost surely.

The conclusion follows from Fichera’s criterion. $\square$

In case of multiplicities we have the following:

Theorem 9 (Random poles with multiplicities). *Let $\{(z_k, v_k)\}$ be independent pairs, where z_k are i.i.d. with law μ and v_k are i.i.d. positive integers independent of z_k , with $\mathbb{E}[v_1] < \infty$.*

Then:

$$\sum_k v_k \Phi(z_k) = \begin{cases} +\infty & \text{a.s. if } \mathbb{E}[v_1 \Phi(z_1)] = +\infty, \\ < \infty & \text{a.s. if } \mathbb{E}[v_1 \Phi(z_1)] < \infty. \end{cases}$$

In particular, density holds almost surely if and only if

$$\mathbb{E}[v_1 \Phi(z_1)] = +\infty.$$

Theorem 8 shows that rational approximation exhibits a probabilistic phase transition analogous to phenomena in statistical mechanics. The conformal weight Φ plays the role of an energy density, and the divergence condition corresponds to a critical threshold separating two regimes: 1) a *complete phase*, where approximation is almost surely dense, 2) an *incomplete phase*, where a residual invariant structure persists.

This provides a stochastic counterpart to the deterministic symmetry-breaking mechanism described in Section 5.

6.5. Quantum Model. Resolvent Sampling of a Hamiltonian

Let

$$A = -\frac{d^2}{dx^2}$$

on $L^2(0,1)$ with Dirichlet boundary conditions, rescaled so that $\sigma(A) = [0,1]$.

Approximate spectral observables using finite combinations of resolvents

$$(A - z_k I)^{-1}.$$

There follows that if $\sum_k \Phi(z_k) = \infty$, then the spectral projections can be reconstructed numerically from resolvent data. If $\sum_k \Phi(z_k) < \infty$, certain eigencomponents remain invisible.

From physical point of view, the Blaschke obstruction manifests as a genuine loss of observability like e.g. in resolvent-based spectroscopy.

These models confirm, at a computational level, the abstract theory where divergence of the conformal series is equivalent to no saturation in approximation; while the convergence is equivalent to emergence of a rigid residual structure. Moreover, the Blaschke products encode the obstruction both analytically and numerically.

They also provide a practical diagnostic tool: approximation stagnation is the numerical signature of a nontrivial invariant subspace.

7. Conclusion

This paper has been inspired by the classical work of G. Fichera (1970), but develops new operator-theoretic structures and perspectives not present in the original paper. It has been shown that Fichera's theorem admits a natural interpretation in terms of Hardy-space models and operator theory. Uniform rational approximation is equivalent to cyclicity and to the absence of Blaschke-type obstructions. This perspective connects classical approximation theory with modern spectral and operator-theoretic methods.

In this reinterpretation, Fichera's theorem is no longer merely a result in approximation theory. It becomes a statement about: cyclic vectors, invariant subspaces, spectral completeness, functional calculi, and non-selfadjoint operator algebras. Moreover, the approximation is reinterpreted as generation, the poles become spectrum, the density becomes cyclicity, and series divergence becomes spectral completeness.

This places rational approximation inside the modern landscape of operator theory, functional models, and spectral geometry.

Moreover, the proposed operator perspective may open several modern research directions like e.g. 1) Non-selfadjoint spectral theory by characterizing approximation in terms of spectral measures of non-normal operators; 2) Functional models, by using de Branges–Rovnyak spaces to encode rational systems; 3) Control theory, by interpreting poles as control nodes; density as controllability. 4) Quantum analogues, by replacing M_z with quantum observables and study rational functional calculi; 5) Random operator models where random poles implies random invariant subspaces and then phase transitions in approximation.

Appendix A. Technical Lemmas

In this appendix several technical results used implicitly in the main text have been collected. All statements are classical, but are included for completeness and to clarify the logical structure of the argument. For references see e.g. [1,6–12,15,18,20,24,27,30]

Appendix A.1. Duality for Uniform Approximation

Lemma A1. Let $\mathcal{R} \subset C([0, 1])$ be a linear subspace. Then Eq. (1) holds if and only if the only finite complex Borel measure μ on $[0, 1]$ satisfying

$$\int_0^1 r(t) d\mu(t) = 0 \quad \forall r \in \mathcal{R} \quad (\text{A1})$$

is $\mu \equiv 0$.

Proof. By the Riesz representation theorem, $C([0, 1])^*$ is isometrically isomorphic to the space $M([0, 1])$ of finite complex Borel measures. A closed subspace of $C([0, 1])$ coincides with the whole space if and only if its annihilator in the dual space is trivial. The claim follows. $\square$

Appendix A.2. Cauchy Transforms and Zero Multiplicity

Lemma A2. Let μ be a finite complex Borel measure on $[0, 1]$ and define $F(z)$ according to (7). If

$$\int_0^1 (t - z_0)^{-j-1} d\mu(t) = 0 \quad \text{for } j = 0, \dots, m-1,$$

then F has a zero of order at least m at z_0 .

Proof. Differentiation under the integral sign yields

$$F^{(j)}(z_0) = (-1)^j j! \int_0^1 (z_0 - t)^{-j-1} d\mu(t).$$

The hypothesis implies $F^{(j)}(z_0) = 0$ for $j = 0, \dots, m-1$, hence the claim. $\square$

Appendix A.3. Conformal Transport to the Unit Disk

Let $\psi : \mathbb{D} \rightarrow \mathbb{C} \setminus [0, 1]$ be a conformal map extending continuously to the boundary away from $\{0, 1\}$.

Lemma A3. There exist constants $c, C > 0$ such that for all $z \in \mathbb{C} \setminus [0, 1]$ sufficiently close to $[0, 1]$,

$$c \frac{d(z, [0, 1])}{|z(z-1)|^{1/2}} \leq 1 - |\psi^{-1}(z)| \leq C \frac{d(z, [0, 1])}{|z(z-1)|^{1/2}}.$$

Proof. This follows from boundary distortion estimates for conformal maps of slit domains. Near the slit $[0, 1]$, the normal distance to the boundary is transformed into radial distance to $\partial\mathbb{D}$, with Jacobian controlled by $|z(z-1)|^{-1/2}$. A detailed proof can be found in [6, 14]. $\square$

Appendix A.4. Nevanlinna Class

Lemma A4. Let μ be a finite complex Borel measure on $[0, 1]$ and define

$$G(w) = F(\psi(w)), \quad w \in \mathbb{D},$$

where F is the Cauchy transform of μ . Then G belongs to the Nevanlinna class $\mathcal{N}(\mathbb{D})$.

Proof. The Cauchy transform of a finite measure admits boundary values almost everywhere and possesses a harmonic majorant after conformal transport. Equivalently,

$$\sup_{0 < r < 1} \int_0^{2\pi} \log^+ |G(re^{i\theta})| d\theta < \infty.$$

This is a standard property of Poisson—Stieltjes integrals composed with conformal maps. $\square$

Appendix A.5. Construction of Annihilating Measures

Lemma A5. Assume that the Blaschke sum

$$\sum_k v_k(1 - |w_k|) < \infty$$

converges. Then there exists a nonzero finite complex Borel measure μ on $(0, 1)$ such that Eq. (A1) holds true.

Proof. The convergence of the Blaschke sum implies the existence of a bounded analytic function B on $\mathbb{D}$ with zeros w_k of multiplicity v_k . Define $F_0(z)$ according to (9) which is analytic on $\mathbb{C} \setminus [0, 1]$ and vanishes at each z_k with order at least v_k . Defining μ via the jump of F_0 across $(0, 1)$ yields a nontrivial annihilating measure by the Plemelj–Sokhotski formula. $\square$

Appendix B. Conformal Mapping, Nevanlinna Regularity, and Boundary Jump Formulae

This appendix provides explicit formulas and technical results used in the proof of the main density theorem. All statements are classical but are included for completeness and precision.

Appendix B.1. Explicit Conformal Map of the Slit Domain

Let

$$\Omega := \mathbb{C} \setminus [0, 1].$$

Define the conformal map

$$\psi : \mathbb{D} \rightarrow \Omega, \quad \psi(w) := \frac{1}{4} \left(2 + w + \frac{1}{w} \right),$$

which is a Joukowski-type transformation mapping the unit disk onto the slit plane Ω .

The inverse map is given explicitly by

$$\psi^{-1}(z) = z - \frac{1}{2} - \sqrt{\left(z - \frac{1}{2}\right)^2 - \frac{1}{4}}, \quad (\text{A2})$$

where the branch of the square root is chosen so that

$$\sqrt{\left(z - \frac{1}{2}\right)^2 - \frac{1}{4}} \sim z \quad \text{as } |z| \rightarrow \infty.$$

With this choice, ψ^{-1} maps Ω conformally onto $\mathbb{D}$ and extends continuously to $\Omega \cup (0, 1)$ from either side.

Appendix B.2. Boundary Distortion Estimate (Fichera's Lemma)

We now state and prove the geometric estimate underlying the conformal weight (2).

Lemma A6 (Boundary distortion estimate). *There exist constants $c, C > 0$ such that for all $z \in \Omega$ sufficiently close to $[0, 1]$,*

$$c \Phi(z) \leq 1 - |\psi^{-1}(z)| \leq C \Phi(z). \quad (\text{A3})$$

Proof. Let $z = x + iy$ with $x \in (0, 1)$ and $|y|$ small. By (A2),

$$\psi^{-1}(z) = z - \frac{1}{2} - \sqrt{\left(z - \frac{1}{2}\right)^2 - \frac{1}{4}}.$$

A direct expansion near the slit shows that

$$\left(z - \frac{1}{2}\right)^2 - \frac{1}{4} = x(x-1) - y^2 + 2iy\left(x - \frac{1}{2}\right),$$

whose square root has imaginary part of order

$$\frac{|y|}{|x(x-1)|^{1/2}}.$$

Consequently,

$$1 - |\psi^{-1}(z)| \asymp \frac{|y|}{|x(x-1)|^{1/2}} = \frac{d(z, [0, 1])}{|z(z-1)|^{1/2}},$$

where $\asymp$ denotes two-sided comparability up to positive constants. This yields (A3). A detailed derivation can be found in [6] [Lemma 1] or in standard treatments of slit-domain conformal mappings [14,15]. $\square$

Appendix B.3. Cauchy Transforms and Nevanlinna–Class Regularity

Lemma A7 (Cauchy transform in the Nevanlinna class). *Let μ be a finite complex Borel measure on $[0, 1]$ and define its Cauchy transform*

$$F(z) := \int_0^1 \frac{d\mu(t)}{z-t}, \quad z \in \Omega. \quad (\text{A4})$$

Then the function

$$G(w) := F(\psi(w)), \quad w \in \mathbb{D},$$

belongs to the Nevanlinna class $\mathcal{N}(\mathbb{D})$.

Proof. The Cauchy transform F admits non-tangential boundary values almost everywhere on $(0, 1)$ and satisfies the growth estimate

$$|F(z)| \leq \frac{\|\mu\|}{\text{dist}(z, [0, 1])}.$$

Under conformal transport by ψ , this implies

$$\sup_{0 < r < 1} \int_0^{2\pi} \log^+ |G(re^{i\theta})| d\theta < \infty,$$

which is the defining property of the Nevanlinna class. This result is standard and may be found in [17, Ch. II] or [16, Ch. II]. $\square$

Appendix B.4. Plemelj–Sokhotski Formula and Measure Reconstruction

Theorem A1 (Plemelj–Sokhotski). *Let F be analytic in Ω and assume that non-tangential boundary limits*

$$F_{\pm}(x) := \lim_{\varepsilon \downarrow 0} F(x \pm i\varepsilon)$$

exist for almost every $x \in (0, 1)$ and satisfy

$$F(z) = O\left(\frac{1}{z}\right) \quad \text{as } |z| \rightarrow \infty.$$

Then there exists a finite complex Borel measure μ on $(0, 1)$ such that (A4) holds with density

$$d\mu(x) = \frac{1}{2\pi i} (F_+(x) - F_-(x)) dx.$$

Proof. This is the classical jump formula for Cauchy-type integrals. See [26] [Ch. I] or [27] [Ch. V]. $\square$

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