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Article

Algebraic Solution of the Riemann Hypothesis

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Abstract

A proof of the Riemann Hypothesis is obtained using the properties of the Γ -function, the Bernoulli function and limitations imposed by the rational relations between the complex roots of the ζ -function.

Keywords: Riemann hypothesis; Gamma function; Bernoulli numbers

1. Introduction

Most of the information about the ζ -function is well known and I will not go into a lengthy detail about the particulars of theorems and foundational work that concern the Riemann Hypothesis. The Riemann Zeta function is defined as

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} n^{-s}, s \in \mathbb{C} \tag{1}$$

I will use the convention, $s = \sigma + it$, where, $\tau \in \mathfrak{R}$ (reals).

Euler proved that the function $\zeta(s), s = \sigma + it, \sigma > 1$ can be represented in terms of primes, p . $\zeta(s)$ is analytic for $\sigma > 1$ and satisfies in this half-plane the identity:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{2}$$

Here, p is a prime. Except for a pole at $s = 1$, $\zeta(s)$ behaves properly and can be easily extended using the Gamma Function. The extension of $\zeta(s)$ to the entire complex plane can be obtained by consideration the entirety and the general definition of the Gamma function:

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \tag{3}$$

Change variables by the substitution $t = n^2\pi x, z = \frac{s}{2}$, in (3),

$$\Gamma\left(\frac{s}{2}\right) = (n^2\pi)^{\frac{s}{2}} \int_0^{\infty} e^{-n^2\pi x} x^{\frac{s}{2}-1} dx \tag{4}$$

Extracting $\zeta(s)$ from (4),

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right) = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-n^2\pi x} x^{\frac{s}{2}-1} dx \tag{5}$$

The convergence of the series,

$$S(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x} \tag{6}$$

in the interval $[0, \infty]$ gives the relation:

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right)=\int_0^\infty S(x)x^{\frac{s}{2}-1}dx \tag{7}$$

This can be split into two separate integrals,

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right)=\int_0^1 S(x)x^{\frac{s}{2}-1}dx+\int_1^\infty S(x)x^{\frac{s}{2}-1}dx \tag{8}$$

Note that the sum (6) is related to the Jacobi Theta function. See Ref. [1]

$$\theta(x)=\sum_{n=-\infty}^\infty e^{-n^2\pi x} \tag{9}$$

$$2S(x)=2\sum_{n=1}^\infty e^{-n^2\pi x}=\sum_{n=-\infty}^\infty e^{-n^2\pi x}-1=\theta(x)-1 \tag{10}$$

The Jacobi theta function obeys the symmetry

$$x^{\frac{1}{2}}\theta(x)=\theta(x^{-1}) \tag{11}$$

Thus

$$x^{\frac{1}{2}}(2S(x)+1)=2S(x^{-1})+1 \tag{12}$$

Obviously, this leads to the reflection formula as follows.

$$S(x^{-1})=-\frac{1}{2}+\frac{1}{2}x^{\frac{1}{2}}+x^{\frac{1}{2}}S(x) \tag{13}$$

The integral (8.0) now becomes

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right)=\int_1^\infty S(x^{-1})x^{\frac{s}{2}-1}dx+\int_1^\infty S(x)x^{\frac{s}{2}-1}dx \tag{14}$$

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right)=\int_1^\infty x^{\frac{s}{2}-1}\left(-\frac{1}{2}+\frac{1}{2}x^{\frac{1}{2}}+x^{\frac{1}{2}}S(x)\right)dx+\int_1^\infty S(x)x^{\frac{s}{2}-1}dx \tag{15}$$

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right)=\frac{1}{s(s-1)}+\int_1^\infty S(x)\left(x^{\frac{s}{2}-1}+x^{\frac{s}{2}-\frac{1}{2}}\right)dx \tag{16}$$

The right side of the relation (16.) is invariant to the substitution $s\rightarrow 1-s$. This gives the reflection formula for $\zeta(s)$:

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right)=\pi^{-\frac{1-s}{2}}\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right) \tag{17}$$

The reflection formula indicates that the roots should obey a reflection and a conjugate symmetry if they lie on the $\frac{1}{2}$ -line.

2. Method

This paper uses the relationships between the Gamma function and the roots of the Riemann Zeta function. This simple method leads to a simple restriction based on the rational relations of the roots. As expected, the Riemann hypothesis should be a fundamental property of the mathematical functions relating the Riemann Zeta function to the Gamma function when the Riemann Zeta function vanishes.

As expected, *no new theorems* are required since in its fundamental form, the result is as fundamental as the arithmetic that describes numbers. The hypothesis is proved to be a fundamental

property of Gamma functions, their relation to the vanishing of the Riemann Zeta function, and the pole $z = 1$, of the Riemann Zeta function.

3. Results

By expanding the Riemann Zeta function $\zeta(z)$, in terms of Bernoulli relations, a rational form of the roots is obtained that can only be true if the real part of the root z of the zeta function is $\frac{1}{2}$.

3.1. The Relationship Between $\zeta(s)$, s , Bernoulli Numbers, and Gamma Functions with the Rational Functions of the Roots

Consider the rational forms of the Riemann Zeta function $\zeta(z)$.

Jensen formula [1]. (p 1036),

$$\zeta(z) = \left(\frac{2^{z-1}}{2^z - 1} \right) \left(\frac{z}{z-1} \right) + \frac{2}{2^z - 1} \int_0^\infty \left(\frac{\left(\frac{1}{4} + t^2 \right)^{-\frac{z}{2}} \sin(z \tan^{-1} 2t)}{e^{2\pi t} - 1} \right) dt \quad (18)$$

Substituting $\tan \theta = 2t$,

$$\zeta(z) = \left(\frac{2^{z-1}}{2^z - 1} \right) \left(\frac{z}{z-1} \right) + \frac{2}{2^z - 1} \int_0^{\frac{\pi}{2}} \left(\frac{\left(\frac{1}{2} \sec \theta \right) \sin(z\theta) \left(\frac{1}{2} (\sec \theta)^2 \right)^{\frac{z}{2}}}{e^{\pi \tan \theta} - 1} \right) d\theta \quad (19)$$

This reduces to

$$\zeta(z) = \left(\frac{2^{z-1}}{2^z - 1} \right) \left(\frac{z}{z-1} \right) + \frac{2^z}{2^z - 1} \int_0^{\frac{\pi}{2}} \left(\frac{\sin(z\theta) ((\sec \theta)^{2-z})}{e^{\pi \tan \theta} - 1} \right) d\theta \quad (20)$$

Using the ChebyshevT function [1] (p 993, 8.94),

$$\cos(z \cos^{-1}(x)) = \frac{1}{2} \left[\left(x + i\sqrt{1-x^2} \right)^z + \left(x - i\sqrt{1-x^2} \right)^z \right]^2 \quad (21)$$

Putting $x = \cos \theta$, and expand for $\sin(z\theta)$,

$$\zeta(z) = \left(\frac{2^{z-1}}{2^z - 1} \right) \left[\left(\frac{z}{z-1} \right) + 2z \int_0^{\frac{\pi}{2}} \left(\frac{(\cos \theta)^{z-2} \left(1 - \frac{1}{2} \left((\cos \theta + i(1 - \cos \theta^2)^{\frac{1}{2}} \right))^z + \frac{1}{2} \left((\cos \theta - i(1 - \cos \theta^2)^{\frac{1}{2}} \right))^z \right)}{e^{\frac{\pi \sqrt{1-\cos \theta^2}}{\cos \theta}} - 1} \right) d\theta \right]$$

$$\zeta(z) = \left(\frac{2^{z-1}}{2^z - 1} \right) \left[\left(\frac{z}{z-1} \right) + 2s \int_0^{\frac{\pi}{2}} \left(\frac{(\cos \theta)^{z-2} \left(1 - \frac{1}{2} \left((\cos \theta + i(1 - \cos \theta^2)^{\frac{1}{2}} \right))^z + \frac{1}{2} \left((\cos \theta - i(1 - \cos \theta^2)^{\frac{1}{2}} \right))^z \right)}{e^{\frac{\pi \sqrt{1-\cos \theta^2}}{\cos \theta}} - 1} \right) d\theta \right]$$

Which can be reduced again by the substitution, $x = \tan \theta$,

$$\zeta(z) = \left(\frac{2^{z-1}}{2^z - 1} \right) \left[\left(\frac{z}{z-1} \right) + 2z \int_0^\infty \left(\frac{\left(\frac{1}{\sqrt{1+x^2}} \right)^{2z} \left(\left(\frac{1}{\sqrt{1+x^2}} \right)^{-2z} - \left(\frac{1}{2} (1+ix)^z + \frac{1}{2} (1-ix)^z \right)^{\frac{1}{2}} \right)}{e^{\pi x} - 1} \right) dx \right]$$

And finally, one arrives at the Abel Plena form [1]:

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\left(\frac{s}{s-1} \right) + 2 \int_0^\infty \left(\frac{i(1+ix)^{-s} + (1+ix)^{-s}}{e^{\pi x} - 1} \right) dx \right] \quad (22)$$

The function (22) can now be reduced to a series form as follows:

$$(1+ix)^{-s} = \sum_{n=0}^{\infty} \frac{(ix)^n \Gamma(1-s)}{\Gamma(-s-n+1)n!}, (1-ix)^{-s} = \sum_{n=0}^{\infty} \frac{(-ix)^n \Gamma(1-s)}{\Gamma(-s-n+1)n!} \quad (23)$$

It is convenient at this point to note that the odd terms vanish and one is left with:

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\left(\frac{s}{s-1} \right) + \int_0^{\infty} \left(2i \sum_{n=1}^{\infty} \frac{(ix)^{2n-1} \Gamma(1-s)}{2\Gamma(-s-2n+2)(2n-1)! e^{\pi x} - 1} \right) dx \right] \quad (24)$$

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\left(\frac{s}{s-1} \right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \Gamma(1-s)}{\Gamma(-s-2n+2)(2n-1)!} \int_0^{\infty} \left(\frac{x^{2n-1}}{e^{\pi x} - 1} \right) dx \right] \quad (25)$$

Using the relation [1] (p 1038)

$$\zeta(2n) \Gamma(2n) \left[\frac{1}{\pi^{2n}} \right] = \int_0^{\infty} \frac{x^{2n-1}}{e^{\pi x} - 1} dx \quad (26)$$

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\left(\frac{s}{s-1} \right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2\Gamma(1-s) \zeta(2n) \Gamma(2n)}{\pi^{2n} \Gamma(-s-n+2)(2n-1)!} dx \right] \quad (27)$$

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\left(\frac{s}{s-1} \right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2\Gamma(1-s) \zeta(2n)}{\pi^{2n} \Gamma(-s-n+2)} dx \right] \quad (28)$$

Using the relation,

$$\zeta(2n) = \frac{(-1)^{n+1} 2^{2n-1} \pi^{2n} B_{2n}}{(2n)!} \quad (29)$$

One arrives at the desired form:

$$\zeta(z) = \left(\frac{2^{z-1}}{2^z - 1} \right) \left[\left(\frac{z}{z-1} \right) + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} \Gamma(1-z)}{\Gamma(2-z-2n)(2n)!} \right] \quad (30)$$

When the Zeta function vanishes at a root ρ ,

$$\frac{\rho}{\rho-1} = - \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} \Gamma(1-\rho)}{\Gamma(2-\rho-2n)(2n)!} \quad (31)$$

3.2. The Role of the Gamma Function in the Riemann Hypothesis

The Weierstrass Gamma function has some restrictions for the validity of complex arguments [1], (p 895 2.11):

$$\Gamma(1-\rho) = \frac{e^{-\gamma(1-\rho)}}{1-\rho} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{1-\rho}{k}}}{1 + \frac{1-\rho}{k}} \right), \quad \Re(1-\rho) > 0 \quad (32)$$

where, γ , is the Euler constant.

The relation (32) is only valid for $\Re(\rho) > 0$, i.e., $\Re(\rho) = \Re(-\rho - 2n + 1) > 0$.

Let us expand $\Gamma(1-\rho)$ using the functional relation (32):

$$\frac{\rho}{\rho-1} = \frac{1}{1-\rho} \left(e^{-\gamma(1-\rho)} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{1-\rho}{k}}}{1 + \frac{1-\rho}{k}} \right) \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(2-\rho-2n)(2n)!} \right) \quad (33)$$

One finds that from the numerator, a root ρ , can be expressed as

$$\rho = -e^{-\gamma(1-\rho)} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{1-\rho}{k}}}{1 + \frac{1-\rho}{k}} \right) \left[\sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(2-\rho-2n)(2n)!} \right] \quad (34)$$

This relation must be satisfied by the equation 41 when we substitute, $\rho \rightarrow 1-\rho$ in (34), hence,

$$1-\rho = -e^{-\gamma\rho} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{\rho}{k}}}{1 + \frac{\rho}{k}} \right) \left[\sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(-\rho-2n+2)(2n)!} \right] \quad (35)$$

and so an equivalent representation of the rational form $\frac{\rho}{\rho-1}$ is obtained shown in bold text:

$$\frac{\rho}{\rho-1} = \frac{e^{-\gamma(1-\rho)} \prod_{k=1}^{\infty} \left(\left(1 + \frac{1-\rho}{k}\right)^{-1} e^{\frac{1-\rho}{k}} \right) \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(2-\rho-2n)(2n)!}}{e^{-\gamma\rho} \prod_{k=1}^{\infty} \left(\left(1 + \frac{\rho}{k}\right)^{-1} e^{\frac{\rho}{k}} \right) \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(-\rho-2n+2)(2n)!}} \quad (36)$$

One can simplify (36) as follows:

$$\frac{\rho}{\rho-1} = e^{-(\gamma(1-2\rho)+(1-2\rho)\sum_{k=1}^{\infty}(\frac{1}{k}))} \prod_{k=1}^{\infty} \left(\frac{(k+\rho)}{(k+1-\rho)} \right) \left[\frac{\sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(2-\rho-2n)(2n)!}}{\sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(-\rho-2n+2)(2n)!}} \right] \quad (37)$$

$$\frac{\rho}{(\rho-1)\Gamma(1-\rho)} = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(2-\rho-2n)(2n)!}$$

$$\frac{(1-\rho)}{\rho\Gamma(\rho)} = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(-\rho-2n+2)(2n)!}$$

$$\frac{\rho}{\rho-1} = e^{-(\gamma(1-2\rho)+(1-2\rho)\sum_{k=1}^{\infty}(\frac{1}{k}))} \prod_{k=1}^{\infty} \left(\frac{(k+\rho)}{(k+1-\rho)} \right) \left[\frac{\frac{\rho}{(\rho-1)\Gamma(1-\rho)}}{\frac{(1-\rho)}{\rho\Gamma(\rho)}} \right] \quad (38)$$

$$\frac{\rho}{\rho-1} = e^{-(\gamma(1-2\rho)+(1-2\rho)\sum_{k=1}^{\infty}(\frac{1}{k}))} \left[\frac{\Gamma(\rho)}{\Gamma(1-\rho)} \right] \prod_{k=1}^{\infty} \left(\frac{(k+\rho)}{(k+1-\rho)} \right) \left[\left(\frac{\rho}{\rho-1} \right)^2 \right] \quad (39)$$

and finally, one gets the rational form of the roots,

$$\frac{\rho-1}{\rho} = e^{-(\gamma(1-2\rho)+(1-2\rho)\sum_{k=1}^{\infty}(\frac{1}{k}))} \left[\frac{\Gamma(\rho)}{\Gamma(1-\rho)} \right] \prod_{k=1}^{\infty} \left(\frac{(k+\rho)}{(k+1-\rho)} \right) \quad (40)$$

Putting $\rho = x - iy$, $\{x, y\} > 0 \in \Re$ in (40),

$$\frac{x-iy-1}{x+iy} = e^{-\gamma(1-(2x+2iy))\sum_{k=1}^{\infty}(\frac{1}{k})} \left[\frac{\Gamma(x+iy)}{\Gamma(1-x-iy)} \right] \prod_{k=1}^{\infty} \left(\frac{(k+\rho)}{(k+1-\rho)} \right) \quad (41)$$

The complex representation of the Γ -function provides a way to express the rational functions of the roots (41) in terms of the Gamma functions [1] (p 895 2¹¹).

$$\Gamma(x+iy) = \Gamma(x) \left[\frac{x e^{-iy\gamma}}{x+iy} \right] \prod_{k=1}^{\infty} \left(\frac{e^{\frac{iy}{k}}}{1 + \frac{iy}{(x+k)}} \right), \quad x > 0, \quad (42)$$

$$(x+iy) = \frac{\Gamma(x) x e^{-iy\gamma}}{\Gamma(x+iy)} \prod_{k=1}^{\infty} \left(e^{\frac{iy}{k}} \left(\frac{x+k}{x+k+iy} \right) \right) \quad (43)$$

$$(x + iy) = \frac{\Gamma(x + 1)e^{-iy\gamma}}{\Gamma(x + iy)} \prod_{k=1}^{\infty} \left(e^{\frac{iy}{k}} \left(\frac{x + k}{x + k + iy} \right) \right) \quad (44)$$

$$(1 - x - iy) = \frac{\Gamma(2 - x)e^{-iy\gamma}}{\Gamma(1 - x - iy)} \prod_{k=1}^{\infty} \left(e^{\frac{-iy}{k}} \left(\frac{1 - x + k}{1 - x - iy + k} \right) \right) \quad (45)$$

One can now express the rational form of the variables as follows:

$$\frac{(1 - x - iy)}{x + iy} = \frac{\frac{\Gamma(2 - x)e^{-iy\gamma}}{\Gamma(1 - x - iy)} \prod_{k=1}^{\infty} \left(e^{\frac{iy}{k}} \left(\frac{1 - x + k}{1 - x - iy + k} \right) \right)}{\frac{\Gamma(x + 1)e^{-iy\gamma}}{\Gamma(x + iy)} \prod_{k=1}^{\infty} \left(e^{\frac{iy}{k}} \left(\frac{x + k}{x + k + iy} \right) \right)} \quad (46)$$

$$\frac{(1 - x - iy)}{x + iy} = \frac{\Gamma(2 - x)}{\Gamma(x + 1)} \frac{\Gamma(x + iy)}{\Gamma(1 - x - iy)} e^{2iy\gamma} \prod_{k=1}^{\infty} \left(\left(\frac{(1 - x + k)}{x + k} \right) \left(\frac{x + k + iy}{1 - x + k - iy} \right) \right) \quad (47)$$

From (41),

$$\frac{x + iy - 1}{x + iy} = e^{-\gamma(1 - (2x + 2iy)) \sum_1^{\infty} \left(\frac{1}{k} \right)} \left[\frac{\Gamma(x + iy)}{\Gamma(1 - x - iy)} \right] \prod_{k=1}^{\infty} \left(\frac{x + k + iy}{1 - x + k - iy} \right) \quad (48)$$

$$(49)$$

Dividing (48) by (47), one gets: using the absolute value of $\left| \frac{\Gamma(2-x)}{\Gamma(x+1)} \right|$:

$$1 = \frac{\left| \frac{\Gamma(2-x)}{\Gamma(x+1)} \right| \frac{\Gamma(x + iy)}{\Gamma(1 - x - iy)} e^{2iy\gamma \sum_1^{\infty} \left(\frac{1}{k} \right)} \prod_{k=1}^{\infty} \left(\left(\frac{(1 - x + k)}{x + k} \right) \left(\frac{x + k + iy}{1 - x + k - iy} \right) \right)}{e^{-\gamma(1 - (2x + 2iy)) \sum_1^{\infty} \left(\frac{1}{k} \right)} \left[\frac{\Gamma(x + iy)}{\Gamma(1 - x - iy)} \right] \prod_{k=1}^{\infty} \left(\frac{x + k + iy}{1 - x + k - iy} \right)} \quad (50)$$

$$\left| \frac{\Gamma(2-x)}{\Gamma(x+1)} \right| e^{\gamma(1-2x) \sum_1^{\infty} \left(\frac{1}{k} \right)} \prod_{k=1}^{\infty} \left(\left(\frac{(1 - x + k)}{x + k} \right) \right) = 1 \quad (51)$$

3.3. The Riemann Hypothesis

From (50), it is clear that the relations of the Gamma function as reflections of unity can be expressed as

$$1 = e^{-\gamma(1-2x) \sum_1^{\infty} \left(\frac{1}{k} \right) + \log \left(\left| \frac{\Gamma(2-x)}{\Gamma(x+1)} \right| \prod_{k=1}^{\infty} \left(\left(\frac{(1-x+k)}{x+k} \right) \right) \right)} \quad (52)$$

Now the left-hand side of (52) is unity. However, on the right-hand side of (52) can blow up due to the fact that the sum

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} \right) = \zeta(1) = \infty$$

except when $x = \frac{1}{2}$ and one finds that all the terms in (51) reduce to unity making the relation (51) only valid for the $\frac{1}{2}$ -line and the roots of the Zeta function.

This proves the Riemann hypothesis.

4. Discussion

It is clear that the restrictions for the validity of the complex and real Gamma functions and the reflection formula for complex variables restrict the roots to the half-line.

Conclusions

As expected, *no new theorems* are required since the result is as fundamental as the arithmetic that describes it. The hypothesis is proved to be a fundamental property of Gamma functions, their relation to the vanishing of the Zeta function, at $\rho = \frac{1}{2} + iy$. from (41),

$$\frac{x + iy - 1}{x + iy} = e^{-\gamma(1-(2x+2iy))} \left[\frac{\Gamma(x + iy)}{\Gamma(1 - x - iy)} \right] \prod_{k=1}^{\infty} \left(\frac{x + k + iy}{1 - x + k - iy} \right) \quad (53)$$

$$\frac{\frac{1}{2} - iy}{\frac{1}{2} + iy} = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2} + 1\right)} \frac{\Gamma\left(\frac{1}{2} + iy\right)}{\Gamma\left(\frac{1}{2} - iy\right)} e^{2i\gamma} \prod_{k=1}^{\infty} \left(e^{-\frac{2iy}{k}} \left(\frac{\left(\frac{1}{2} + k\right)}{\frac{1}{2} + k} \right) \left(\frac{\frac{1}{2} + k + iy}{\frac{1}{2} + k - iy} \right) \right) \quad (54)$$

Put $x = \frac{1}{2}$ and $y = \frac{1}{2} \tan \theta$,

$$\frac{\frac{1}{2} - i\frac{1}{2} \tan \theta}{\frac{1}{2} + i\frac{1}{2} \tan \theta} = e^{\gamma(i \tan \theta)} \left[\frac{\Gamma\left(\frac{1}{2} + i\frac{1}{2} \tan \theta\right)}{\Gamma\left(\frac{1}{2} - i\frac{1}{2} \tan \theta\right)} \right] \prod_{k=1}^{\infty} \left(e^{\frac{i \tan \theta}{k}} \frac{\left(k + \frac{1}{2} + i\frac{1}{2} \tan \theta\right)}{k + \frac{1}{2} - i\frac{1}{2} \tan \theta} \right) \quad (55)$$

Using the exponential form:

$$\frac{\frac{1}{2} + i\frac{1}{2} \tan \theta}{\frac{1}{2} - i\frac{1}{2} \tan \theta} = e^{2i\theta} = e^{\gamma(i \tan \theta)} \left[\frac{\Gamma\left(\frac{1}{2} + i\frac{1}{2} \tan \theta\right)}{\Gamma\left(\frac{1}{2} - i\frac{1}{2} \tan \theta\right)} \right] \prod_{k=1}^{\infty} \left(e^{\frac{i \tan \theta}{k}} \frac{\left(k + \frac{1}{2} + i\frac{1}{2} \tan \theta\right)}{k + \frac{1}{2} - i\frac{1}{2} \tan \theta} \right) \quad (56)$$

$$e^{2i\theta} = e^{\gamma(i \tan \theta)} \left[\frac{\Gamma\left(\frac{1}{2} + i\frac{1}{2} \tan \theta\right)}{\Gamma\left(\frac{1}{2} - i\frac{1}{2} \tan \theta\right)} \right] \prod_{k=1}^{\infty} \left(e^{\frac{i \tan \theta}{k}} \frac{\left(k + \frac{1}{2} + i\frac{1}{2} \tan \theta\right)}{k + \frac{1}{2} - i\frac{1}{2} \tan \theta} \right) \quad (57)$$

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