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Posted Date: 24 June 2024

doi: 10.20944/preprints202406.1672.v1

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On Certain Rough Marcinkiewicz Integral Operators with Grafakos-Stefanov Kernels on Product Spaces

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Abstract: In this paper, several classes of rough Marcinkiewicz integral operators along surfaces of revolution on product spaces are investigated. We prove the L^p boundedness of these operators when their kernels functions belong to a class of functions related to a class of functions introduced by Grafakos-Stefanov. The results in this paper extend and improve several known results on Marcinkiewicz integrals over a symmetric space.

Keywords: L^p bound; rough Marcinkiewiz integrals; surfaces of revolution; product domains; Grafakos-Stefanov class

1. Introduction

Let $\mathbb{R}^q$ ($q = m$ or n) be the $2 \leq q$-Euclidean space and $\mathbb{S}^{q-1}$ be the unit sphere in $\mathbb{R}^q$ equipped with the normalized Lebesgue surface measure $d\sigma_q(\cdot) \equiv d\sigma$. Also, let $z' = z/|z|$ for $z \in \mathbb{R}^q \setminus \{0\}$.

Let $\mho$ be a measurable function defined on $\mathbb{R}^m \times \mathbb{R}^n$, integrable over $\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}$ and satisfy the following:

$$\mho(rx, sy) = \mho(x, y), \ \forall r, s > 0, \tag{1}$$

$$\int_{\mathbb{S}^{m-1}} \mho(x, y) d\sigma(x) = \int_{\mathbb{S}^{n-1}} \mho(x, y) d\sigma(y) = 0. \tag{2}$$

For an appropriate mapping $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, we consider the parametric Marcinkiewicz integral operator $\mathfrak{M}_{\mho,\Phi}$ along the surface of revolution $\Lambda_\Phi(u, v) = (u, v, \Phi(|u|, |v|))$ defined, initially for $g \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$, by

$$\mathfrak{M}_{\mho,\Phi}(g)(u, v, w) = \left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} |F_{r,s}(g)(u, v, w)|^2 \frac{drds}{rs} \right)^{1/2}, \tag{3}$$

where

$$F_{r,s}(g)(u, v, w) = \frac{1}{rs} \int_{|y| \leq s} \int_{|x| \leq r} f(u - x, v - y, w - \Phi(|x|, |y|)) \frac{\mho(x, y)}{|x|^{m-1} |y|^{n-1}} dxdy.$$

We point out her that the Marcinkiewicz integral $\mathfrak{M}_{\mho,\Phi}$ is a natural generalization of the Marcinkiewicz inegral $\mathfrak{M}_{\mho}^{\phi}$ along surface of revolution $\Lambda_\Phi(u) = (u, \Phi(|u|))$ in the one parameter setting which is given by

$$\mathfrak{M}_{\mho}^{\phi}(g)(u, u_{m+1}) = \left(\int_{\mathbb{R}_+} \left| \frac{1}{r} \int_{|x| \leq r} g(u - x, u_{m+1} - \phi(|x|)) \frac{\mho(x)}{|x|^{m-1}} dx \right|^2 \frac{dr}{r} \right)^{1/2}. \tag{4}$$

The investigation of the L^p boundedeness of $\mathfrak{M}_{\mho}^{\phi}$ under certain assumptions on $\mho$ and ϕ has received a large amount of attention by many mathematicians. For a sample of some known results relevant to our study, we refer the readers to see [1–7].

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Let $G_\alpha(\mathbb{S}^{m-1})$ (for $\alpha > 0$) be the class of all functions $\mho \in L^1(\mathbb{S}^{m-1})$ that satisfy the condition

$$\sup_{\xi \in \mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} |\mho(x)| \log^\alpha(|\xi \cdot x|^{-1}) d\sigma(x) < \infty. \tag{5}$$

It is worth mentioning that the class $G_\alpha(\mathbb{S}^{m-1})$ was defined by Walsh in [8] and developed by Grafakos and Stefanov in [9].

An interesting result related to our work in the one parameter setting was obtained in [10] as described in the following theorem:

Theorem A. Let $\mho \in G_\alpha(\mathbb{S}^{m-1})$ for some $\alpha > 1/2$. Then $\mathfrak{M}_\mho^\phi$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R})$ for all $p \in (\frac{1+2\alpha}{2\alpha}, 1+2\alpha)$ if one of the following conditions is satisfied:

(a) $\phi \in C^1(\mathbb{R}_+)$, ϕ' is increasing and convex function with either $m = 2$ or $m \geq 3$ and $\phi'(0) = 0$.

(b) ϕ is a polynomial with either $m = 2$ or $m \geq 3$ and $\phi'(0) = 0$.

Our focus will be on the operator $\mathfrak{M}_{\mho,\Phi}$. If $\Phi \equiv 0$, we denote $\mathfrak{M}_{\mho,\Phi}$ by $\mathfrak{M}_\mho$ which is the classical Marcinkiewicz integral on product domains. The study of the boundedness of $\mathfrak{M}_\mho$ started by Ding in [11] and then continued by many authors. For a sample of past studies and for more information about the applications and development of the operator $\mathfrak{M}_\mho$, we refer the readers to consult [14–21] and the references therein.

For $\alpha > 0$, let $G_\alpha(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ be the set of all functions $\mho \in L^1(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ which satisfy Grafakos and Stefanov condition:

$$\sup_{(\xi,\zeta) \in \mathbb{S}^{m-1} \times \mathbb{S}^{n-1}} \iint_{\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}} |\mho(x,y)| \log^\alpha(|\xi \cdot x|^{-1}) \log^\alpha(|\zeta \cdot y|^{-1}) d\sigma(x) d\sigma(y) < \infty. \tag{6}$$

By following the same arguments as those in [9], we get

$$\bigcup_{\kappa > 1} L^\kappa\left(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}\right) \nsubseteq G_\alpha\left(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}\right) \ for\ any\ \alpha > 0, \tag{7}$$

$$\bigcap_{\alpha > 0} G_\alpha\left(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}\right) \nsubseteq L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \nsubseteq \bigcup_{\alpha > 0} G_\alpha\left(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}\right). \tag{8}$$

Very recently, in [23] the authors studied the related singular integral operators along surfaces of revolution $\mathcal{T}_{\mho,\Phi}$ which are given by

$$\mathcal{T}_{\mho,\Phi}(g)(u,v,w) = \iint_{\mathbb{R}^m \times \mathbb{R}^n} f(u-x, v-y, w-\Phi(|x|,|y|)) \frac{\mho(x,y)}{|x|^m |y|^n} dxdy, \tag{9}$$

where $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is an appropriate mapping. When $\mho \in G_\alpha(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ for some $\alpha > 1/2$, the authors of [23] showed that $\mathcal{T}_{\mho,\Phi}$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$ for all $p \in (\frac{1+2\alpha}{2\alpha}, 1+2\alpha)$ under various assumptions on Φ.

In light of the assumptions imposed on Φ in [23] and of the results in [10] concerning the boundedness of Marcinkiewicz operator $\mathfrak{M}_\mho^\phi$ in the one parameter setting whenever $\mho \in G_\alpha(\mathbb{S}^{m-1})$ as well as of the results in [21] concerning the boundedness of Marcinkiewicz operator $\mathfrak{M}_{\mho,\Phi}$ whenever $\mho \in L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \cup B_\kappa^{(0,0)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$, a question arises naturally is the following:

Question: Under the same conditions on Φ in [23], does the operator $\mathfrak{M}_{\mho,\Phi}$ satisfy the L^p boundedness provided that $\mho \in G_\alpha(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ for some $\alpha > 1/2$?

Our main purpose in this paper is on giving a positive answer to this question. Indeed, we obtain the following:

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Theorem 1. *Assume that $\mho$ belongs to the space $G_\alpha(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $\alpha > 1/2$ and satisfies the conditions (1)-(2). Assume that $\Phi \in C^1(\mathbb{R}_+\times\mathbb{R}_+)$. If one of the following conditions holds, then $\mathfrak{M}_{\mho,\Phi}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ for all $p \in (\frac{1+2\alpha}{2\alpha}, 1+2\alpha)$.*
(i) $m = 2 = n$, $D_t\Phi(t,\ell)$ and $D_\ell\Phi(t,\ell)$ are convex increasing functions,
(ii) $m = 2$, $n \geq 3$, $D_t\Phi(t,\ell)$ and $D_\ell\Phi(t,\ell)$ are convex increasing functions with $D_\ell(t,0) = 0$,
(iii) $n = 2$, $m \geq 3$, $D_t\Phi(t,\ell)$ and $D_\ell\Phi(t,\ell)$ are convex increasing functions with $D_t\Phi(0,\ell) = 0$,
(iv) $m \geq 3$, $n \geq 3$, $\Phi_t(t,\ell)$ and $D_\ell\Phi(t,\ell)$ are convex increasing functions with $D_t\Phi(0,\ell) = 0 = D_\ell\Phi(t,0)$.

Theorem 2. *Assume that $\mho \in G_\alpha(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $\alpha > 1/2$ and satisfies the conditions (1)-(2). Assume that $\Phi(t,\ell) = \sum_{i=0}^{d_1}\sum_{j=0}^{d_2} a_{j,i}t^{\alpha_i}\ell^{\beta_j}$ is a generalized polynomial on $\mathbb{R}^2$, where the exponents in each set $\{\alpha_i : 0 \leq i \leq d_1\}$ and $\{\beta_j : 0 \leq j \leq d_2\}$ are distinct positive numbers. Then $\mathfrak{M}_{\mho,\Phi}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ for all $p \in (\frac{1+2\alpha}{2\alpha}, 1+2\alpha)$ whenever one of the following conditions holds:*
(i) $\alpha_i \neq 1$ and $\beta_j \neq 1$ for all $0 \leq i \leq d_1$ and $0 \leq j \leq d_2$,
(ii) There is $\alpha_{i_0} = 1$ for some $0 \leq i_0 \leq d_1$ and $\beta_j \neq 1$ for all $0 \leq j \leq d_2$; and if $m \geq 3$, we need $D_t\Phi(0,\ell) = 0$,
(iii) There is $\beta_{j_0} = 1$ for some $0 \leq j_0 \leq d_2$ and $\alpha_i \neq 1$ for all $0 \leq i \leq d_1$; and if $n \geq 3$, we need $D_\ell\Phi(t,0) = 0$,
(iv) There exists $\alpha_{i_0} = 1$ for some $0 \leq i_0 \leq d_1$ and there exists $\beta_{j_0} = 1$ for some $0 \leq j_0 \leq d_2$. Moreover, whenever $m \geq 3$ we need $D_t\Phi(0,\ell) = 0$ and whenever $n \geq 3$ we need $D_\ell\Phi(t,0) = 0$.

Theorem 3. *Assume that $\mho$ is given as in Theorem 1. Assume that $\Phi(t,\ell) = \phi(t)P(\ell)$, where $\phi(t)$ is in $C^1(\mathbb{R}_+)$, ϕ' is increasing and convex function; and P is a generalized polynomials given by $P(\ell) = \sum_{j=0}^{d_2} a_j\ell^{\beta_j}$. Then $\mathfrak{M}_{\mho,\Phi}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ for all $p \in (\frac{1+2\alpha}{2\alpha}, 1+2\alpha)$ if one of the following conditions is satisfied:*
(i) $\beta_j \neq 1$ for all $0 \leq j \leq d_2$, and if $m \geq 3$ we need $\phi'(0) = 0$,
(ii) There is $\beta_{j_0} = 1$ for some $0 \leq j_0 \leq d_2$, and if $m, n \geq 3$ we need $\phi'(0) = 0 = P(0)$.

Theorem 4. *Assume $\Phi(t,\ell) = \phi_1(t) + \phi_2(\ell)$, where $\phi_j(\cdot)$ $(j = 1,2)$ is either a generalized polynomial or is in $C^1(\mathbb{R}_+)$, ϕ_j' is increasing and convex function. If $\mho \in G_\alpha(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $\alpha > 1/2$ and one of the following conditions is held, then $\mathfrak{M}_{\mho,\Phi}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ for all $p \in (\frac{1+2\alpha}{2\alpha}, 1+2\alpha)$.*
Case 1: $\phi_j(\cdot)$ $(j = 1,2)$ is in $C^1(\mathbb{R}_+)$, ϕ_j' is increasing and convex function.
(i) $m = 2$, $n = 2$,
(ii) $m = 2$, $n \geq 3$ and $\phi_1'(0)$,
(iii) $n = 2$, $m \geq 3$ and $\phi_2'(0)$,
(iv) $m \geq 3$, $n \geq 3$ and $\phi_1'(0) = 0 = \phi_2'(0)$.
Case 2: $\phi_1(\cdot)$ is in $C^1(\mathbb{R}_+)$, $\phi_2(\cdot)$ is increasing and convex function; $\phi_2(\cdot)$ is a generalized polynomial on $\mathbb{R}$ given by $\phi_2(\ell) = \sum_{j=0}^{d_2} a_j\ell^{\beta_j}$, where the exponents in the set $\{\beta_j : 0 \leq j \leq d_2\}$ are distinct positive numbers.
(i) $\beta_j \neq 1$ for all $0 \leq j \leq d_2$, and if $n \geq 3$ we need $\phi_1'(0) = 0$,
(ii) There is $\beta_{j_0} = 1$ for some $0 \leq j_0 \leq d_2$, and if $m, n \geq 3$ we need $\phi_1'(0) = 0 = \phi_2'(0)$.

Theorem 5. *Assume that $\Phi(t,\ell) = \phi(t\ell)$ for all $t,\ell > 0$, where $\phi \in C^1(\mathbb{R}_+)$. If $\mho \in G_\alpha(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $\alpha > 1/2$ and one of the following conditions is satisfied, then $\mathfrak{M}_{\mho,\Phi}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ for all for all $p \in (\frac{1+2\alpha}{2\alpha}, 1+2\alpha)$.*
(i) $m = 2 = n$ and $\phi'(\cdot)$ is an increasing and convex function,
(ii) If m or $n \geq 3$ and $\phi'(\cdot)$ is an increasing and convex function with $\phi'(0) = 0$.

We point out that the surfaces of revolution $\Lambda_\Phi(u,v) = (u, v, \Phi(|u|,|v|))$ considered in Theorems 1–5 cover several important natural classical surfaces. For example, our theorems allow surfaces of the type Λ_Φ, where $\Phi(t,\ell) = t^2\ell^2(e^{-1/t} + e^{-1/\ell}), t,\ell > 0)$; $\Phi(t,\ell) = t^\alpha\ell^\beta$ with $\alpha,\beta > 0$; $\Phi(t,\ell) = P(t,\ell)$

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is a polynomial; $\Phi(t,\ell) = \phi_1(t)\phi_2(\ell)$, where for $i \in \{1,2\}$, each $\phi_i \in C^2(\mathbb{R}_+)$ is a convex increasing function with $\phi_i(0) = 0$.

Henceforward, the constant C denotes a positive real constant which is not necessary the same at each occurrence but it is independent of the essential variables.

2. Some Lemmas

For a suitable mapping $\Phi(r,s)$ on $\mathbb{R}_+ \times \mathbb{R}_+$, we define the family of measures $\{\lambda^{k,j}_{\mho,\Phi,r,s} := \lambda^{k,j}_{r,s} : k,j \in \mathbb{Z}, r,s \in \mathbb{R}_+\}$ and its related maximal operator λ^* on $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ by

$$\iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} g d\lambda^{k,j}_{r,s} = \frac{1}{2^{k+j} rs} \int_{2^j s\leq|u|\leq 2^{j+1}s} \int_{2^k r\leq|v|\leq 2^{k+1}r} g(v,u,\Phi(|v|,|u|))K_{\mho,h}(v,u)dvdu$$

and

$$\lambda^*(g)(x,y,w) = \sup_{k,j\in\mathbb{Z}} \sup_{r,s\in\mathbb{R}_+} ||\lambda^{k,j}_{r,s}| * g(x,y,w)|,$$

where $|\lambda^{k,j}_{r,s}|$ is defined in the same way as $\lambda^{k,j}_{r,s}$ but with replacing $\mho$ by $|\mho|$.

The following lemma can be proved directly by the results in [24,25], see also [23].

Lemma 1. *Let $P : \mathbb{R}^+ \to \mathbb{R}^q$ be a function given by $P(t) = (c_1 t^{\gamma_1}, \cdots, c_n t^{\gamma_q})$, where c_j's are real numbers and γ_j's are distinct positive exponents (not necessarily integer). Define the maximal function related to P by*

$$\mathcal{M}_P g(z) = \sup_{h>0} \frac{1}{h} \int_0^h |g(z-P(t))|dt.$$

Then, for all $1 < p \leq \infty$, there exists a constant $C_p > 0$ (independent of g and c_j's) such that

$$\|\mathcal{M}_P(g)\|_{L^p(\mathbb{R}^q)} \leq C_p |g\|_{L^p(\mathbb{R}^q)}.$$

We shall need the following estimate from [26].

Lemma 2. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a C^1 function such that φ' is convex and increasing with $\varphi'(0) = 0$. Then, there exists a positive constant C such that*

$$\left|\int_1^b e^{-i(2^k tr+\eta\varphi(2^k t))}dt\right| \leq C\left|2^k r\right|^{-1/2}$$

holds for all $b \geq 1$, $r,\eta \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Lemma 3. *[27] Let $\Gamma(x) = \sum_{|\alpha|\leq d} b_\alpha x^\alpha$, where $b_\alpha \in \mathbb{R}$. Then*

$$\left|\int_{[0,1]^q} e^{-i\Gamma(x)}dx\right| \leq C_{d,q}\left(\sum_{0<|\alpha|\leq d} |b_\alpha|\right)^{-1/d}.$$

Now we need to prove the following lemma which will play a key role on proving our main results.

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Lemma 4. *Let $\mho \in G_\alpha(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ for $\alpha > 1/2$ and satisfy the conditions (1)-(2). Suppose that Φ is given as in any of Theorems 1-5. Then there is a real number $C > 0$ such that the inequalities*

$$\begin{aligned}
\left\|\lambda_{r,s}^{k,j}\right\| &\leq C, &(10)\\
\left|\hat{\lambda}_{r,s}^{k,j}(\xi,\zeta,\eta)\right| &\leq C\min\left\{\left|2^k\xi\right|, \left(\log|2^k\xi|\right)^{-\alpha}\right\}, &(11)\\
\left|\hat{\lambda}_{r,s}^{k,j}(\xi,\zeta,\eta)\right| &\leq C\min\left\{\left|2^j\zeta\right|, \left(\log|2^j\zeta|\right)^{-\alpha}\right\} &(12)
\end{aligned}$$

hold for all $k, j \in \mathbb{Z}$ and $r, s > 0$, where $\left\|\lambda_{r,s}^{k,j}\right\|$ stands for the total variation of $\lambda_{r,s}^{k,j}$.

Proof. By the definition of $\lambda_{r,s}^{k,j}$, it is easy to prove the inequality (10). Next, by Hölder's inequality, we have

$$\begin{aligned}
\left|\hat{\lambda}_{r,s}^{k,j}(\xi,\zeta,\eta)\right| &\leq C\int_1^2\int_1^2\left|\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}} e^{-i\{2^k rtx\cdot\xi+2^j s\ell y\cdot\zeta+\Phi(2^k rt,2^j s\ell)\eta\}}\right.\\
&\times \left.\mho(x,y)d\sigma(x)d\sigma(y)\right|\frac{dtd\ell}{t\ell}\\
&\leq C\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}|\mho(x,y)|\int_1^2 I_k(\xi,\eta,x,t)\frac{d\ell}{\ell}d\sigma(x)d\sigma(y), \qquad (13)
\end{aligned}$$

where

$$I_k(\xi,\eta,x,t) = \int_1^2 e^{-i\{2^k rtx\cdot\xi+\Phi(2^k rt,2^j s\ell)\eta\}}\frac{dt}{t}.$$

Hence, we have

$$I_k(\xi,\eta,x,t) = \left|\int_1^2 e^{-i\{2^k rtx\cdot\xi+\eta 2^k rtD_t\Phi(0,2^j s\ell)+[\Phi(2^k rt,2^j s\ell)-2^k rtD_t\Phi(0,2^j s\ell)]\eta\}}\frac{dt}{t}\right|.$$

Now we need to consider several cases.

Case 1. If Φ is given as in Theorem 1, then by Lemma 2, we deduce that

$$I_k(\xi,\eta,x,t) \leq \left|2^k r|\xi||x\cdot\xi'+|\eta||\xi|^{-1}D_t\Phi(0,2^j s\ell)\right|^{-1/2}. \qquad (14)$$

Let $\delta = \min\left\{2, \eta|\xi|^{-1}D_t\Phi(0,2^j s\ell)\right\}sgn(\eta D_t\Phi(0,2^j s\ell))$. Combine the inequality (14) with the trivial estimate $|I_k(\xi,\eta,x,t)| \leq 1$ and use the fact that $(t/\log^\alpha t)$ is increasing over the interval $(2^\alpha,\infty)$, we get that

$$|I_k(\xi,\eta,x,t)| \leq C\frac{\log^\alpha(2|\xi'\cdot x+\delta|^{-1})}{\log^\alpha|2^k r\xi|} \quad if\ |2^k r\xi| > 2^\alpha. \qquad (15)$$

Thus, if $n \geq 3$, then by the additional assumption $D_t\Phi(0,2^j s\ell) = 0$, we get $\delta = 0$. Hence,

$$|I_k(\xi,\eta,x,t)| \leq C\frac{\log^\alpha(2|\xi'\cdot x|^{-1})}{\log^\alpha|2^k r\xi|} \quad if\ |2^k r\xi| > 2^\alpha. \qquad (16)$$

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Therefore, by (13) we acquire that

$$\begin{aligned}\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| &\leq C\left(\log\left|2^k r\xi\right|\right)^{-\alpha}\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}|\mho(x,y)|\log^{\alpha}(2|\xi'\cdot x|^{-1})d\sigma(x)d\sigma(y)\\ &\leq C\left(\log\left|2^k r\xi\right|\right)^{-\alpha}\quad if\ |2^k r\xi|>2^{\alpha}.\end{aligned}\tag{17}$$

Now, if $n=2$, then follow the argument similar to that in [28]; we may assume that $\delta>0$. Set $\delta'=\min\{1,\delta\}$ and $\theta=arcsin(\delta')$, and let e_+ denote the vector obtained by rotating ξ' by angles θ and e_- denote the vector obtained by rotating ξ' by angles $-\theta$. Then we conclude that a constant $C_0\in(0,1)$ exists such that

$$\xi'\cdot x+\delta'\geq C_0\min\{|x\cdot e_-|^2,|x\cdot e_+|^2\}$$

for $x\in\mathbb{S}^1$. So, the estimate in (16) holds and therefore (17) holds for the case $n=2$.

We notice her that the conclusions of this lemma when $n=2, m=2$ are proved without the additional assumptions $D_t\Phi(0,\ell)=0$ and $D_\ell\Phi(t,o)=0$.

In the same manner, we can prove that

$$\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| \leq C\left(\log\left|2^j s\zeta\right|\right)^{-\alpha}\quad if\ |2^j s\zeta|>2^{\alpha}.\tag{18}$$

Now by using the cancellation conditions on $\mho$, we get

$$\begin{aligned}\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| &\leq C\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}|\mho(x,y)|\int_1^2|J_k(\xi,\eta,x,t)|\frac{d\ell}{\ell}d\sigma(x)d\sigma(y)\\ &\leq C\left|2^k r\xi\right|,\end{aligned}\tag{19}$$

where

$$J_k(\xi,\eta,x,t)=\int_1^2 e^{-i\{2^k rtx\cdot\xi+\Phi(2^k rt,2^j s\ell)\eta\}}-e^{\Phi(2^k rt,2^j s\ell)\eta}\frac{dt}{t}.$$

Similarly, we obtain that

$$\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| \leq C\left|2^k r\xi\right|.\tag{20}$$

Consequently, by (17)–(20), we finish the proof of this lemma for the first case.

We notice here that the proofs of (19)–(20) do not depend on Φ.

Case 2. If Φ is given as in Theorem 2. Notice that

$$\Phi(t,\ell)=\sum_{j=1}^{d_2}\sum_{i=1}^{d_1}a_{j,i}t^{\alpha_i}\ell^{\beta_j}.$$

Hence, $\Phi(t,\ell)$ can be written as

$$\Phi(t,\ell)=P_\ell(t)=\sum_{i=0}^{d_1}b_i(\ell)t^{\alpha_i}$$

and

$$\Phi(t,\ell)=Q_t(\ell)=\sum_{j=0}^{d_2}c_j(t)\ell^{\beta_j},$$

where

$$b_i(\ell)=\sum_{j=0}^{d_2}a_{j,i}\ell^{\beta_j}\quad and\quad c_j(t)=\sum_{i=0}^{d_1}a_{j,i}t^{\alpha_i}.$$

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If $\alpha_i \neq 1$ for all $1 \leq i \leq d_1$, then by Lemma 3, we get that

$$I_k(\xi,\eta,x,t) \leq \left|2^k|\xi|(\xi'\cdot x)\right|^{-1/d_1}. \tag{21}$$

So, by following the same argument as above, we get

$$\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| \leq \left(\log\left|2^k r\xi\right|\right)^{-\alpha} \quad if\ |2^k r\xi| > 2^\alpha. \tag{22}$$

If $\alpha_{i_0} = 1$ for some $1 \leq i_0 \leq d_1$. Since

$$I_k(\xi,\eta,x,t) = \left|\int_1^2 e^{-i2^k rt|\xi|\{x\cdot\xi'+\eta|\xi|^{-1}b_{i_0}(2^j s\ell)+\eta\sum_{i=0,i\neq i_0}^{d_1} b_i(2^j s\ell)(2^k rt))^{\alpha_i}\}}\frac{dt}{t}\right|,$$

then by invoking Lemma 3 we obtain

$$|I_k(\xi,\eta,x,t)| \leq C\left|2^k r|\xi|\left(x\cdot\xi'+\eta|\xi|^{-1}b_{i_0}(2^j s\ell)\right)\right|^{-1/D_t}. \tag{23}$$

Set $Y = \min\left\{2, \eta|\xi|^{-1}b_{i_0}(2^j s\ell)\right\}sgn(b_{i_0}(2^j s\ell)\eta)$. By combining (23) with the trivial estimate $|I_k(\xi,\eta,x,t)| \leq 1$ and using the fact that $(t/\log^\alpha t)$ is increasing over the interval $(2^\alpha,\infty)$, we get

$$|I_k(\xi,\eta,x,t)| \leq C\frac{\log^\alpha(2|\xi'\cdot x+Y|^{-1})}{\log^\alpha|2^k r\xi|} \quad if\ |2^k r\xi| > 2^\alpha. \tag{24}$$

When $n \geq 3$, by the additional condition $D_t\Phi(0,2^j s\ell) = 0$, we deduce that $Y = 0$; which yields that

$$|I_k(\xi,\eta,x,t)| \leq C\frac{\log^\alpha(2|\xi'\cdot x|^{-1})}{\log^\alpha|2^k r\xi|} \quad if\ |2^k r\xi| > 2^\alpha. \tag{25}$$

By the last estimate and (13) we get

$$\begin{aligned}\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| &\leq C\left(\log\left|2^k r\xi\right|\right)^{-\alpha}\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}|\mho(x,y)|\log^\alpha(2|\xi'\cdot x|^{-1})d\sigma(x)d\sigma(y)\\ &\leq C\left(\log\left|2^k r\xi\right|\right)^{-\alpha} \quad if\ |2^k r\xi| > 2^\alpha. \end{aligned}\tag{26}$$

When $n = 2$, we follow the same arguments similar to those in Case 1, we obtain (26). Similarly, we have that

$$\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| \leq C\left(\log\left|2^j s\zeta\right|\right)^{-\alpha} \quad if\ |2^j s\zeta| > 2^\alpha. \tag{27}$$

Case 3. This lemma can be proved for the other cases of Φ by following the same argument as in the proof adopted for the cases 1 and 2. We omit the details. □

Lemma 5. *Suppose that $\mho \in L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ and satisfies the conditions (1)-(2). Let Φ be given as in any one of Theorems 1-5. Then there exists $C_p > 0$ such that*

$$\|\lambda^*(g)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq C_p\|g\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}. \tag{28}$$

We can easily prove this lemma by using iterated integration and using at least one of results in Corollary 5.3 in [26] or Lemma 1.

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3. Proof of Main Theorems

Assume that $\mho \in G_\alpha(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ for some $\alpha > 1/2$ and satisfies the conditions (1)-(2), and let Φ be given as in any of Theorems 1-5. Then by Minkowski's inequality, we have

$$\begin{aligned}
\mathfrak{M}_{\mho,\Phi}(g)(u,v,w) &= \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left| \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{-1} 2^{j+k} \left(\lambda_{r,s}^{k,j} * g\right)(u,v,w) \right|^2 \frac{drds}{rs} \right)^{1/2} \\
&\leq \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{-1} 2^{j+k} \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left| \left(\lambda_{r,s}^{0,0} * g\right)(u,v,w) \right|^2 \frac{drds}{rs} \right)^{1/2} \\
&= \left(\int_1^2 \int_1^2 \left| \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left(\lambda_{r,s}^{k,j} * g\right)(u,v,w) \right|^2 \frac{drds}{rs} \right)^{1/2} \\
&\leq \left(\int_1^2 \int_1^2 \left| \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left(\lambda_{r,s}^{k,j} * g\right)(u,v,w) \right|^2 drds \right)^{1/2}. \qquad (29)
\end{aligned}$$

Choose two radial Schwartz functions $\Psi_1 \in \mathcal{S}(\mathbb{R}^m)$ and $\Psi_2 \in \mathcal{S}(\mathbb{R}^n)$ which satisfy the following properties:

$(i)\, 0 \leq \Psi_1 \leq 1, \;\; 0 \leq \Psi_2 \leq 1,$

$(ii)\, \mathrm{supp}(\Psi_1) \subseteq \{u \in \mathbb{R}^m : \; 1/4 \leq |u| \leq 4\}, \;\; \mathrm{supp}\,(\Psi_2) \subseteq \{v \in \mathbb{R}^n : \; 1/4 \leq |v| \leq 4\}.$

$(iii)\, \sum_{a_1\in\mathbb{Z}} (\Psi_1(2^{a_1}r))^2 = 1, \;\; \sum_{a_2\in\mathbb{Z}} (\Psi_2(2^{a_2}s))^2 = 1.$

Define the multiplier operators $\{T_{a_1,a_2}\}$ on $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ by

$$\widehat{T_{a_1,a_2}(g)}\;(\xi,\zeta,\eta) = \Psi_1(2^{a_1}|\xi|)\Psi_2(2^{a_2}|\zeta|)\hat{g}(\xi,\zeta,\eta).$$

Thus for any $g \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$, we have

$$g(u,v,w) = \sum_{a_2\in\mathbb{Z}} \sum_{a_1\in\mathbb{Z}} T^2_{a_1,a_2}(g)(u,v,w),$$

which in turn by (29) leads to

$$\mathfrak{M}_{\mho,\Phi}(g)(u,v,w) \;\; \leq \;\; \mathcal{I}(g)(u,v,w), \qquad (30)$$

where

$$\mathcal{I}(g)(u,v,w) = \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \int_1^2 \int_1^2 \left| \sum_{a_1\in\mathbb{Z}} \sum_{a_2\in\mathbb{Z}} T_{j-a_1,k-a_2}\left(\lambda_{r,s}^{k,j} * \left(T_{j-a_1,k-a_2}(g)\right)\right)(u,v,w) \right|^2 drds \right)^{1/2}.$$

Therefore, by Lemmas 4–5 and employing the same argument as in [[22], p. 301], we obtain

$$\|\mathcal{I}(g)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \qquad (31)$$

for all $p \in (1 + 1/(2\alpha), 1 + 2\alpha)$. Now the proof is complete by the last inequality and using (30).

Author Contributions: Formal analysis and writing—original draft preparation: H.A.-Q and M.A. All authors have read and agreed to the published version of the manuscript.

Funding: This work is partially supported by the deanship of research at Jordan University of Science and Technology (Research Grant No. 20230654).

Acknowledgments: The authors are grateful to the editor for handling the full submission of the manuscript.

Conflicts of Interest: The authors declare that they have no conflict of interest.

References

1. Stein, E. On the function of Littlewood–Paley, Lusin and Marcinkiewicz. *Trans. Am. Math. Soc.* **1958**, *88*, 430–466.
2. Benedek, A.; Calderón, A.; Panzones, R. Convolution operators on Banach space valued functions. *Proc. Natl Acad. Sci. USA.* **1962**, *48*, 356–365.
3. Ding, Y.; Fan, D.; Pan, Y. L^p boundedness of Marcinkiewicz integrals with Hardy space function kernel. *Acta Math.* **2000**, *16*, 593–600.
4. Wu, H. L^p bounds for Marcinkiewicz integrals associated to surfaces of revolution. *J. Math. Anal. Appl.* **2006**, *321*, 811–827.
5. Ding, Y.; Fan, D.; Pan, Y. On the L^p boundedness of Marcinkiewicz integrals. *Michigan Math. J.* **2002**, *50*, 17–26.
6. Al-Qassem, H.; Pan, Y. On rough maximal operators and Marcinkiewicz integrals along submanifolds. *Studia Math.* **2009**, *190*, 73–98.
7. Ali, M.; Al-Refai, O. Boundedness of Generalized Parametric Marcinkiewicz Integrals Associated to Surfaces. *Mathematics.* **2019**, *7*, 886. http://doi.org/10.3390/math7100886.
8. Walsh, T. On the function of Marcinkiewicz. *Studia Math.* **1972**, 44, 203–217.
9. Grafakos, L.; AStefanov, A. L^p bounds for singular integrals and maximal singular integrals with rough kernel. *India. Univ. Math. J.* **1998**, *47*, 455–469.
10. Ding, Y.; Pan, Y. L^p bounds for Marcinkiewicz integral. *Proce. Edinburgh Math. Soc.* **2003**, *46*, 669–677.
11. Ding, Y. L^2-boundedness of Marcinkiewicz integral with rough kernel. *Hokk. Math. J.* **1998**, 27, 105–115.
12. Al-Qassem, A.; Al-Salman, A.; Cheng, L.; Pan, Y. Marcinkiewicz integrals on product spaces. *Studia Math.* **2005**, 167, 227–234.
13. Al-Qassem, H. Rough Marcinkiewicz integral operators on product spaces. *Collec. Math.* **2005**, 36, 275–297.
14. Choi, Y. Marcinkiewicz integrals with rough homogeneous kernel of degree zero in product domains. *J. Math. Anal. Appl.* **2001**, 261, 53–60.
15. Chen, J.; Fan, D.; Ying, Y. Rough Marcinkiewicz integrals with $L(\log L)^2$ kernels. *Adv. Math.* **2001**, 30, 179–181.
16. Fan, D.; Pan, Y.; Yang, D. A weighted norm inequality for rough singular integrals. *Tohoku Math. J.* **1999**, *51*, 141–161.
17. Fan, D.; Pan, Y. Singular integral operators with rough kernels supported by subvarieties. *Amer J. Math.* **1997**, *119*, 799–839.
18. Kim, W.; Wainger, S.; Wright, J.; Ziesler, S. Singular Integrals and Maximal Functions Associated to Surfaces of Revolution. *Bull. Lond. Math. Soc.* **1996**, *28*, 291–296.
19. Chen, J.; Ding, Y.; Fan, D. L^p boundedness of the rough Marcinkiewicz integral on product domains. *Chinese J. Contemp Math.* **2000**, 21, 47–54.
20. Ali, M.; Al-Qassem, H. On certain estimates for parabolic Marcinkiewicz integrals related to surfaces of revolution on product spaces and extrapolation. *Axioms.* **2023**, *12*, 35. https://doi.org/10.3390/axioms12010035.
21. Al-Qassem, H.; Ali, M. On the functions of Marcinkiewicz integrals along surfaces of revolution on product domains via extrapolation. *Symmetry.* **2023**, *15*, 1814. https://doi.org/10.3390/sym15101814.
22. Wu, H.; Xu, J. Rough Marcinkiewicz integrals associated to surfaces of revolution on product domains. *Acta Math. Sci.* **2009**, *29*, 294–304.
23. Al-Qassem, H.; Cheng, L.; Pan, Y. Rough singular integrals associated to surfaces of revolution on product spaces. *submitted.*
24. Stein, E.; Wainger, S. Problems in harmonic analysis related to curvature. *Bull. Amer. Math. Soc.* **1978,** 84, 1239–1295.
25. Ricci, F.; Stein, E. Multiparameter singular integrals and maximal functions. *Ann. Inst. Fourier.* **1992,** 42, 637–670.
26. Duoandikoetxea, J.; Rubio de Francia, J. Maximal and singular integral operators via Fourier transform estimates. *Invent Math.* **1986**, 84, 541–561.

 doi:10.20944/preprints202406.1672.v1

27. Christ, A.; Wright, J. Multidimensional van der Corput and sublevel set estimates. *J. Amer. Math. Soc.* **1999**, 12, 981–1015.
28. Cheng, L.; Pan, Y. L^p bounds for singular integrals associated to surfaces of revolution. *J. Math. Anal. Appl.* **2002**, 265, 163–169.