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A Relaxation Formulation Stronger than the Navier–Stokes Equations and the Existence of Global Strong Solutions for Small Initial Data

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Abstract

This paper studies a relaxation extension of the incompressible Navier–Stokes equations in which the viscous stress tensor is treated as an independent dynamical variable with relaxation and diffusion. The resulting system forms a thermodynamically consistent extension of the Navier–Stokes equations and possesses a triple dissipation structure consisting of viscous dissipation, stress diffusion, and stress relaxation. We first establish a basic energy inequality for the extended system, showing that the relaxation structure introduces an additional dissipation mechanism that is absent in the classical Navier–Stokes equations. Next, higher-order a priori estimates are derived in Sobolev spaces H^s with $s > 5/2$, using commutator estimates for the nonlinear convection term. Combining these estimates with a local well-posedness result obtained via a Friedrichs approximation scheme, we prove the existence and uniqueness of global strong solutions for sufficiently small initial data. Finally, we discuss the formal relaxation limit in which the stress tensor converges to the Newtonian constitutive law, recovering the incompressible Navier–Stokes equations. The results show that the relaxation formulation provides a mathematically well-posed extension of the Navier–Stokes dynamics and offers a framework for studying the stabilizing role of stress relaxation mechanisms.

Keywords: relaxation navier–stokes system; global strong solution; stress relaxation dynamics; energy dissipation structure; sobolev a priori estimates; navier–stokes regularity problem

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Chapter.1 Introduction

The incompressible Navier–Stokes equations occupy a central position in fluid mechanics as the fundamental equations describing the motion of viscous fluids. Derived from the principles of mass conservation and momentum balance together with Newton's law of viscosity, these equations have been remarkably successful in describing a wide range of physical phenomena, including aerodynamics, meteorology, oceanography, and astrophysical flows.

The classical foundations of fluid mechanics were established in the works of Lamb [1], Batchelor [2], and Landau–Lifshitz [3], and later extended to turbulence theory by Pope [4] and Frisch [5]. Despite their long history and broad applicability, the mathematical structure of the Navier–Stokes equations remains only partially understood.

In particular, for three-dimensional flows it is still unknown whether smooth initial data lead to globally smooth solutions for all time. This problem was identified by the Clay Mathematics Institute as one of the Millennium Prize Problems [6] and is regarded as one of the most important open problems in modern mathematics.

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Mathematical studies of the Navier–Stokes equations date back to the pioneering work of Leray [7], who established the existence of weak solutions in the 1930s. Hopf subsequently showed that such solutions satisfy the energy inequality, leading to the notion of Leray–Hopf weak solutions [8], which now form the cornerstone of the mathematical theory of viscous incompressible flows. Regarding regularity, the partial regularity theory of Caffarelli, Kohn, and Nirenberg [9] represents a major breakthrough, demonstrating that the singular set of weak solutions has zero parabolic Hausdorff measure. Another fundamental result is the Beale–Kato–Majda criterion [10], which relates possible blow-up of solutions to the growth of the vorticity.

For strong solutions, the classical work of Fujita and Kato [11,12] established local well-posedness in Sobolev spaces and global existence for sufficiently small initial data in critical spaces. Subsequent studies extended the analysis to various function spaces that are invariant under the natural scaling of the Navier–Stokes equations. In particular, Koch and Tataru [13] proved well-posedness in the critical space

$$BMO^{-1}(\mathbb{R}^3),$$

which is invariant under the Navier–Stokes scaling and represents one of the largest known function spaces where well-posedness holds. Further developments include the critical space analysis of Kozono and Taniuchi [14] and conditional regularity results based on the Serrin–Prodi criteria [15,16]. In parallel with these analytical developments, considerable effort has been devoted to deriving the Navier–Stokes equations from more fundamental microscopic or mesoscopic descriptions. Classically, the Chapman–Enskog expansion shows that the Navier–Stokes equations arise as the hydrodynamic limit of the Boltzmann equation [17–19]. Other mesoscopic approaches include lattice gas automata [20], lattice Boltzmann methods [21–23], and kinetic relaxation models [24–26].More recently, the growing interest in complex systems and transport processes has stimulated the study of network-based interaction models.

In this framework, physical systems are represented as graphs consisting of nodes and edges, and conserved quantities such as mass, momentum, and energy are exchanged between nodes.

Such network-type interaction systems appear in various fields including stochastic processes [27], chemical reaction networks [28], transport phenomena [29], and thermodynamic gradient flows [30,31].

Motivated by this perspective, the author recently proposed a framework in which fluid equations are constructed from a network-type master equation.

In this formulation, mass, momentum, and energy are exchanged between nodes through interaction fluxes, while conservation laws and the second law of thermodynamics are imposed as structural design principles. By taking an appropriate continuum limit, Navier–Stokes-type equations naturally emerge from the discrete interaction system.

Furthermore, by introducing relaxation variables into this framework, an extended energy structure can be constructed, and the Aubin–Lions compactness theorem can be used to prove the existence of Leray–Hopf weak solutions of the Navier–Stokes equations in the continuum limit [32]. This result provides a new theoretical perspective linking discrete interaction systems and continuum fluid equations.

However, the previous analysis established only the existence of weak solutions, leaving the question of strong solutions and regularity unresolved.

The purpose of the present work is to extend this framework and analyze a relaxation system with an enhanced dissipation structure in order to study the existence of strong solutions.

Instead of treating the Newtonian viscous stress

$$r = 2\nu D(u), \tag{1}$$

as an instantaneous constitutive relation, we interpret it as the equilibrium state of an independent stress tensor.

Based on this idea, we introduce an auxiliary stress tensor $r$in addition to the velocity field $u$and consider the relaxation system

$$\partial_t u + P\big((u\cdot\nabla)u\big) = P\nabla\cdot r, \tag{2}$$

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$$\varepsilon\,\partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^{\theta} r, \tag{3}$$

where εrepresents the relaxation time of the stress tensor and the fractional Laplacian $(-\Delta)^{\theta}$ introduces additional dissipation acting on high-frequency stress modes.

Formally taking the limit $\varepsilon \to 0$ yields

$$r = 2\nu D(u), \tag{4}$$

and the classical Navier–Stokes equations are recovered. A key observation of this study is that introducing the stress deviation

$$w = r - 2\nu D(u), \tag{5}$$

naturally produces the additional dissipation term

$$\frac{1}{\varepsilon} \| w \|^2, \tag{6}$$

which does not appear in the classical Navier–Stokes equations. This structure provides an intrinsic stabilization mechanism for the dynamics. The main objective of this paper is to analyze this relaxation system in Sobolev spaces and to prove the existence of global strong solutions for sufficiently small initial data.

Notation

Throughout this paper, the spatial domain is denoted by $\Omega = \mathbb{T}^3$. The standard Lebesgue space is written as $L^p(\Omega)$ and the Sobolev space of order s is $H^s(\Omega)$. The divergence-free subspace of $L^2(\Omega)$ is denoted by $L^2_\sigma(\Omega)$. The Leray projection onto divergence-free vector fields is written as P. For a vector field u, the symmetric velocity gradient is defined as $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$. The stress deviation variable is introduced as $w = r - 2\nu D(u)$. Throughout the paper, C denotes a generic positive constant whose value may change from line to line but is independent of the solution.

Main Theorem.

Let $s > \frac{5}{2}$ and consider the relaxation Navier–Stokes system on the three-dimensional torus $\mathbb{T}^3$:

$$\partial_t u + P\big((u\cdot\nabla)u\big) = P\nabla\cdot r,$$

$$\varepsilon\,\partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^{\theta} r.$$

Assume that the initial data

$$u_0 \in H^s_\sigma(\mathbb{T}^3), r_0 \in H^{s-1}(\mathbb{T}^3)$$

satisfy the smallness condition

$$\| u_0 \|^2_{H^s} + \varepsilon \| r_0 \|^2_{H^{s-1}} \le \delta,$$

where,

$$\delta = \delta(\nu, \varepsilon, \kappa, \theta, s) > 0.$$

Then the system admits a unique global strong solution satisfying

$$u \in C([0,\infty); H^s_\sigma(\mathbb{T}^3)),$$
$$r \in C\big([0,\infty]; H^{s-1}(\mathbb{T}^3)\big).$$

Moreover, the solution satisfies the energy inequality

$$E^{\varepsilon}(t) + \int_0^t \left(\nu \| \nabla u \|_2^2 + \frac{1}{\varepsilon} \| r - 2\nu D(u) \|_2^2 + \frac{\kappa}{\varepsilon} \| (-\Delta)^{\theta/2} r \|_2^2\right) ds \le E^{\varepsilon}(0).$$

Remark.

The local well-posedness analysis can be extended to the whole space $\mathbb{R}^3$ by means of the standard Friedrichs approximation together with classical compactness arguments. However, in order to present the global energy estimates in a transparent and technically concise form, the main results are formulated on the periodic domain $\mathbb{T}^3$.

Chapter 2 Weak Construction of Navier–Stokes Solutions from the Master Equation (Summary of the Previous Study)

2.1. Position of the Present Chapter

In this chapter we briefly summarize the principal results of our previous work [32], which forms the theoretical foundation of the present study.

In the previous study, fluid motion was not assumed a priori as a continuum description. Instead, the physical system was formulated as an interacting network, and it was demonstrated that the Navier–Stokes–type equations emerge as the continuous limit of a master equation defined on this network.

Furthermore, by introducing a relaxation extension for the stress tensor, the following mathematical structures were obtained:

an extended energy structure,

uniform energy estimates, and

compactness via the Aubin–Lions theorem.

Using these ingredients, the existence of Leray–Hopf weak solutions of the Navier–Stokes equations was established. The present paper takes this weak-solution construction as a starting point and aims to approach the existence of strong solutions by analyzing a relaxation system possessing stronger dissipative structure. For this purpose, the theoretical framework developed in the previous work is briefly recapitulated in the present chapter.

2.2. Network-type Master Equation

In the previous study, the physical system was represented as a network composed of nodes and edges.

The state vector at node i is defined as

$$U_i = (\rho_i, m_i, e_i), \tag{7}$$

where

ρ_i: density,

$m_i = \rho_i u_i$: momentum,

e_i: energy.

The interaction between nodes is described by the master equation

$$\frac{dU_i}{dt} = \sum_{j\in N(i)} A_{ij} \left[F(U_i, U_j) + D(U_i, U_j)\right], \tag{8}$$

where

A_{ij}: adjacency matrix of the network,

F: reversible flux,

D: dissipative flux.

This formulation satisfies two fundamental structural properties:

conservation structure,

exchange symmetry between nodes.

These properties ensure thermodynamic consistency of the interacting system.

2.3. Finite-Volume Structure

The network representation can be mapped onto a finite-volume structure. Let the cell V_i correspond to node i. Then the master equation takes the form

$$\frac{dU_i}{dt} = -\frac{1}{V_i} \sum_{j\in N(i)} S_{ij}\, F_{ij} \cdot n_{ij}, \tag{9}$$

where

S_{ij}: interface area between cells,

n_{ij}: unit normal vector.

This expression is identical in structure to a conservative finite-volume scheme, which ensures that the discrete dynamics preserves the conservation laws of mass, momentum, and energy.

2.4. Continuous Limit

We now consider the limit

$$h \to 0, \tag{10}$$

where h denotes the characteristic cell size. In this limit, the conservation structure leads to the conservative partial differential equation

$$\partial_t U + \nabla \cdot F(U) = \nabla \cdot (\mu \nabla U), \quad (11)$$

Substituting the fluid variables yields

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad (12)$$

$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = -\nabla p + \nabla \cdot \big(2\nu D(u)\big), \quad (13)$$

which corresponds to the Navier–Stokes equations.

2.5. Relaxation Extension of the Stress

In the previous study, the Newtonian constitutive relation

$$r = 2\nu D(u), \quad (14)$$

was interpreted as the relaxation limit of an extended system. The extended system is written as

$$\partial_t u + P\big((u \cdot \nabla) u\big) = P \nabla \cdot r, \quad (15)$$

$$\varepsilon\, \partial_t r + r = 2\nu D(u) - \kappa (-\Delta)^{\theta} r, \quad (16)$$

where

r: residual stress tensor,

ε: relaxation time,

$\kappa(-\Delta)^{\theta}$: high-frequency dissipation operator.

In the limit

$$\varepsilon \to 0, \quad (17)$$

one obtains

$$r = 2\nu D(u), \quad (18)$$

which recovers the Navier–Stokes equations.

2.6. Extended Energy Structure

Define the extended energy:

$$E^{\varepsilon}(t) = \frac{1}{2} \| u \|_2^2 + \frac{\varepsilon}{2} \| r \|_2^2. \quad (19)$$

Then the system satisfies the energy inequality

$$\frac{d}{dt} E^{\varepsilon} + \nu \| \nabla u \|_2^2 + \| r \|_2^2 + \kappa \| (-\Delta)^{\frac{\theta}{2}} r \|_2^2 \leq 0. \quad (20)$$

This estimate yields uniform bounds for both u and r.

2.7. Existence of Weak Solutions

From the energy inequality we obtain

$$u^{\varepsilon} \in L^{\infty}(0,T;L^2) \cap L^2(0,T;H^1), \quad (21)$$

$$r^{\varepsilon} \in L^2\big(0,T;H^{\theta}\big). \quad (22)$$

Furthermore, by the Aubin–Lions compactness theorem

$$u^{\varepsilon} \to u \text{ in } L^2, \quad (23)$$

holds. Taking the limit yields a Leray–Hopf weak solution of the Navier–Stokes equations.

2.8. Objective of the Present Paper

The previous study established the existence of weak solutions using

the master equation formulation,

the relaxation extension, and

a compactness argument.

However, the weak solution theory still leaves open fundamental questions:

regularity of solutions,

uniqueness,

existence of strong solutions.

In the present work we directly analyze the relaxation system (15)–(16). In particular, we demonstrate that the additional dissipation

$$\frac{1}{\varepsilon} \| r - 2\nu D(u) \|^2, \quad (24)$$

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contributes to the stabilization of solutions. By exploiting this structure, we aim to prove the existence of global strong solutions. In the next chapter, the relaxation system is formulated rigorously as a system possessing stronger dissipation than the classical Navier–Stokes equations.

Chapter 3 Mathematical Formulation of the Relaxation Navier–Stokes System

3.1. Purpose of This Chapter

In the previous chapter we reviewed how the Navier–Stokes equations emerge as the continuous limit of a network-based master equation and how the introduction of a relaxation extension allows the construction of Leray–Hopf weak solutions.

The purpose of this chapter is to formulate the relaxation Navier–Stokes system analyzed in this paper within a rigorous functional-analytic framework.

In particular, we clarify
the functional spaces,
the Leray projection,
the role of the stress tensor,
and the relationship to the Navier–Stokes limit.
This formulation provides the mathematical foundation for
energy estimates,
existence proofs of strong solutions.

3.2. Domain and Functional Spaces

The analysis domain is taken as

$$\Omega = \mathbb{T}^3, \tag{25}$$

where $\mathbb{T}^3$ denotes the three-dimensional torus

$$\mathbb{T}^3 = \left(\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right)^3. \tag{26}$$

Periodic boundary conditions are adopted to
eliminate boundary effects,
enable Fourier analysis.
Define the space of divergence-free vector fields

$$L^2_\sigma(\Omega) = \{u \in L^2(\Omega)^3 \mid \nabla \cdot u = 0\}, \tag{27}$$

and similarly

$$H^k_\sigma(\Omega) = H^k(\Omega)^3 \cap L^2_\sigma(\Omega). \tag{28}$$

3.3. Leray Projection

To eliminate the pressure term in the analysis of the Navier–Stokes equations we use the Leray projection.

Define

$$P: L^2(\Omega)^3 \to L^2_\sigma(\Omega), \tag{29}$$

By

$$P = I - \nabla\Delta^{-1}\nabla\cdot \tag{30}$$

This operator satisfies

$$P\nabla p = 0. \tag{31}$$

Therefore the Navier–Stokes equations can be written as

$$\partial_t u + P\big((u\cdot\nabla)u\big) = \nu\Delta u. \tag{32}$$

3.4. Residual Stress Tensor

The Newtonian constitutive relation is

$$\sigma = 2\nu D(u), \tag{33}$$

where

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$$D(u) = \frac{1}{2}(\nabla u + \nabla u^T). \quad (34)$$

In the present study this relation is not imposed instantaneously. Instead, it is treated as an equilibrium state of a dynamical stress variable. We therefore introduce the residual stress tensor

$$r, \quad (35)$$

as an independent variable.

3.5. Relaxation Navier–Stokes System

The governing system studied in this paper is

$$\partial_t u + P\big((u \cdot \nabla)u\big) = P\nabla \cdot r, \quad (36)$$

$$\nabla \cdot u = 0, \quad (37)$$

$$\varepsilon\, \partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^\theta r, \quad (38)$$

where

u: velocity field

r: stress tensor

$\varepsilon > 0$: relaxation time

$\kappa > 0$: stress diffusion coefficient

$$0 < \theta \le 1$$

3.6. Navier–Stokes Limit

Formally taking the limit

$$\varepsilon \to 0, \quad (39)$$

Yields

$$r = 2\nu D(u). \quad (40)$$

Substituting this into (36) gives

$$\partial_t u + P\big((u \cdot \nabla)u\big) = \nu\Delta u, \quad (41)$$

which reduces to the Navier–Stokes equations. Thus the system (36)–(38) can be interpreted as a relaxation extension of Navier–Stokes dynamics.

3.7. Stress Deviation

Define stress deviation

$$w = r - 2\nu D(u). \quad (42)$$

Equation (38) can then be written as

$$\varepsilon\, \partial_t w + w = -2\nu\varepsilon D(\partial_t u) - \kappa(-\Delta)^\theta r. \quad (43)$$

This expression shows that the deviation $w$possesses relaxation dissipation.

3.8. Function Spaces for Strong Solutions

For the analysis of strong solutions we assume

$$u \in H^s_\sigma(\Omega), \quad (44)$$

$$r \in H^{s-1}(\Omega)^{3\times 3}, \quad (45)$$

With

$$s > 5/2. \quad (46)$$

Under this condition the Sobolev embedding

$$H^s(\Omega) \subset C^1, \quad (47)$$

holds, ensuring that the nonlinear term

$$(u \cdot \nabla)u, \quad (48)$$

is well defined.

3.9. Fundamental Structure of the Present System

The most important feature of system (36)–(38) is the presence of an additional dissipative structure. Using the stress deviation w, the system contains a dissipation term corresponding to

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$$\frac{1}{\varepsilon} \parallel w \parallel^2. \quad (49)$$

This term does not appear in the classical Navier–Stokes equations. The additional dissipation enables the control of high-frequency modes, which plays a crucial role in establishing strong solutions.

3.10. Content of the Next Chapter

In the next chapter we introduce the extended energy

$$E^{\varepsilon}, \quad (50)$$

for system (36)–(38) and derive an energy inequality. This estimate provides the basis for

a priori bounds,

the proof of global strong solutions.

Chapter 4 Extended Energy Structure and the Fundamental Energy Inequality

4.1. Purpose of This Chapter

In this chapter, for the relaxation Navier–Stokes system introduced in Chapter 3,

$$\partial_t u + P\big((u \cdot \nabla)u\big) = P\nabla \cdot r, \quad (51)$$

$$\varepsilon\, \partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^{\theta} r, \quad (52)$$

We introduce an extended energy structure and derive the corresponding fundamental energy inequality. We show that this energy structure possesses a triple dissipation mechanism, consisting of

viscous dissipation,

stress-diffusion dissipation, and

stress-relaxation dissipation.

This triple dissipative structure constitutes the key mathematical advantage of the relaxation system over the classical Navier–Stokes equations.

4.2. Introduction of the Stress Deviation

As the deviation from the Newtonian viscous stress

$$2\nu D(u), \quad (53)$$

we define

$$w := r - 2\nu D(u). \quad (54)$$

The quantity $w$represents stress deviation.

4.3. Rewriting of the Momentum Equation

Substituting (54) into (51), we obtain

$$\partial_t u + P((u \cdot \nabla)u) = \nu\Delta u + P\nabla \cdot w. \quad (55)$$

Accordingly, the relaxation system may be interpreted as the Navier–Stokes equations supplemented by a forcing term generated by the stress deviation.

4.4. Preservation of the Symmetry of the Stress Tensor

In the relaxation equation, the stress tensor r physically represents the Cauchy stress. It is therefore natural to assume that the initial data are symmetric tensors. This symmetry is preserved under the time evolution generated by the relaxation equation.

Lemma 4.1 (Preservation of the Symmetry of the Stress Tensor)

Assume that the initial data satisfies

$$r_0 = r_0^T. \quad (56)$$

Then every solution of the relaxation stress equation

$$\varepsilon\, \partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^{\theta} r,$$

satisfies, for all times,

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$$r(t) = r(t)^T. \tag{57}$$

Remark

Moreover, the trace-free property of the stress tensor is also preserved in time. Since the incompressibility condition $\nabla \cdot u = 0$ implies $\operatorname{tr} D(u) = 0$, taking the trace of equation (38) yields

$$\varepsilon\, \partial_t(\operatorname{tr} r) + \operatorname{tr} r + \kappa(-\Delta)^\theta(\operatorname{tr} r) = 0.$$

Therefore, if the initial stress satisfies $\operatorname{tr} r_0 = 0$, then

$$\operatorname{tr} r(t) = 0 \text{ for all } t \geq 0.$$

Proof

Define the antisymmetric part

$$A = r - r^T. \tag{58}$$

Taking the transpose of the relaxation equation gives

$$\varepsilon\, \partial_t r^T + r^T = 2\nu D(u) - \kappa(-\Delta)^\theta r^T.$$

Subtracting this from the original equation, we obtain

$$\varepsilon\, \partial_t A + A = -\kappa(-\Delta)^\theta A. \tag{59}$$

Multiplying this equation by A and integrating over space, we find

$$\frac{\varepsilon}{2}\frac{d}{dt} \| A \|_2^2 + \| A \|_2^2 + \kappa \| (-\Delta)^{\theta/2} A \|_2^2 = 0. \tag{60}$$

Since the initial condition gives $A(0) = 0$, it follows that $A(t) = 0$ for all t. Hence $r(t) = r(t)^T$ for all times. On the Trace-Free Structure Furthermore, the Newtonian stress $\sigma = 2\nu D(u)$ satisfies $\operatorname{tr} D(u) = 0$ under the incompressibility condition $\nabla \cdot u = 0$. Therefore, the relaxation equation also yields an independent evolution equation for the trace component. In particular, if the initial data satisfy

$$\operatorname{tr} r_0 = 0, \tag{61}$$

then $\operatorname{tr} r(t) = 0$ is preserved.

4.5. Velocity Energy Estimate

Multiplying (55) by u and integrating over space, we obtain

$$\langle \partial_t u, u \rangle + \langle (u \cdot \nabla) u, u \rangle = \nu \langle \Delta u, u \rangle + \langle \nabla \cdot w, u \rangle. \tag{62}$$

Here,

$$\langle \partial_t u, u \rangle = \frac{1}{2}\frac{d}{dt} \| u \|_2^2$$

holds.

4.6. Nonlinear Term

By the incompressibility condition $\nabla \cdot u = 0$, we have

$$\langle (u \cdot \nabla) u, u \rangle = 0. \tag{63}$$

4.7. Viscous Dissipation

By integration by parts,

$$\langle \Delta u, u \rangle = -\| \nabla u \|_2^2. \tag{64}$$

4.8. Stress-Deviation Term

Similarly,

$$\langle \nabla \cdot w, u \rangle = -\langle w, D(u) \rangle, \tag{65}$$

holds.

4.9. Velocity Energy Identity

Combining the above relations, we obtain

$$\frac{1}{2}\frac{d}{dt} \| u \|_2^2 + \nu \| \nabla u \|_2^2 = -\langle w, D(u) \rangle. \tag{66}$$

4.10. Transformation of the Stress Equation

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Substituting the definition:

$$r = w + 2\nu D(u), \tag{67}$$

into (52), we obtain

$$\varepsilon\, \partial_t w + w = -2\nu\varepsilon D(\partial_t u) - \kappa(-\Delta)^{\theta} r. \tag{68}$$

4.11. Energy of the Stress Deviation

Multiplying (68) by w and integrating, we obtain

$$\frac{\varepsilon}{2}\frac{d}{dt} \| w \|_2^2 + \| w \|_2^2 = -2\nu\varepsilon\langle D(\partial_t u), w\rangle - \kappa\langle(-\Delta)^{\theta} r, w\rangle. \tag{69}$$

4.12. Dissipation by the Fractional Laplacian

From the Fourier representation,

$$\langle(-\Delta)^{\theta} r, r\rangle = \| (-\Delta)^{\frac{\theta}{2}} r \|_2^2, \tag{70}$$

holds. This term represents the high-frequency dissipation of the stress field.

4.13. Extended Energy

We now introduce the following extended energy:

$$E_{\varepsilon}(t) = \frac{1}{2} \| u \|_2^2 + \frac{\alpha}{2} \| w \|_2^2 + \frac{\varepsilon}{2} \| r \|_2^2, \tag{71}$$

where $\alpha > 0$ is a sufficiently small constant.

4.14. Fundamental Energy Inequality

Combining (66), (69), (70), and (71), we obtain, for suitable positive constants c_1, c_2, c_3,

$$E_{\varepsilon}(t) + \int_0^t \left(\nu \| \nabla u \|_2^2 + \frac{1}{\varepsilon} \| w \|_2^2 + \frac{\kappa}{\varepsilon} \| (-\Delta)^{\theta/2} r \|_2^2\right) ds \le E_{\varepsilon}(0). \tag{72}$$

4.15. Important Consequences

Lemma 4.1 yields the following energy bounds. For the velocity field,

$$u \in L^{\infty}(0,T;L^2) \cap L^2(0,T;H^1).$$

For the stress field,

$$r \in L^2(0,T;H^{\theta}).$$

Furthermore, for the stress deviation

$$w = r - 2\nu D(u),$$

we have

$$w \in L^2(0,T;L^2).$$

In particular, the dissipation corresponds to

$$\frac{1}{\varepsilon} \| w \|^2$$

constitutes an additional dissipation mechanism absent from the classical Navier–Stokes equations.

4.16. Conclusion of This Chapter

We have thus shown that the relaxation Navier–Stokes system possesses a triple dissipative structure consisting of

viscous dissipation,
stress-diffusion dissipation, and
stress-relaxation dissipation.

This structure provides a stabilization mechanism stronger than that of the classical Navier–Stokes equations.

In the next chapter, using this energy estimate, we derive a priori bounds in the Sobolev space H^s.

Chapter 5 A Priori Estimates in Sobolev Spaces

5.1. Purpose of This Chapter

In Chapter 4 we derived the fundamental energy inequality for the relaxation Navier–Stokes system and obtained L^2-level stability for both the velocity field and the stress field. However, in order to discuss the existence of strong solutions, control of Sobolev norms with higher regularity is indispensable.

In this chapter we derive a priori estimates in Sobolev spaces for the velocity field u and the stress tensor r, and show that the relaxation system, even in the presence of the nonlinear convective term, admits a closed higher-order energy estimate.

5.2. Definition of the Higher-Order Energy

Fix the Sobolev index

$$s > \frac{5}{2}.$$

This condition guarantees, in three dimensions, the Sobolev embedding $H^s(\Omega) \hookrightarrow C^1(\Omega)$. We define the higher-order energy by

$$E_s(t) = \| u(t) \|_{H^s}^2 + \varepsilon \| r(t) \|_{H^{s-1}}^2. \quad (73)$$

Here the Sobolev norm is given by

$$\| u \|_{H^s}^2 = \sum_{|\alpha| \le s} \| \partial^\alpha u \|_2^2. \quad (74)$$

5.3. Higher-Order Differentiation of the Equations

For a multi-index α with $|\alpha| \le s$, differentiating the velocity equation

$$\partial_t u + P((u \cdot \nabla) u) = P \nabla \cdot r$$

yields

$$\partial^\alpha \partial_t u + \partial^\alpha P((u \cdot \nabla) u) = \partial^\alpha P \nabla \cdot r. \quad (75)$$

Similarly, from the stress equation

$$\varepsilon \partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^\theta r,$$

we obtain

$$\varepsilon \partial^\alpha \partial_t r + \partial^\alpha r = 2\nu \partial^\alpha D(u) - \kappa(-\Delta)^\theta \partial^\alpha r. \quad (76)$$

5.4. Higher-Order Energy Estimate for the Velocity Equation

Multiplying (75) by $\partial^\alpha u$ and integrating over space, we obtain

$$\frac{1}{2}\frac{d}{dt} \| \partial^\alpha u \|_2^2 + \int \partial^\alpha P((u \cdot \nabla) u) \cdot \partial^\alpha u \, dx = \int \partial^\alpha \nabla \cdot r \cdot \partial^\alpha u \, dx. \quad (77)$$

To handle the nonlinear term, we use the commutator decomposition

$$\partial^\alpha ((u \cdot \nabla) u) = (u \cdot \nabla) \partial^\alpha u + [\partial^\alpha, u \cdot \nabla] u. \quad (78)$$

Since

$$\int (u \cdot \nabla) \partial^\alpha u \cdot \partial^\alpha u \, dx = 0,$$

the remaining contribution is the commutator term. To treat fractional Sobolev regularity, we use the estimate

$$\| [\Lambda^s, u \cdot \nabla] u \|_{L^2} \le C \| \nabla u \|_{L^\infty} \| u \|_{H^s},$$

which follows from the classical Kato–Ponce commutator inequality. Hence,

$$\| [\partial^\alpha, u \cdot \nabla] u \|_2 \le C \| \nabla u \|_{L^\infty} \| u \|_{H^s}. \quad (79)$$

5.5. Estimate of the Nonlinear Term

Using (79), we obtain

$$\left| \int \partial^\alpha ((u \cdot \nabla) u) \cdot \partial^\alpha u \, dx \right| \le C \| \nabla u \|_{L^\infty} \| u \|_{H^s}^2. \quad (80)$$

5.6. Stress Term

The stress term is

$$\int \partial^\alpha \nabla \cdot r \cdot \partial^\alpha u \, dx. \quad (81)$$

By integration by parts,

$$\int \partial^\alpha \nabla \cdot r \cdot \partial^\alpha u \, dx = -\int \partial^\alpha r : \partial^\alpha D(u) \, dx. \quad (82)$$

5.7. Energy Estimate for the Stress Equation

Multiplying (76) by $\partial^\alpha r$ and integrating, we obtain

$$\frac{\varepsilon}{2}\frac{d}{dt} \| \partial^\alpha r \|_2^2 + \| \partial^\alpha r \|_2^2 + \kappa \| (-\Delta)^{\theta/2} \partial^\alpha r \|_2^2 = 2\nu \int \partial^\alpha r : \partial^\alpha D(u) \, dx. \quad (83)$$

5.8. Cancellation of the Cross Terms

Adding (82) and (83), the terms involving

$$\int \partial^\alpha r : \partial^\alpha D(u) \, dx$$

cancel. Therefore, we obtain

$$\frac{d}{dt} E_s + \| r \|_{H^{s-1}}^2 + \kappa \| r \|_{H^{s-1+\theta}}^2 \le C \| \nabla u \|_{L^\infty} \| u \|_{H^s}^2. \quad (84)$$

5.9. Sobolev Embedding

By the Sobolev embedding

$$H^s(\Omega) \hookrightarrow L^\infty(\Omega) (s > 3/2), \quad (85)$$

we have

$$\| \nabla u \|_{L^\infty} \le C \| u \|_{H^s}. \quad (86)$$

Therefore,

$$\frac{d}{dt} E_s \le C E_s^{3/2}. \quad (87)$$

5.10. A Priori Bound

The differential inequality (87) yields

$$E_s(t) \le \frac{E_s(0)}{\left(1 - Ct\sqrt{E_s(0)}\right)^2}. \quad (88)$$

Hence, if the initial energy is sufficiently small,

$$E_s(0) \ll 1, \quad (89)$$

the solution does not blow up in finite time.

5.11. Conclusion of This Chapter

We have therefore established the following result.

Proposition 5.1 (Sobolev A Priori Bound)

If the initial data

$$u_0 \in H_\sigma^s, \; r_0 \in H^{s-1}, \quad (90)$$

are sufficiently small, then

$$u(t) \in H^s, \; r(t) \in H^{s-1}$$

hold on the time interval $[0, T]$.

5.12. Connection to the Next Chapter

In this chapter we derived an a priori bound by combining
higher-order Sobolev energy,
commutator estimates, and
control of the nonlinear term.
In the next chapter, we use this estimate to prove the local existence theorem for strong solutions.

Chapter 6 Local Existence and Uniqueness of Strong Solutions

6.1. Purpose of This Chapter

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In the previous chapter we derived an a priori bound in the Sobolev space H^s for the relaxation Navier–Stokes system. In this chapter, using that estimate, we prove the local existence and uniqueness of strong solutions for the initial value problem

$$u(0) = u_0, r(0) = r_0. \tag{91}$$

The analysis is carried out in the Sobolev spaces

$$u \in H^s_\sigma(\Omega),\ \ r \in H^{s-1}(\Omega)^{3\times 3}, \tag{92}$$

With

$$s > \frac{5}{2}. \tag{93}$$

Under this condition, the Sobolev embedding

$$H^s(\Omega) \subset C^1(\Omega), \tag{94}$$

holds, and products appearing in the nonlinear terms are therefore well defined.

6.2. Construction of Local Solutions by Friedrichs Approximation

To construct local strong solutions, we employ the Friedrichs cut-off approximation. Define the low-frequency projection in Fourier space by

$$J_N f = \mathcal{F}^{-1}(1_{|\xi|\le N}\,\hat{f}(\xi)). \tag{95}$$

We then consider the approximate system

$$\partial_t u_N + PJ_N((u_N\cdot\nabla)u_N) = P\nabla\cdot r_N, \tag{96}$$

$$\varepsilon\,\partial_t r_N + r_N = 2\nu D(u_N) - \kappa(-\Delta)^\theta r_N. \tag{97}$$

This system is a finite-dimensional system of ordinary differential equations, and therefore local solutions exist by standard theory.

1) Uniform Energy Estimate

The energy estimates derived in Chapters 4 and 5 remain valid for this approximate system. In particular,

$$E_s^{(N)}(t) \le C,$$

holds with a constant independent of N. Consequently, u_N is uniformly bounded in $L^\infty(0,T;H^s)$.

2) Compactness

By the Aubin–Lions lemma, one obtains the convergence

$$u_N \to u\text{in } C([0,T];H^{s-1}).$$

Hence the limit function (u, r) is a local strong solution of the relaxation Navier–Stokes system.

6.3. Linearized Problem

We first consider the linearized system in which the nonlinear term is treated with a given velocity field:

$$\partial_t u + P((v\cdot\nabla)u) = P\nabla\cdot r, \tag{98}$$

$$\varepsilon\,\partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^\theta r, \tag{99}$$

where v is a prescribed velocity field. This is a linear evolution system, and solutions exist by the standard energy method.

6.4. Iterative Construction

Define the iteration sequence by

$$u^{(0)} = u_0, r^{(0)} = r_0. \tag{100}$$

Then, for $k \ge 0$, solve

$$\partial_t u^{(k+1)} + P((u^{(k)}\cdot\nabla)u^{(k+1)}) = P\nabla\cdot r^{(k+1)}, \tag{101}$$

$$\varepsilon\,\partial_t r^{(k+1)} + r^{(k+1)} = 2\nu D(u^{(k+1)}) - \kappa(-\Delta)^\theta r^{(k+1)}. \tag{102}$$

This is a linear system at each step, so a solution can be constructed iteratively.

6.5. Uniform Energy Bound

Applying the estimate of Chapter 5, the higher-order energy

$$E_s^{(k)}(t) = \| u^{(k)} \|_{H^s}^2 + \varepsilon \| r^{(k)} \|_{H^{s-1}}^2, \tag{103}$$

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Satisfies

$$\frac{d}{dt}E_s^{(k)} \le C(E_s^{(k)})^{3/2}. \tag{104}$$

Therefore, if the initial data are sufficiently small, then $E_s^{(k)}(t) \le C$.

6.6. Convergence

Consider the differences

$$\delta u^{(k)} = u^{(k+1)} - u^{(k)}, \delta r^{(k)} = r^{(k+1)} - r^{(k)}. \tag{105}$$

These satisfy

$$\frac{d}{dt} \| \delta u^{(k)} \|_{H^{s-1}}^2 + \| \delta r^{(k)} \|_{H^{s-2}}^2 \le C \| \delta u^{(k-1)} \|_{H^{s-1}}^2. \tag{106}$$

By Grönwall's inequality,

$$\delta u^{(k)} \to 0, \delta r^{(k)} \to 0, \tag{107}$$

follows. Therefore,

$$(u^{(k)}, r^{(k)}) \to (u, r). \tag{108}$$

Moreover, by the Aubin–Lions compactness theorem,

$$u^{(k)} \to u \text{in } C([0,T]; H^{s-1}). \tag{109}$$

6.7. Local Existence of Solutions

We thus obtain the following theorem.

Theorem 6.1 (Local Existence of Strong Solutions)

For initial data

$$u_0 \in H_\sigma^s(\Omega), r_0 \in H^{s-1}(\Omega), \tag{110}$$

there exists $T > 0$ such that a solution satisfying

$$u \in C([0,T]; H_\sigma^s), \tag{111}$$

$$r \in C([0,T]; H^{s-1}), \tag{112}$$

exists.

6.8. Uniqueness

Consider two solutions

$$(u_1, r_1), (u_2, r_2). \tag{113}$$

For the differences

$$\tilde{u} = u_1 - u_2, \tilde{r} = r_1 - r_2, \tag{114}$$

an energy estimate gives

$$\frac{d}{dt}(\| \tilde{u} \|_{H^{s-1}}^2 + \| \tilde{r} \|_{H^{s-2}}^2) \le C(\| \tilde{u} \|_{H^{s-1}}^2 + \| \tilde{r} \|_{H^{s-2}}^2). \tag{115}$$

Here, to estimate the nonlinear term, we use

$$\| (u_1 \cdot \nabla)\tilde{u} \| \le \| \nabla u_1 \|_{L^\infty} \| \tilde{u} \|,$$

together with the Sobolev embedding

$$H^s(\Omega) \hookrightarrow W^{1,\infty}(\Omega) (s > 5/2),$$

which implies

$$\| \nabla u \|_{L^\infty} \le C \| u \|_{H^s}.$$

Applying Grönwall's inequality, we conclude that $\tilde{u} = \tilde{r} = 0$, and uniqueness follows.

6.9. Conclusion of This Chapter

In this chapter we established:

local existence of solutions in Sobolev spaces,

uniqueness of solutions, and

continuous dependence on the initial data.

Therefore, the relaxation Navier–Stokes system is locally well posed in the space H^s. In the next chapter, by combining the a priori bound of Chapter 5 with the local existence result obtained here, we prove the existence of global strong solutions for small initial data.

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Chapter 7 Global Strong Solutions for Small Initial Data

7.1. Purpose of This Chapter

In this chapter, by combining the results established in Chapters 4 through 6, we prove the existence of global strong solutions for small initial data for the relaxation Navier–Stokes system

$$\partial_t u + P\big((u \cdot \nabla)u\big) = P\nabla \cdot r, \tag{36}$$

$$\varepsilon\,\partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^{\theta} r. \tag{38}$$

The central idea is that the higher-order a priori estimate derived previously prevents finite-time blow-up when the initial energy is sufficiently small.

7.2. Restatement of the Higher-Order Energy

We recall the higher-order energy introduced in Chapter 5:

$$E_s(t) = \| u(t) \|_{H^s}^2 + \varepsilon \| r(t) \|_{H^{s-1}}^2. \tag{116}$$

Here we assume

$$s > \frac{5}{2}. \tag{117}$$

Under this condition, the Sobolev embedding

$$H^s(\Omega) \subset C^1(\Omega), \tag{118}$$

holds.

7.3. Differential Inequality for the Energy

From the computation in Chapter 5, we have

$$\frac{d}{dt}E_s + \| r \|_{H^{s-1}}^2 + \kappa \| r \|_{H^{s-1+\theta}}^2 \le C \| u \|_{H^s}^3. \tag{119}$$

By the relation between the Sobolev norm and the higher-order energy,

$$\| u \|_{H^s}^3 \le CE_s^{3/2}, \tag{120}$$

and therefore, we obtain

$$\frac{d}{dt}E_s \le CE_s^{3/2}. \tag{121}$$

This differential inequality is the key mechanism governing the long-time behavior of the solution.

7.4. Small-Data Condition

Assume that the initial energy

$$E_s(0) = \| u_0 \|_{H^s}^2 + \varepsilon \| r_0 \|_{H^{s-1}}^2, \tag{122}$$

is sufficiently small. Namely, we assume

$$E_s(0) \le \delta, \tag{123}$$

where $\delta > 0$ is a sufficiently small constant.

7.5. Boundedness of the Energy

Using the differential inequality (121), we have

$$\frac{d}{dt}E_s \le CE_s^{3/2}. \tag{124}$$

Integrating this inequality yields

$$E_s(t) \le \frac{E_s(0)}{\left(1 - Ct\sqrt{E_s(0)}\right)^2}. \tag{125}$$

If the initial data are sufficiently small, then

$$C\sqrt{E_s(0)} \ll 1. \tag{126}$$

Hence no finite-time blow-up can occur. More precisely, the higher-order energy remains bounded on any finite time interval, and the local solution can therefore be continued beyond every finite time horizon.

7.6. Extension to a Global Solution

By Chapter 6, there exists a local solution satisfying

$$u \in C([0,T]; H^s), \quad (127)$$

$$r \in C([0,T]; H^{s-1}). \quad (128)$$

On the other hand, (125) yields

$$E_s(t) \leq C, \quad (129)$$

for all times within the interval of existence. Therefore, the blow-up condition

$$\lim_{t \to T^*} \| u(t) \|_{H^s} = \infty, \quad (130)$$

cannot occur. Consequently, the solution can be extended up to

$$T^* = \infty. \quad (131)$$

Thus, the local strong solution obtained in Chapter 6 extends globally in time.

7.7. Main Theorem

Summarizing the above argument, we obtain the following result.

Main Theorem

For sufficiently small initial data

$$u_0 \in H^s_\sigma(\Omega), r_0 \in H^{s-1}(\Omega),$$

the relaxation Navier–Stokes system (36)–(38) admits a global strong solution satisfying

$$u \in C([0,\infty); H^s_\sigma), \quad (132)$$

$$r \in C([0,\infty); H^{s-1}). \quad (133)$$

Moreover, the solution is unique.

7.8. Relation to the Navier–Stokes Equations

Taking the relaxation limit

$$\varepsilon \to 0, \quad (134)$$

formally yields

$$r = 2\nu D(u), \quad (135)$$

and the system reduces to the Navier–Stokes equations

$$\partial_t u + (u \cdot \nabla)u = \nu\Delta u - \nabla p. \quad (136)$$

Accordingly, the present result establishes the existence of global strong solutions for a system possessing stronger dissipation than the classical Navier–Stokes equations.

This should be interpreted not as a direct solution to the Navier–Stokes regularity problem itself, but as evidence that the introduction of a physically meaningful relaxation structure can fundamentally enhance stability.

7.9. Conclusion of This Chapter

In this chapter we have shown:

boundedness of the higher-order energy under a small-data condition,

global continuation of the local solution, and

existence of a global strong solution.

These results suggest that the relaxation structure may suppress the nonlinear instability inherent in the Navier–Stokes equations.

Chapter 8 Conclusion

8.1. Objective and Background of the Present Study

In this study, we revisited the mathematical structure of the incompressible Navier–Stokes equations and analyzed a relaxation Navier–Stokes system in which the stress tensor is introduced as an independent variable.

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This approach is based on the framework of the previous study [32], in which fluid equations were derived from the master equation of an interacting system, and its purpose is to extend the analysis from the theory of weak solutions to that of strong solutions.

Instead of treating the Newtonian constitutive relation

$$r = 2\nu D(u), \tag{137}$$

as an instantaneous law, we represent it as a process in which stress relaxes toward equilibrium. This leads to the following extended system:

$$\partial_t u + P\big((u\cdot\nabla)u\big) = P\nabla\cdot r, \tag{138}$$

$$\varepsilon\,\partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^{\theta} r. \tag{139}$$

This system reduces to the Navier–Stokes equations in the relaxation limit

$$\varepsilon \to 0. \tag{140}$$

8.2. Principal Results of the Present Study

In this paper, we obtained the following results for the relaxation Navier–Stokes system. First, by introducing extended energy

$$E^{\varepsilon}(t) = \frac{1}{2}\parallel u \parallel_2^2 + \frac{\varepsilon}{2}\parallel r \parallel_2^2, \tag{141}$$

we derived the fundamental energy inequality

$$E^{\varepsilon}(t) + \int_0^t \left(\nu \parallel \nabla u \parallel_2^2 + \frac{1}{\varepsilon}\parallel r - 2\nu D(u) \parallel_2^2 + \kappa \parallel (-\Delta)^{\theta/2} r \parallel_2^2\right) ds \le E^{\varepsilon}(0). \tag{142}$$

This estimate reveals a triple dissipative structure consisting of

viscous dissipation of the fluid,

high-frequency dissipation induced by stress diffusion, and

relaxation dissipation of the stress deviation.

Furthermore, by introducing the higher-order energy in the Sobolev space H^s,

$$E_s(t) = \parallel u \parallel_{H^s}^2 + \varepsilon \parallel r \parallel_{H^{s-1}}^2, \tag{143}$$

We obtained a priori estimate

$$\frac{d}{dt}E_s \le C E_s^{3/2}. \tag{144}$$

Using this estimate, we proved the local existence of strong solutions and, under a small-data condition, established the existence of global strong solutions.

8.3. Theoretical Interpretation

The results of the present study provide a new perspective for interpreting the nonlinear structure of the Navier–Stokes equations.

Introducing stress deviation

$$w = r - 2\nu D(u), \tag{145}$$

We find that the energy inequality (142) contains an additional dissipation term corresponding to

$$\frac{1}{\varepsilon}\parallel w \parallel^2. \tag{146}$$

This term is absent from the classical Navier–Stokes equations and acts as an additional stabilization mechanism that suppresses amplification of high-frequency modes. From this viewpoint, the present result suggests the following interpretation of the singularity problem for the Navier–Stokes equations: the formation of singularities may be understood as a breakdown of the stress-relaxation structure. Although this interpretation remains heuristic at the present stage, it provides a conceptually meaningful direction for further investigation.

8.4. Relation to the Previous Study

In the previous study [32], the Navier–Stokes equations were derived as the continuous limit of a network-type master equation, and the existence of Leray–Hopf weak solutions was proved by means of a relaxation extension.

The present study further develops that framework by integrating

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weak-solution construction,
higher-order energy estimates, and
local strong-solution theory.

In this sense, the present work establishes a theoretical continuity of the form

discrete interacting system $\longrightarrow$ weak solution $\longrightarrow$ strong solution (for small data).

This continuity clarifies how the relaxation framework can serve as a bridge between microscopic interaction models and macroscopic regularity theory.

8.5. Future Problems

The present study does not prove the existence of strong solutions for the Navier–Stokes equations themselves. However, it does show that global strong solutions exist for a relaxation system endowed with a stronger dissipative structure.

The following problems remain important directions for future research:

rigorous analysis of the relaxation limit

$$\varepsilon \to 0, \tag{147}$$

more refined estimates for the stress-deviation energy, and

clarification of the relationship to singularity formation in the Navier–Stokes equations.

In particular, understanding how the relaxation structure influences the regularity problem of the Navier–Stokes equations remain a central challenge.

8.6. Positioning of the Present Study

The above results present a new perspective in which fluid equations are understood as relaxation processes of interacting systems. This framework occupies an intersection between fluid mechanics and nonequilibrium statistical mechanics, and is expected to open new possibilities for future research.

Its significance lies not only in the derivation of a mathematically tractable extended system, but also in the conceptual proposal that constitutive relaxation may play a fundamental role in the stabilization of nonlinear fluid motion.

Appendix

A.1 Discrete Interacting Systems and the Continuous Limit (Related to Chapter 2, Equations (8)–(13))

In this section, we summarize the limiting structure through which the network-type master equation gives rise to a conservative partial differential equation. Let the state variable at node i be

$$U_i = (\rho_i, m_i, e_i). \tag{A1}$$

Here, ρ_i denotes the density, m_i the momentum, and e_i the energy. The master equation is given by

$$\frac{dU_i}{dt} = \sum_{j\in N(i)} A_{ij}\ [F(U_i, U_j) + D(U_i, U_j)]. \tag{A2}$$

Finite-Volume Form:

Introducing the cell volume V_i and the interface area S_{ij}, we obtain

$$\frac{dU_i}{dt} = -\frac{1}{V_i}\sum_{j\in N(i)} S_{ij}\ F_{ij}\cdot n_{ij}. \tag{A3}$$

This coincides with the finite-volume discretization of the conservative PDE

$$\partial_t U + \nabla\cdot F(U) = 0. \tag{A4}$$

Continuous Limit:

In the limit where the cell size satisfies

$$h \to 0, \tag{A5}$$

we have

$$\nabla\cdot F(U) = \lim_{h\to 0}\frac{1}{V_i}\sum S_{ij}F_{ij}\cdot n_{ij}. \tag{A6}$$

Through this limiting procedure, one obtains

$$\partial_t U + \nabla\cdot F(U) = \nabla\cdot(\mu\nabla U). \tag{A7}$$

This relation formally connects the interacting discrete system to the macroscopic conservation law with diffusion.

A.2 Derivation of the Fundamental Energy Inequality (Related to Chapter 4, Lemma 4.1, Equation (72))

In this section, we provide a more detailed derivation of the fundamental energy inequality (72) established in Chapter 4.

Consider the relaxation system

$$\partial_t u + P\big((u\cdot\nabla)u\big) = P\nabla\cdot r, \tag{A8}$$

$$\varepsilon\,\partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^{\theta} r. \tag{A9}$$

Velocity Energy:

Multiply (A8) by u. Then

$$\int u\cdot\partial_t u\,dx + \int u\cdot P((u\cdot\nabla)u)\,dx = \int u\cdot P\nabla\cdot r\,dx. \tag{A10}$$

The first term becomes

$$\int u\cdot\partial_t u\,dx = \frac{1}{2}\frac{d}{dt}\parallel u\parallel_2^2. \tag{A11}$$

Nonlinear Term:

By the incompressibility condition

$$\nabla\cdot u = 0, \tag{A12}$$

we have

$$\int (u\cdot\nabla)u\cdot u\,dx = 0. \tag{A13}$$

Stress Term:

By integration by parts,

$$\int u\cdot\nabla\cdot r\,dx = -\int r{:}\,D(u)\,dx. \tag{A14}$$

Stress Equation:

Multiply (A9) by r. Then

$$\varepsilon\int r\;\partial_t r\,dx + \parallel r\parallel_2^2 = 2\nu\int r{:}\,D(u)\,dx - \kappa\int r(-\Delta)^{\theta} r\,dx. \tag{A15}$$

The first term becomes

$$\varepsilon\int r\;\partial_t r\,dx = \frac{\varepsilon}{2}\frac{d}{dt}\parallel r\parallel_2^2. \tag{A16}$$

Fractional Laplacian:

By the Fourier representation,

$$\int r(-\Delta)^{\theta} r\,dx = \parallel (-\Delta)^{\theta/2} r\parallel_2^2. \tag{A17}$$

Introducing the stress-deviation variable $w = r - 2\nu D(u)$, and combining the above estimates with Young's inequality, one obtains the triple-dissipation energy inequality (72) derived in Chapter 4.

A.3 Sobolev Product Estimate (Related to Chapter 5, Equations (79)–(86))

On the three-dimensional torus, the embedding

$$H^s(\Omega)\subset L^{\infty}(\Omega), s > \frac{3}{2}, \tag{A18}$$

holds.

Product Estimate:

We have the estimate

$$\parallel fg\parallel_{H^s} \le C(\parallel f\parallel_{H^s}\parallel g\parallel_{L^{\infty}} + \parallel g\parallel_{H^s}\parallel f\parallel_{L^{\infty}}). \tag{A19}$$

Application for the Nonlinear Term:

Applying this to

$$(u\cdot\nabla)u, \tag{A20}$$

we obtain

$$\parallel (u\cdot\nabla)u\parallel_{H^s} \le C\parallel u\parallel_{H^s}\parallel \nabla u\parallel_{H^s}. \tag{A21}$$

This estimate is one of the basic ingredients in the higher-order energy analysis.

A.4 Aubin–Lions Compactness (Related to Chapter 6, Equation (109))

Let

$$X_0 \subset X \subset X_1, \tag{A22}$$

be Banach spaces such that

$$X_0 \Subset X, \tag{A23}$$

is a compact embedding. Then

$$L^2(0,T;X_0) \cap H^1(0,T;X_1), \tag{A24}$$

is compactly embedded into

$$L^2(0,T;X). \tag{A25}$$

Application for the Present Study:

In the present work, we take

$$X_0 = H^1, X = L^2, X_1 = H^{-1}. \tag{A26}$$

Then we obtain

$$u^\varepsilon \to u \text{ in } L^2. \tag{A27}$$

This compactness plays a central role in the passage from approximate solutions to strong solutions.

Nomenclature

Roman symbols

A_{ij}— Coupling strength (adjacency matrix element) between nodes $i$and j
C— Generic positive constant independent of the solution
$D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$— Rate-of-deformation tensor
$D(U_i, U_j)$— Dissipative interfacial flux between nodes
$E_i^{tot} = k_i + e_i$— Total energy density at node i
$E^\varepsilon(t)$— Extended energy functional of the relaxation system
$E_s(t)$— High-order Sobolev energy
$F(U_i, U_j)$— Conservative exchange flux between nodes
H^s— Sobolev space of order s
k_i— Kinetic energy density at node i
$N(i)$— Set of nodes adjacent to node i
P— Leray projection operator onto divergence-free vector fields
p— Pressure field
r— Stress tensor treated as an independent dynamical variable
S_{ij}— Interface area between cells $i$and j
t— Time variable
T— Final time of the time interval
$U_i = (\rho_i, m_i, e_i)$— State vector at node i
u— Velocity field
u_0— Initial velocity field
V_i— Volume of the cell corresponding to node i
$w = r - 2\nu D(u)$— Stress deviation from Newtonian equilibrium

Greek symbols
α— Multi-index for partial derivatives
ε— Stress relaxation parameter
θ— Fractional Laplacian exponent
κ— Stress diffusion coefficient
μ— Diffusion coefficient in the continuum limit
ν— Kinematic viscosity
ρ— Fluid density
σ— Newtonian viscous stress tensor

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Ω— Spatial domain
∇— Gradient operator
Δ— Laplacian operator

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