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Unified 4D Spinor Space for Dirac and Weyl Spinors: A Vector-Sum Decomposition of the Dirac Spinor

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Abstract

We introduce a novel four-dimensional spinor representation of the Lorentz group in which both Dirac and Weyl spinors are realized as four-component objects living in a common vector space. Furthermore, Dirac spinors can be expressed as vector sum -rather than a direct sum- of left- and right-chiral four-component Weyl spinors. In this representation, Dirac spinors and their left and right components transform under the same spinor space, permitting an unambiguous identification of their chiral constituents. This formalism provides a symmetric and geometrically transparent reinterpretation of Weyl and Dirac spinors and may offer new insights into extended spinor models and relativistic field theories.

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1. Introduction

In the chiral (Weyl) basis, a Dirac spinor ψ is conventionally expressed as a direct sum of the following form:

$$\psi = \xi_L \oplus \chi_R \tag{1}$$

In this equation, ξ_L and χ_R are two-component Weyl spinors, each of which transforms under the respective two-dimensional spinor representation of the Lorentz group [1–5]:

$$\xi_L \to \xi'_L = L\xi_L, \qquad \chi_R \to \chi'_R = \dot{L}\chi_R \tag{2}$$

Where $\dot{L} = (L^{-1})^\dagger$. To isolate the left- and right-chiral parts of ψ, one uses the standard projection operators, defined in terms of γ^5:

$$P_L = \frac{1}{2}(I_4 - \gamma^5), \qquad P_R = \frac{1}{2}(I_4 + \gamma^5) \tag{3}$$

Action of these projectors on ψ yield the four-component parts ψ_L and ψ_R

$$\psi_L = \xi_L \oplus 0, \qquad \psi_R = 0 \oplus \chi_R, \qquad \psi = \psi_L + \psi_R \tag{4}$$

But, ψ_L and ψ_R are distinct from the original two-component Weyl spinors ξ_L and χ_R. In this traditional framework, the projection operators do not truncate the remaining empty spinor components, they instead produce objects with extended structures, The situation will be even worse in a representation other than the chiral one. Therefore, although a Dirac spinor can be considered as a direct sum of two-component Weyl spinors, once assembled, it cannot be decomposed back into its original components by using the standard projection operators.

In what follows, we propose an alternative formulation based on a new set of matrices g^μ, which obeys the Clifford algebra of spacetime. Although, there exists a similarity transformation between

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g^μ and traditional basis γ^μ, the new representation is structurally distinct, and it serves as a bridge between the Clifford algebra and the formalism of four-component Weyl spinors.

In the new framework, all spinors live in the same unified four-dimensional space, and Dirac spinors can be expressed as vector sum of left- and right-chiral four-component Weyl spinors. Furthermore, left- and right-projection operators defined in terms of g^5 retrieves back the full left- and right-chiral Weyl components directly.

2. A Four-Dimensional Spinor Representation for the Lorentz Group

It is well known that a Lorentz transformation matrix Λ can be written as a direct product of left-and right-handed spinor representations of the Lorentz group:

$$\Lambda = L \otimes L^* \tag{5}$$

Let us write this equation in a different way:

$$L \otimes L^* = (L \otimes I_2)(I_2 \otimes L^*) \tag{6}$$

and define two new matrices Z and Z^* [6,7]:

$$Z = L \otimes I_2, \qquad Z^* = I_2 \otimes L^* \tag{7}$$

These definitions allow us to express Λ as a *matrix* product:

$$\Lambda = ZZ^* \tag{8}$$

Since Z and Z^* commute we also have $\Lambda = Z^*Z$.

The significance of these rather trivial re-definitions is that Z and Z^* now can be interpreted as new four-dimensional left and right *irreducible* representations for the Lorentz group acting on left- and right-chiral four-component Weyl spinors, respectively. Formally, Z and Z^* are four-dimensional analogues of L and L^*, but in the new framework the spinor space is also four-dimensional.

The expression given in Equation (5) does not return the traditional (real) Lorentz transformation matrix. To recover the familiar real form a change of basis is required:

$$\Lambda \to U\Lambda U^{-1} \tag{9}$$

Explicit form of U is given in the Appendix A. Accordingly, we re-define Z and Z^* matrices. In the new basis,

$$Z = U(L \otimes I_2)U^{-1}, \qquad Z^* = U(I_2 \otimes L^*)U^{-1} \tag{10}$$

We also define four-dimensional analogues of the Pauli matrices:

$$\Sigma^\mu = U(\sigma^\mu \otimes \sigma^0)U^{-1}, \qquad \sigma^0 = I_2 \tag{11}$$

$$\Sigma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Sigma^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \Sigma^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \Sigma^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

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Let us choose the eigenvectors of Σ^3 as a basis for our unified four-dimensional spinor space [1]:

$$e_u^p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_d^p = \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}, \quad e_u^q = \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad e_d^q = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{13}$$

where e_u^p and e_u^q correspond to $+1$ eigenvalue, and thus represent spin-up states; e_d^p and e_d^q correspond to -1 eigenvalue, and represent spin-down states.

We now use this four-dimensional structure to define four-component Weyl-spinors. In the new formalism Weyl spinors will be solutions to the four-dimensional version of the Weyl equations expressed in the Σ^μ basis:

$$(i\Sigma^\mu \partial_\mu)\mathcal{X} = 0, \qquad (i\bar{\Sigma}^\mu \partial_\mu)\mathcal{Y} = 0 \tag{14}$$

where

$$\Sigma^\mu = (\Sigma^0, \vec{\Sigma}), \qquad \bar{\Sigma}^\mu = (\Sigma^0, -\vec{\Sigma}) \tag{15}$$

In a similar way to the two-dimensional framework, right-handed four-component Weyl spinors can be defined via a spinor-Minkowski metric g for the four-dimensional spinor representation of the Lorentz group:

$$\mathcal{X}_R = (g\mathcal{X}_L)^* \tag{16}$$

where

$$g = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \\ -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad g^{-1} = g^\dagger \tag{17}$$

Left- and right-chiral Weyl spinors transform under the four-dimensional left and right representations respectively:

$$\mathcal{X}_L \to \mathcal{X}'_L = Z\mathcal{X}_L, \qquad \mathcal{X}_R \to \mathcal{X}'_R = \dot{Z}\mathcal{X}_R, \qquad \text{where} \quad \dot{Z} = (Z^{-1})^\dagger \tag{18}$$

Z preserves both the spinor metric g and the Minkowski metric η [2].

$$Z^T g Z = g, \qquad Z^T \eta Z = \eta \tag{19}$$

Since η is real, $Z^T\eta Z = \eta$ directly entails $\Lambda^T\eta\Lambda = \eta$. We also have the following relation:

$$g\Sigma_i g^{-1} = -\Sigma_i^* \tag{20}$$

And the dot product of four-component Weyl spinors is given as:

$$\mathcal{X} \cdot \mathcal{X} = \mathcal{X}^T g \mathcal{X} \tag{21}$$

which is Lorentz invariant:

$$\mathcal{X}^T g \mathcal{X} \to \mathcal{X}^T (Z^T g Z)\mathcal{X} = \mathcal{X}^T g \mathcal{X} \tag{22}$$

As an immediate consequence of extension of the spinor space, there are two eigenvectors for spin-up and two for spin-down: the number of Weyl spinors doubles. That is, we have *four* four-component left-chiral Weyl spinors: two spin-up $\mathcal{U}_L^p, \mathcal{U}_L^q$, and two spin-down $\mathcal{D}_L^p, \mathcal{D}_L^q$. Similarly, we have four right-chiral four-component Weyl spinors, $\mathcal{U}_R^p, \mathcal{U}_R^q, \mathcal{D}_R^p$ and $\mathcal{D}_R^q$. These basic forms can be

1 Similarly, the eigenvectors of $(\Sigma^3)^*$ span the corresponding dual space.
2 Also, $(Z^{-1})^\dagger$ preserves g and η.

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obtained by the action of Z and $\dot{Z} = (Z^{-1})^{\dagger}$ on basis $e^{p,q}_{u,d}$ and $\dot{e}^{p,q}_{u,d} = (ge^{p,q}_{u,d})^{*}$, respectively. [3]. A list of them is given in the Appendix A.

All these basic forms will play a central role in defining Dirac spinors and their vector decomposition into four-component Weyl spinors in the g^{μ} basis.

3. Dirac Algebra in the New Framework

There is no nontrivial fourth 2×2 matrix that anticommutes with all Pauli matrices. To define Dirac γ^{μ} matrices an extension of the basis is required:

$$\sigma^{\mu\nu} = \sigma^{\mu} \otimes \sigma^{\nu} \tag{23}$$

This is a direct product extension of the spinor space in which four-component Dirac spinors live. A subset of $\sigma^{\mu\nu}$ satisfying the Clifford algebra of spacetime is identified as Dirac γ^{μ} matrices [8]:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu} I_4 \tag{24}$$

The Dirac equation in γ^{μ} basis is

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0 \tag{25}$$

Here ψ is the traditional four-component Dirac spinor. It transforms under the representation $S_{\gamma}[\Lambda]$, for which the boost and rotation generators K_i and J_i are defined by

$$K_i = \frac{i}{4}[\gamma^0, \gamma^i], \qquad J_i = \frac{i}{4}[\gamma^i, \gamma^j] \tag{26}$$

These generators satisfy the Lorentz algebra:

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \qquad [J_i, K_j] = i\varepsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\varepsilon_{ijk}J_k \tag{27}$$

In terms of the parameters and generators

$$S_{\gamma}[\Lambda] = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \tag{28}$$

where

$$S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}] \tag{29}$$

In any representation $S_{\gamma}[\Lambda]$ obeys the Lorentz covariance condition:

$$S_{\gamma}[\Lambda]^{-1}\gamma^{\mu}S_{\gamma}[\Lambda] = \Lambda^{\mu}{}_{\nu}\gamma^{\nu} \tag{30}$$

In the extended formalism introduced here, the Σ^i matrices play a role analogous to the Pauli matrices, satisfying similar algebraic relations, and there is no nontrivial 4×4 matrix that anticommutes with all Σ^i. Therefore, by themselves, they cannot generate a full Clifford algebra. To resolve this issue we simply employ set of 16 matrices $\Sigma^{\mu}(\Sigma^{\nu})^{*}$, which span the full four-dimensional matrix space.

[3] Similarly, explicit forms of dual-right- and dual-left-chiral four-component Weyl spinors can be obtained by acting on the corresponding basis with Z^{*}, $(Z^{-1})^{T}$.

Within this new framework, we can identify six distinct subsets of the set $\Sigma^\mu(\Sigma^\nu)^*$ that satisfy the Clifford algebra of spacetime. One such set, g^μ, is compatible with the Z matrix formalism we introduced earlier:

$$g^0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad g^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad g^2 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad g^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} \tag{31}$$

with the property

$$\{g^\mu, g^\nu\} = 2\eta^{\mu\nu} I_4 \tag{32}$$

There exists a similarity transformation between the g^μ and γ^μ. If γ^μ is given in the chiral basis, then g^μ can be obtained from γ^μ by the following transformation:

$$g^\mu = A\gamma^\mu A^{-1} \tag{33}$$

where

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ i & 0 & 0 & -i \\ 0 & -1 & 1 & 0 \end{pmatrix}, \qquad A^{-1} = A^\dagger \tag{34}$$

Therefore, in this sense, our representation is not new. In fact, when we apply this similarity transformation on $S_\gamma[\Lambda]$ we get $\widetilde{S}_g[\Lambda]$

$$\widetilde{S}_g[\Lambda] = AS_\gamma[\Lambda]A^{-1} \tag{35}$$

But, due to the different conventions adopted in our formalism, $\widetilde{S}_g[\Lambda]$ is a not the correct transformation for the already defined Ψ in the g^μ basis [4]. Hence, we will build $S_g[\Lambda]$ from scratch by beginning with the generators defined in the g^μ basis:

$$K_i = \frac{i}{4}[g^0, g^i], \qquad J_i = \frac{i}{4}[g^i, g^j] \tag{36}$$

After exponentiation $S_g[\Lambda]$ reads

$$S_g[\Lambda] = \frac{1}{2} \begin{pmatrix} a + a^* & b - b^* & ib + ib^* & a - a^* \\ c - c^* & d + d^* & id - id^* & c + c^* \\ -ic - ic^* & -id + id^* & d + d^* & -ic + ic^* \\ a - a^* & b + b^* & ib - ib^* & a + a^* \end{pmatrix} \tag{37}$$

Here, parameters a, b, c, d are the elements of L:

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{38}$$

$S_g[\Lambda]$ satisfies the required Lorentz covariance condition:

$$S_g[\Lambda]^{-1} g^\mu S_g[\Lambda] = \Lambda^\mu{}_\nu g^\nu \tag{39}$$

where $S_g[\Lambda]^{-1}$ can be obtained by simply exchanging $a \leftrightarrow d$, and changing the signs of b and c, i.e., $b \to -b$ and $c \to -c$.

As a result, within the context of the Clifford algebra, g^μ basis is equivalent to γ^μ. Therefore, in this sense, there is nothing new. But, *structurally*, the set $\Sigma^\mu(\Sigma^\nu)^*$ obtained by a matrix product of Σ^μ and $(\Sigma^\nu)^*$ is distinct from the set $\sigma^\mu \otimes \sigma^\nu$ which is a direct product of Pauli matrices. This structural

[4] Actually, $\widetilde{S}_g[\Lambda]$ is a Lorentz transformation on $g^0\Psi$. This point will be discussed in the Appendix A.

distinction allows us to construct a new framework based on the four-dimensional unified spinor space, in which g^μ basis connects the Dirac algebra to four-component Weyl spinors, enabling symmetric decomposition and reconstruction.

4. Decomposition of Dirac Spinors into Four-Component Weyl Spinors

First of all, g^μ obeys the Clifford algebra, hence an associated Dirac equation for Ψ can be written in the new basis:

$$(ig^\mu \partial_\mu - m)\Psi = 0 \tag{40}$$

Where we use Ψ to distinguish it from the traditional ψ. The projection operators for left- and right-chiral Weyl components of Ψ can be defined in terms of g^5:

$$g^5 = -ig^0 g^1 g^2 g^3 \tag{41}$$

In this definition, the minus sign is just a convention.

To proceed further, we first set $\vec{p} = 0$ and apply the usual ansatz to find the plane-wave solutions in the g^μ basis. Apart from the phase factors a set of solutions is:

$$\Psi_0^1 = \begin{pmatrix} 1 \\ 1 \\ i \\ 1 \end{pmatrix}, \quad \Psi_0^2 = \begin{pmatrix} 1 \\ 1 \\ -i \\ -1 \end{pmatrix}, \quad \Psi_0^3 = \begin{pmatrix} 1 \\ -1 \\ -i \\ 1 \end{pmatrix}, \quad \Psi_0^4 = \begin{pmatrix} 1 \\ -1 \\ i \\ -1 \end{pmatrix} \tag{42}$$

In terms of the basis vectors given in Equation (13), we can also write:

$$\Psi_0^1 = e_u^p + e_u^q, \qquad \Psi_0^2 = e_d^p + e_d^q, \qquad \Psi_0^3 = e_u^p - e_u^q, \qquad \Psi_0^4 = -e_d^p + e_d^q, \tag{43}$$

Then, we apply the Lorentz transformation $S_g[\Lambda]$ on Ψ_0^a to obtain general solutions:

$$\Psi^a = S_g[\Lambda]\Psi_0^a, \tag{44}$$

where Ψ^a, $a \in \{1,2,3,4\}$ is a more general solution to the Dirac equation. Due to the linearity of $S_g[\Lambda]$:

$$\Psi^1 = S_g[\Lambda]e_u^p + S_g[\Lambda]e_u^q, \quad etc. \tag{45}$$

Now, it is crucial to observe that

$$S_g[\Lambda]e_u^p = Ze_u^p = \mathcal{U}_L^p, \qquad S_g[\Lambda]e_d^p = Ze_d^p = \mathcal{D}_L^p \tag{46}$$

and

$$S_g[\Lambda]e_u^q = \dot{Z}\dot{e}_d^q = \mathcal{U}_R^q, \qquad S_g[\Lambda]e_d^q = -\dot{Z}\dot{e}_u^q = -\mathcal{D}_R^q \tag{47}$$

Hence, Ψ^a can be written as vector sum of four-component Weyl spinors:

$$\Psi^1 = \mathcal{U}_L^p + \mathcal{U}_R^q, \quad \Psi^2 = \mathcal{D}_L^p - \mathcal{D}_R^q, \quad \Psi^3 = \mathcal{U}_L^p - \mathcal{U}_R^q, \quad \Psi^4 = -\mathcal{D}_L^p - \mathcal{D}_R^q \tag{48}$$

These are not direct sums in the traditional sense (as in block forms), but rather sums of vectors (spinors) within the same four-dimensional spinor space. In this sense, new formalism is distinct from the the traditional direct sum representation given in Equation (1).

In the new framework, basic spinors embedded in the same representation space are distinguished only by their transformation properties: $\mathcal{U}_L^p$ and $\mathcal{D}_L^p$ transform under the left-handed representation Z, while $\mathcal{U}_R^q$ and $\mathcal{D}_R^q$ transform under the right-handed one, $\dot{Z}$.

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We can also decompose left-chiral Dirac spinors Ψ into left- and right-chiral Weyl components by means of the projection operators P_L and P_R defined in the g^μ basis:

$$P_L = \frac{1}{2}(I_4 - g^5), \qquad P_R = \frac{1}{2}(I_4 + g^5) \tag{49}$$

It is enough to examine the action of these projectors on the basis $e^{p,q}_{u,d}$:

$$P_L e^p_{u,d} = e^p_{u,d}, \qquad P_L e^q_{u,d} = 0, \qquad P_R e^p_{u,d} = 0, \qquad P_R e^q_{u,d} = e^q_{u,d} \tag{50}$$

These relations immediately yield the left- and right-chiral Weyl components of Ψ^a:

$$P_L \Psi^1 = \Psi^1_L = \mathcal{U}^p_L, \qquad P_R \Psi^1 = \Psi^1_R = \mathcal{U}^q_R, \qquad etc. \tag{51}$$

As a result,

$$\Psi^a = \Psi^a_L + \Psi^a_R \tag{52}$$

But, now, in contrast to the Dirac formalism in γ^μ basis, in the new framework, Ψ_L and Ψ_R can be directly associated with Weyl spinors. The new formalism not only clarifies the internal structure of Dirac spinors but also establishes an explicit and symmetric embedding of left- and right-chiral Weyl components in a unified spinor space.

Now we can write the Dirac equation as

$$(ig^\mu \partial_\mu - m)(\Psi_L + \Psi_R) = 0 \tag{53}$$

This equation does not split into two uncoupled equations unless $m = 0$. If $m \neq 0$, it splits into two coupled equations:

$$(ig^\mu \partial_\mu)\Psi_L = m\Psi_R, \qquad (ig^\mu \partial_\mu)\Psi_R = m\Psi_L \tag{54}$$

When $m = 0$ they reduce to uncoupled equations:

$$(ig^\mu \partial_\mu)\Psi_L = 0, \qquad (ig^\mu \partial_\mu)\Psi_R = 0 \tag{55}$$

In other words, when $m = 0$, Ψ_L and Ψ_R satisfy massless Dirac equations and Weyl equations given in Equation (14) simultaneously.

5. Conclusions

In this work, we introduce a novel four-dimensional spinor representation of the Lorentz group, in which both the Dirac spinors and Weyl spinors are treated as four-component objects living in the same spinor space.

This framework allows the Dirac spinor to be expressed as a vector sum (not a direct sum) of left- and right-chiral four-component Weyl spinors. Unlike the standard formalism, our formulation permits a direct and symmetric decomposition in a four-dimensional spinor space.

An extension of the SL(2, $\mathbb{C}$) algebra naturally leads to doubling the number of Weyl spinors and enables a more transparent treatment of spin degrees of freedom, with up and down states appearing in pairs. Thus, the left- and right-chiral parts of a Dirac spinor obtained via projection operators defined in terms of g^5 are directly identifiable as the Weyl spinors used in construction of the same spinor.

This approach not only deepens our understanding of spinor structure in relativistic quantum mechanics but also offers potential advantages for exploring extended spinor spaces, alternative representations, and geometrical interpretation of spin.

Appendix A

Appendix A.1. Four-Dimensional Spinor Representation of the Lorentz Group: Extension of the Algebra of SL(2, ℂ)

Here we detail the algebraic structure underlying the Z matrix formalism introduced in the main text by showing that any Lorentz transformation matrix Λ can be expressed as a product of a matrix Z and its complex conjugate, $\Lambda = ZZ^*$ [5].

We begin with the standard real form of the Lorentz transformation matrix in exponential form:

$$\Lambda = e^{(\theta_i J_i + \phi_i K_i)} \tag{A1}$$

Here J_i and K_i are rotation and boost generators, θ_i and ϕ_i are associated parameters. The generators in a fully real representation are given as:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A2}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{A3}$$

These satisfy the Lorentz algebra commutation relations:

$$[J_i, J_j] = \varepsilon_{ijk} J_k, \qquad [K_i, K_j] = -\varepsilon_{ijk} J_k, \qquad [J_i, K_j] = \varepsilon_{ijk} K_k \tag{A4}$$

Now, define new generators Σ_i and Σ_i^*:

$$\Sigma_i = K_i + iJ_i, \qquad \Sigma_i^* = K_i - iJ_i \tag{A5}$$

$$\Sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{A6}$$

$$\Sigma_1^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad \Sigma_2^* = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \Sigma_3^* = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{A7}$$

The matrices Σ_i and Σ_i^* are traceless, Hermitian, and satisfy:

$$[\Sigma_i, \Sigma_j] = i\varepsilon_{ijk}\Sigma_k, \qquad [\Sigma_i^*, \Sigma_j^*] = -i\varepsilon_{ijk}\Sigma_k^*, \qquad [\Sigma_i, \Sigma_j^*] = 0 \tag{A8}$$

We express new Lorentz generators in terms of these matrices:

$$K_i = \frac{1}{2}(\Sigma_i + \Sigma_i^*), \qquad J_i = -\frac{i}{2}(\Sigma_i - \Sigma_i^*) \tag{A9}$$

[5] This matrix product form should not be confused with the polar decomposition or with the tensor product form often used in standard representations.

Substituting into the exponential form, $\Lambda = e^{(\theta_i J_i + \phi_i K_i)}$:

$$\Lambda = e^{[-\frac{i}{2}\theta_i(\Sigma_i - \Sigma_i^*) + \frac{1}{2}\phi_i(\Sigma_i + \Sigma_i^*)]} = e^{[-\frac{i}{2}(\theta_i + i\phi_i)\Sigma_i + \frac{i}{2}(\theta_i - i\phi_i)\Sigma_i^*)]} \tag{A10}$$

Since Σ_i and Σ_j^* commute with each other for all i, j, this exponential splits cleanly into a matrix product:

$$\Lambda = e^{-\frac{i}{2}(\theta_i + i\phi_i)\Sigma_i} e^{\frac{i}{2}(\theta_i - i\phi_i)\Sigma_i^*} = ZZ^* \tag{A11}$$

where, by definition,

$$Z = e^{-\frac{i}{2}(\theta_i + i\phi_i)\Sigma_i}, \qquad Z^* = e^{\frac{i}{2}(\theta_i - i\phi_i)\Sigma_i^*} \tag{A12}$$

These definitions of Z and Z^* can be compared to the exponential forms of L and L^*:

$$L = e^{-\frac{i}{2}(\theta_i + i\phi_i)\sigma_i}, \qquad L^* = e^{\frac{i}{2}(\theta_i - i\phi_i)\sigma_i^*} \tag{A13}$$

In this work, Z and Z^* will be regarded as four-dimensional left- and right-chiral representations of the Lorentz group, acting on four-component Weyl spinors.

Appendix A.2. Parametric Representations

In explicit calculations it is easier to work with the parametric forms of L and Z:

$$L = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} \tag{A14}$$

where α_μ is a complex parameter. Then L can be compactly written in terms of the Pauli matrices:

$$L = \alpha_\mu \sigma^\mu \tag{A15}$$

Analogously, according to Equation (10) parametric form of Z reads

$$Z = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_0 & -i\alpha_3 & i\alpha_2 \\ \alpha_2 & i\alpha_3 & \alpha_0 & -i\alpha_1 \\ \alpha_3 & -i\alpha_2 & i\alpha_1 & \alpha_0 \end{pmatrix} \tag{A16}$$

Or, in a compact form:

$$Z = \alpha_\mu \Sigma^\mu \tag{A17}$$

In order to obtain α_μ in terms of boost and rotation parameters, we write Z in exponential form, $Z = e^M$, where $M = -\frac{i}{2}\vec{\rho}\cdot\vec{\Sigma}, \;\; \rho_i = \theta_i + i\phi_i$. Here $\vec{\rho}$ is a complex vector parameter representing both rotations and boosts. Let $\rho = \sqrt{\rho_1^2 + \rho_2^2 + \rho_3^2}$. Then the exponential form becomes:

$$Z = \Sigma_0 \cos(\rho/2) - \frac{i\sin(\rho/2)}{\rho}\vec{\rho}\cdot\vec{\Sigma} \tag{A18}$$

where

$$\alpha_0 = \cos(\rho/2), \qquad \alpha_1 = -\frac{i\sin(\rho/2)}{\rho}\rho_1, \qquad \alpha_2 = -\frac{i\sin(\rho/2)}{\rho}\rho_2, \qquad \alpha_3 = -\frac{i\sin(\rho/2)}{\rho}\rho_3. \tag{A19}$$

Using this parametrization, we can define various forms of Z

$$Z = \alpha_0\Sigma^0 + \alpha_1\Sigma^1 + \alpha_2\Sigma^2 + \alpha_3\Sigma^3, \quad Z^{-1} = \alpha_0\Sigma^0 - \alpha_1\Sigma^1 - \alpha_2\Sigma^2 - \alpha_3\Sigma^3, \tag{A20}$$

$$Z^* = \alpha_0^*\Sigma^{0*} + \alpha_1^*\Sigma^{1*} + \alpha_2^*\Sigma^{2*} + \alpha_3^*\Sigma^{3*}, \quad Z^\dagger = \alpha_0^*\Sigma^0 + \alpha_1^*\Sigma^1 + \alpha_2^*\Sigma^2 + \alpha_3^*\Sigma^3, \tag{A21}$$

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From the product $\Lambda = ZZ^*$, we obtain the most general form of the real 4×4 Lorentz transformation matrix with entries constructed from bilinear combinations $\alpha_\mu \alpha_\nu^*$:

$$\Lambda = \left(\begin{array}{c|c|c|c} \begin{matrix}\alpha_0\alpha_0^* + \alpha_1\alpha_1^* \\ \alpha_2\alpha_2^* + \alpha_3\alpha_3^*\end{matrix} & \begin{matrix}\alpha_0\alpha_1^* + \alpha_1\alpha_0^* \\ +i(\alpha_3\alpha_2^* - \alpha_2\alpha_3^*)\end{matrix} & \begin{matrix}\alpha_0\alpha_2^* + \alpha_2\alpha_0^* \\ +i(\alpha_1\alpha_3^* - \alpha_3\alpha_1^*)\end{matrix} & \begin{matrix}\alpha_0\alpha_3^* + \alpha_3\alpha_0^* \\ +i(\alpha_2\alpha_1^* - \alpha_1\alpha_2^*)\end{matrix} \\ \hline \begin{matrix}\alpha_0\alpha_1^* + \alpha_1\alpha_0^* \\ -i(\alpha_3\alpha_2^* - \alpha_2\alpha_3^*)\end{matrix} & \begin{matrix}\alpha_0\alpha_0^* + \alpha_1\alpha_1^* \\ -\alpha_2\alpha_2^* - \alpha_3\alpha_3^*\end{matrix} & \begin{matrix}\alpha_1\alpha_2^* + \alpha_2\alpha_1^* \\ +i(\alpha_0\alpha_3^* - \alpha_3\alpha_0^*)\end{matrix} & \begin{matrix}\alpha_1\alpha_3^* + \alpha_3\alpha_1^* \\ +i(\alpha_2\alpha_0^* - \alpha_0\alpha_2^*)\end{matrix} \\ \hline \begin{matrix}\alpha_0\alpha_2^* + \alpha_2\alpha_0^* \\ -i(\alpha_1\alpha_3^* - \alpha_3\alpha_1^*)\end{matrix} & \begin{matrix}\alpha_1\alpha_2^* + \alpha_2\alpha_1^* \\ -i(\alpha_0\alpha_3^* - \alpha_3\alpha_0^*)\end{matrix} & \begin{matrix}\alpha_0\alpha_0^* - \alpha_1\alpha_1^* \\ +\alpha_2\alpha_2^* - \alpha_3\alpha_3^*\end{matrix} & \begin{matrix}\alpha_2\alpha_3^* + \alpha_3\alpha_2^* \\ +i(\alpha_0\alpha_1^* - \alpha_1\alpha_0^*)\end{matrix} \\ \hline \begin{matrix}\alpha_0\alpha_3^* + \alpha_3\alpha_0^* \\ -i(\alpha_2\alpha_1^* - \alpha_1\alpha_2^*)\end{matrix} & \begin{matrix}\alpha_1\alpha_3^* + \alpha_3\alpha_1^* \\ -i(\alpha_2\alpha_0^* - \alpha_0\alpha_2^*)\end{matrix} & \begin{matrix}\alpha_2\alpha_3^* + \alpha_3\alpha_2^* \\ -i(\alpha_0\alpha_1^* - \alpha_1\alpha_0^*)\end{matrix} & \begin{matrix}\alpha_0\alpha_0^* - \alpha_1\alpha_1^* \\ -\alpha_2\alpha_2^* + \alpha_3\alpha_3^*\end{matrix} \end{array}\right) \tag{A22}$$

Appendix A.3. Basic Tools: Two- and Four-Component Weyl Spinors

To simplify the calculations further, we define new parameters:

$$a = \alpha_0 + \alpha_3, \qquad b = \alpha_1 - i\alpha_2, \qquad c = \alpha_1 + i\alpha_2, \qquad d = \alpha_0 - \alpha_3 \tag{A23}$$

Then, L and $\dot{L}$ take their simplest forms

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \dot{L} = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix} \tag{A24}$$

A left-chiral Weyl spinor ξ_L is a two component object that transforms under the $(\frac{1}{2}, 0)$ representation of the Lorentz group. Let $L \in \mathrm{SL}(2, \mathbb{C})$, and let e_u and e_d be a basis for the corresponding two dimensional spinor space:

$$e_u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{A25}$$

These basis vectors are the eigenvectors of σ_3, associated with $+1$ and -1 eigenvalues respectively. When L acts on this basis, we obtain general form of left-chiral spin-up and spin-down states. Let us denote these by u_L and d_L to match our overall notation:

$$u_L = Le_u = \begin{pmatrix} a \\ c \end{pmatrix}, \qquad d_L = Le_d = \begin{pmatrix} b \\ d \end{pmatrix} \tag{A26}$$

It is important to note that, in general, u_L and d_L are not orthogonal, unless L is unitary, which corresponds to pure spatial rotations.

Similarly, two-component right-chiral Weyl spinors transform under the $(0, \frac{1}{2})$ representation. These can be constructed analogously:

$$u_R = \dot{L}\dot{e_u} = \begin{pmatrix} d^* \\ -b^* \end{pmatrix}, \qquad d_R = \dot{L}\dot{e_d} = \begin{pmatrix} -c^* \\ a^* \end{pmatrix} \tag{A27}$$

where $\dot{e}_{u,d} = (\epsilon e_{u,d})^*$, with ϵ being the spinor-Minkowski metric for the two dimensional representation of the Lorentz group:

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A28}$$

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We now extend this structure to define our four-component Weyl-spinors. In the new formalism the spinor space is four-dimensional, and we use a basis set consisting of four vectors $e^{p,q}_{u,d}$, which are the eigenvectors of the matrix Σ^3:

$$e^p_u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad e^p_d = \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}, \quad e^q_u = \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad e^q_d = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{A29}$$

In this basis, e^p_u and e^q_u correspond to $+1$ eigenvalue, and thus represent spin-up states; e^p_d and e^q_d correspond to eigenvalue -1 eigenvalue, and represent spin-down states. As a result, we will have two spin-up left-chiral basic spinors, $\mathcal{U}^p_L$ and $\mathcal{U}^q_L$, and two spin-down left-chiral basic spinors $\mathcal{D}^p_L$ and $\mathcal{D}^q_L$. Their explicit forms are obtained by applying the left-handed four dimensional transformation Z to the basis vectors $e^{p,q}_{u,d}$:

$$\mathcal{U}^p_L = Ze^p_u = \begin{pmatrix} a \\ c \\ -ic \\ a \end{pmatrix}, \quad \mathcal{D}^p_L = Ze^p_d = \begin{pmatrix} b \\ d \\ -id \\ b \end{pmatrix}, \quad \mathcal{U}^q_L = Ze^q_u = \begin{pmatrix} c \\ a \\ ia \\ -c \end{pmatrix}, \quad \mathcal{D}^q_L = Ze^q_d = \begin{pmatrix} d \\ b \\ ib \\ -d \end{pmatrix} \tag{A30}$$

Although the basis vectors $e^{p,q}_{u,d}$ are mutually orthogonal; in general, the spinors need not be, unless Z is unitary. However, spinors carrying different superscripts ($p \neq q$) are always orthogonal. The basis with superscript q is related to the basis with superscript p by g^0:

$$e^q_{u,d} = g^0 e^p_{u,d} \tag{A31}$$

Here g^0 plays the role of γ^0. In the traditional representation γ^0 exchanges upper and lower parts of a Dirac spinor in the chiral basis:

$$\gamma^0 \begin{pmatrix} \xi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} \chi_R \\ \xi_L \end{pmatrix} \tag{A32}$$

Similarly, exchanging p and q does the same transformation on the Dirac spinors in the g^μ basis as we will see in what follows.

We also define right-chiral four-component Weyl spinors by acting on the dotted basis vectors $\dot{e}^{p,q}_{u,d}$ with the right-handed representation $\dot{Z} = (Z^{-1})^\dagger$. The explicit forms are:

$$\mathcal{D}^p_R = \dot{Z}\dot{e}^p_u = \begin{pmatrix} c^* \\ -a^* \\ ia^* \\ c^* \end{pmatrix}, \quad \mathcal{U}^p_R = \dot{Z}\dot{e}^p_d = \begin{pmatrix} d^* \\ -b^* \\ ib^* \\ d^* \end{pmatrix}, \quad \mathcal{D}^q_R = \dot{Z}\dot{e}^q_u = \begin{pmatrix} -a^* \\ c^* \\ ic^* \\ a^* \end{pmatrix}, \quad \mathcal{U}^q_R = \dot{Z}\dot{e}^q_d = \begin{pmatrix} -b^* \\ d^* \\ id^* \\ b^* \end{pmatrix} \tag{A33}$$

where $\dot{e}^{p,q}_{u,d} = (ge^{p,q}_{u,d})^*$.

Appendix A.4. Equivalent Reducible Representations

Within this extended framework there is enough room for the so called left, right, dual-right and dual-left representations for the Dirac spinors. Here, "left" and "right" have nothing to do with the chirality (handedness). They are just labels for equivalent representations. We will denote them by Ψ, $\dot{\Psi}$, Ψ^* and $\dot{\Psi}^*$. These four species transform under the left, right, dual-right and dual-left *reducible* representations, respectively. Furthermore, every Dirac spinor (left, right or dual-right, dual-left) can be expressed as a vector sum -rather than a direct sum- of left- and right-chiral four-component Weyl spinors and their duals, where all forms of Dirac spinors and their chiral constituents live in the same unified four-dimensional space. Left- and right-projection operators can be defined for each version, such that applying these new projection operators on Ψ, $\dot{\Psi}$, etc., retrieves their full left- and right-chiral Weyl components directly.

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We will identify $S_g[\Lambda]$ as the left representation acting on the left spinors Ψ. As mentioned above there also exists a right one, $\dot{S}_g[\Lambda] = (S_g[\Lambda]^{-1})^\dagger$, which acts on right spinors $\dot{\Psi}$. This is equal to what we have called $\widetilde{S}_g[\Lambda]$ previously.

$S_g[\Lambda]$ and $\dot{S}_g[\Lambda]$ are related to each other by a similarity transformation, which is an analogue of the relation $\dot{S}_\gamma[\Lambda] = \gamma^0 S_\gamma[\Lambda]\gamma^0$:

$$\dot{S}_g[\Lambda] = g^0 S_g[\Lambda] g^0 \tag{A34}$$

Once more, g^0 plays the role of γ^0. Hence,

$$\dot{\Psi} = g^0 \Psi \tag{A35}$$

Obviously, $\dot{S}_g[\Lambda]$ is not a *new* representation [6].

Dotted version of Dirac spinors, transform under the dotted representation:

$$\dot{\Psi} \to \dot{\Psi}' = \dot{S}_g[\Lambda]\dot{\Psi} \tag{A36}$$

Explicit form of $\dot{S}_g[\Lambda]$ reads

$$\dot{S}_g[\Lambda] = \frac{1}{2}\begin{pmatrix} d+d^* & c-c^* & -ic-ic^* & -d+d^* \\ b-b^* & a+a^* & -ia+ia^* & -b-b^* \\ ib+ib^* & ia-ia^* & a+a^* & -ib+ib^* \\ -d+d^* & -c-c^* & ic-ic^* & d+d^* \end{pmatrix} \tag{A37}$$

Then it is straightforward to show that $\dot{\Psi}^a$ can be written as a vector sum of four-component Weyl spinors:

$$\dot{\Psi}^1 = \mathcal{U}_L^q + \mathcal{U}_R^p, \quad \dot{\Psi}^2 = \mathcal{D}_L^q - \mathcal{D}_R^p, \quad \dot{\Psi}^3 = \mathcal{U}_L^q - \mathcal{U}_R^p, \quad \dot{\Psi}^4 = -\mathcal{D}_L^q - \mathcal{D}_R^p \tag{A38}$$

Therefore, in order to get $\dot{\Psi}$ we simply swap $p \leftrightarrow q$ in Equation (48). This corresponds to swapping ξ_L and χ_R in Equation (1).

Dual-right and dual-left representations can be obtained by complex conjugating these forms. But, again, they will not be *new* representations.

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[6] In this respect, the relation between $S_g[\Lambda]$ and $\dot{S}_g[\Lambda]$ is not similar to the relation between Z and $\dot{Z}$. It is not possible to obtain $\dot{Z}$ from Z by a similarity transformation. Therefore, Z and $\dot{Z}$ (or Z^*) are distinct *irreducible* representations of the Lorentz group associated with left- and right-chiral four-component Weyl spinors.