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Article

On Constructing H -Irregularity Labeling with Minimum Label for Certain Balloon Graphs

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Abstract: If ϱ and H are simple, connected, undirected graphs, and ϱ can be covered by an H -covering, then for a positive integer p , a total p -labeling φ on ϱ is considered as a total H -irregular p -labeling if, for every subgraph K of ϱ that is isomorphic to H , the weight of K (the sum of the labels of all vertices and edges of K) is a unique number. The smallest integer p for which graph ϱ can be labeled with a total H -irregular p -labeling is called the total H -irregularity strength of graph G . This paper presents the exact total H -irregularity strength values for some particular graphs including balloon graphs, double balloon graphs, and double balloon ladder graphs.

Keywords: total H -irregularity strength; balloon graph; double balloon graph; double balloon ladder graph

MSC: 2000

1. Introduction

Wilson (see [11]) defines a graph ϱ as $(V(\varrho), E(\varrho))$, where $V(\varrho)$ represents a finite set of nonempty vertices and $E(\varrho)$ is a finite set of unordered pairs of distinct elements from $V(\varrho)$, known as edges.

Wallis in [10] describes a labeling on ϱ as a function whose domain is either $V(\varrho)$, $E(\varrho)$, or $V(\varrho) \cup E(\varrho)$, and its codomain is a set of positive or nonnegative integers. When the domain is $V(\varrho)$, it is called vertex labeling; when it is $E(\varrho)$, it is edge labeling; and when it is $V(\varrho) \cup E(\varrho)$, it is total labeling. Various labeling techniques can be employed, one of which is irregular labeling with the domain $V(\varrho) \cup E(\varrho)$ mapping to a set of positive integers. This labeling is discussed in this paper.

Gallian in [4] states that there are about 200 types of labeling techniques, evolving in approximately 3000 papers. These techniques include irregular labeling, introduced by Chartrand et al. (see [3]). In irregular labeling, the weight of a vertex $v \in V(\varrho)$ relative to a total labeling is defined as the sum of the label of v and all labels of edges incident to v . From some types of irregular labeling on a graph ϱ , the concepts of total edge irregularity strength (*tes*) and total vertex irregularity strength (*tvs*) arise.

Ashraf et al. (see [2]) introduced the strength of irregularity- H , an extension of *tes* and *tvs*. A graph ϱ with an H -covering means that for any edge of ϱ , there exists a subgraph of ϱ isomorphic to H that includes the edge. Let $\varphi : V(\varrho) \cup E(\varrho) \rightarrow \{1, 2, \dots, p\}$ be any total p -labeling on ϱ . For any subgraph K of ϱ isomorphic to H , the H -weight of K with respect to φ , denoted by $wt_{\varphi}(K)$, is defined as $wt_{\varphi}(K) = \sum_{v \in V(K)} \varphi(v) + \sum_{e \in E(K)} \varphi(e)$. A total p -labeling on ϱ is called H -irregular if $wt_{\varphi}(K') \neq wt_{\varphi}(K'')$ for every two distinct subgraphs K' and K'' isomorphic to H . The smallest integer p such that ϱ can be labeled with an H -irregular total p -labeling is called the total H -irregularity strength of ϱ , denoted by $tHs(\varrho, H)$.

Recently, $tHs(\varrho, H)$ has been extensively studied. Agustin et al. in [1] obtained results on $tHs(\varrho, H)$ for shackles and amalgamation graphs. Nisviasari et al. in [7] researched $tHs(\varrho, H)$ for triangular ladder and grid graphs. Hidayatul et al. (see [5]) obtained results on $tHs(\varrho, H)$ for grid, butterfly, hexagonal, and diamond graphs. Suni et al. in [9] investigated diamond ladder, circular triple ladder, and prism graphs. Wahyujati et al. (see [12]) studied the total H -irregularity strength of edge comb product graphs. Shulhany et al. (see [8]) examined the same parameter for some classes

of graphs, while Labane et al. (see [6]) have done so for cycles and diamond graphs. In this work, we introduce a new type of graph that has not been previously studied by other researchers, which involves applying specific graph operations. These results not only provide fresh insights but also extend the applicability of $tHs(q, H)$ to a broader spectrum of graph types. The use of graph operations in the investigation further highlights the structural complexity and versatility of the studied graphs. This finding will significantly enrich the theory of $tHs(q, H)$ by deepening our understanding of its behavior in diverse graph classes and paving the way for future studies in this area.

Theorem 1. [2]. Let q be a graph admitting an H -covering and have w subgraphs isomorphic to H . Then

$$tHs(q, H) \geq \left\lceil 1 + \frac{w-1}{|V(H)| + |E(H)|} \right\rceil.$$

2. Main Results

Our research findings will be presented in this section. First, we begin with the following definition.

Definition 1. The balloon graph Bl_η , $\eta \geq 1$, is a graph with

$$V(Bl_\eta) = \{x_q, y_q : q = 1, 2, \dots, \eta\} \cup \{z_q : q = 1, 2, \dots, \eta + 1\}$$

and

$$E(Bl_\eta) = \{x_q y_q, y_q z_q, y_q z_{q+1}, z_q z_{q+1} : q = 1, 2, \dots, \eta\}.$$

The following is provided as an example of balloon graph.

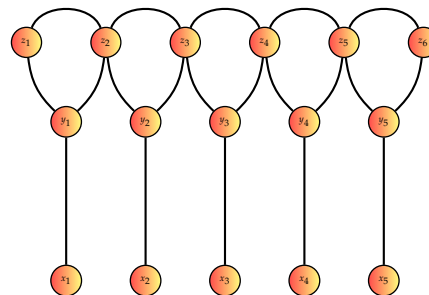


Figure 1. Balloon Graph Bl_5

Theorem 2. Given Bl_η , $\eta \geq 1$ with Bl_ζ -covering. Then, for $1 \leq \zeta \leq \eta$,

$$tHs(Bl_\eta, Bl_\zeta) = \left\lceil \frac{6\zeta + \eta + 1}{7\zeta + 1} \right\rceil.$$

PROOF. The balloon Bl_η , $\eta \geq 1$ admits Bl_ζ -covering with exactly $\eta - \zeta + 1$ balloons Bl_ζ , $1 \leq \zeta \leq \eta$ and η is a positive integer. From Theorem 1, it holds

$$tHs(Bl_\eta, Bl_\zeta) \geq \left\lceil 1 + \frac{(\eta - \zeta + 1) - 1}{7\zeta + 1} \right\rceil = \left\lceil \frac{6\zeta + \eta + 1}{7\zeta + 1} \right\rceil.$$

Next, put $p = \left\lceil \frac{6\zeta + \eta + 1}{7\zeta + 1} \right\rceil$, $tHs(Bl_\eta, Bl_\zeta) \leq p$ will be proven. On balloon graph Bl_η with Bl_ζ -covering, $1 \leq \zeta \leq \eta$, we define

$$\Psi_{\eta, \zeta} : V(Bl_\eta) \cup E(Bl_\eta) \rightarrow \{1, 2, \dots, p\},$$

for $\zeta = 1, 2, \dots, \eta$, in this manner:

for $q = 1, 2, \dots, \eta$,

$$\Psi_{\eta, \zeta}(x_q) = \left\lfloor \frac{q}{7\zeta + 1} \right\rfloor, \Psi_{\eta, \zeta}(y_q) = \left\lfloor \frac{q + 2\zeta}{7\zeta + 1} \right\rfloor, \Psi_{\eta, \zeta}(x_q y_q) = \left\lfloor \frac{q + \zeta}{7\zeta + 1} \right\rfloor,$$

$$\Psi_{\eta, \zeta}(y_q z_q) = \left\lfloor \frac{q + 3\zeta}{7\zeta + 1} \right\rfloor, \Psi_{\eta, \zeta}(y_q z_{q+1}) = \left\lfloor \frac{q + 4\zeta}{7\zeta + 1} \right\rfloor, \Psi_{\eta, \zeta}(z_q z_{q+1}) = \left\lfloor \frac{q + 5\zeta}{7\zeta + 1} \right\rfloor.$$

for $q = 1, 2, \dots, \eta + 1$,

$$\Psi_{\eta, \zeta}(z_q) = \left\lfloor \frac{q + 6\zeta}{7\zeta + 1} \right\rfloor.$$

The largest label is reached when $q = \eta + 1$, that is, $\Psi_{\eta, \zeta}(z_q) = \left\lfloor \frac{6\zeta + \eta + 1}{7\zeta + 1} \right\rfloor$.

For every $r = 1, 2, 3, \dots, \eta - \zeta + 1$, the vertex set is

$$V(Bl_\zeta) = \{x_{r+q}, y_{r+q} : q = 0, 1, 2, \dots, \zeta - 1\} \cup \{z_{r+q} : q = 0, 1, 2, \dots, \zeta\}$$

and edge set is

$$E(Bl_\zeta) = \{x_{r+q}y_{r+q}, y_{r+q}z_{r+q}, y_{r+q}z_{r+q+1}, z_{r+q}z_{r+q+1} : q = 0, 1, 2, \dots, \zeta - 1\}.$$

Furthermore, for every subgraph Bl_ζ^r in Bl_ζ the weight of Bl_ζ^r , $r = 1, 2, \dots, \eta - \zeta + 1$, with respect to $\Psi_{\eta, \zeta} : V(Bl_\zeta) \cup E(Bl_\zeta) \rightarrow \{1, 2, \dots, p\}$ is

$$wt_{\Psi_{\eta, \zeta}}(Bl_\zeta^r) = \sum_{v \in V(Bl_\zeta^r)} \Psi_{\eta, \zeta}(v) + \sum_{e \in E(Bl_\zeta^r)} \Psi_{\eta, \zeta}(e).$$

Then, for $r = 1, 2, \dots, \eta - \zeta$,

$$\begin{aligned} wt_{\Psi_{\eta, \zeta}} Bl_\zeta^{r+1} - wt_{\Psi_{\eta, \zeta}} Bl_\zeta^r &= \Psi_{\eta, \zeta}(x_{r+\zeta}) + \Psi_{\eta, \zeta}(y_{r+\zeta}) + \Psi_{\eta, \zeta}(z_{r+\zeta+1}) \\ &\quad + \Psi_{\eta, \zeta}(x_{r+\zeta} y_{r+\zeta}) + \Psi_{\eta, \zeta}(y_{r+\zeta} z_{r+\zeta}) \\ &\quad + \Psi_{\eta, \zeta}(y_{r+\zeta} z_{r+\zeta+1}) + \Psi_{\eta, \zeta}(z_{r+\zeta} z_{r+\zeta+1}) - \Psi_{\eta, \zeta}(x_r) \\ &\quad - \Psi_{\eta, \zeta}(y_r) - \Psi_{\eta, \zeta}(z_r) - \Psi_{\eta, \zeta}(x_r y_r) - \Psi_{\eta, \zeta}(y_r z_r) \\ &\quad - \Psi_{\eta, \zeta}(y_r z_{r+1}) - \Psi_{\eta, \zeta}(z_r z_{r+1}) \\ &= \left\lfloor \frac{r + \zeta}{7\zeta + 1} \right\rfloor + \left\lfloor \frac{r + 3\zeta}{7\zeta + 1} \right\rfloor + \left\lfloor \frac{r + 7\zeta + 1}{7\zeta + 1} \right\rfloor + \left\lfloor \frac{r + 2\zeta}{7\zeta + 1} \right\rfloor \\ &\quad + \left\lfloor \frac{r + 4\zeta}{7\zeta + 1} \right\rfloor + \left\lfloor \frac{r + 5\zeta}{7\zeta + 1} \right\rfloor + \left\lfloor \frac{r + 6\zeta}{7\zeta + 1} \right\rfloor - \left\lfloor \frac{r}{7\zeta + 1} \right\rfloor \\ &\quad - \left\lfloor \frac{r + 2\zeta}{7\zeta + 1} \right\rfloor - \left\lfloor \frac{r + 6\zeta}{7\zeta + 1} \right\rfloor - \left\lfloor \frac{r + \zeta}{7\zeta + 1} \right\rfloor - \left\lfloor \frac{r + 3\zeta}{7\zeta + 1} \right\rfloor \\ &\quad - \left\lfloor \frac{r + 4\zeta}{7\zeta + 1} \right\rfloor - \left\lfloor \frac{r + 5\zeta}{7\zeta + 1} \right\rfloor \\ &= 1. \end{aligned}$$

For every $r = 1, 2, \dots, \eta - \zeta$ we get $wt_{\Psi_{\eta, \zeta}}(Bl_\zeta^r) < wt_{\Psi_{\eta, \zeta}}(Bl_\zeta^{r+1})$. Hence, $wt_{\Psi_{\eta, \zeta}}(Bl_\zeta^{r+1}) = 1 + wt_{\Psi_{\eta, \zeta}}(Bl_\zeta^r)$. Moreover, all DBL_ζ -weights are distinct. Therefore, $\Psi_{\eta, \zeta}$ is total Bl_ζ -irregular labeling of Bl_η .

Since $tHs(Bl_\eta, Bl_\zeta) \geq p$ and $tHs(Bl_\eta, Bl_\zeta) \leq p$, we have $tHs(Bl_\eta, Bl_\zeta) = p$, which can be expressed as

$$tHs(Bl_\eta, Bl_\zeta) = \left\lceil \frac{6\zeta + \eta + 1}{7\zeta + 1} \right\rceil.$$

□

The following on Figure 2 is an example of Bl_{10} with Bl_3 -covering and we have $tHs(Bl_{10}, Bl_3) = 2$.

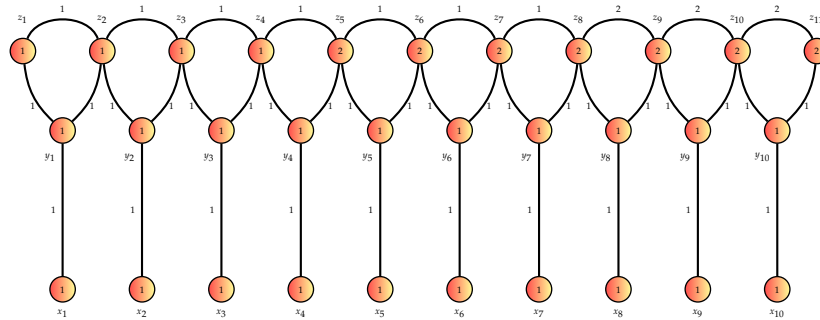


Figure 2. Balloon graph Bl_{10} with Bl_3 -covering.

Now, by double balloon graph, we mean the graph defined in Definition 2 as follows.

Definition 2. A double balloon graph DBl_η , $\eta \geq 1$, is a graph with

$$V(DBl_\eta) = \{x_q, y_q : q = 1, 2, \dots, \eta\} \cup \{w_q, z_q : q = 1, 2, \dots, \eta + 1\}$$

and

$$E(DBl_\eta) = \{w_q w_{q+1}, x_q w_q, x_q w_{q+1}, x_q y_q, y_q z_q, y_q z_{q+1}, z_q z_{q+1} : q = 1, 2, \dots, \eta\}.$$

The following figure is provided as an example of double balloon graph.

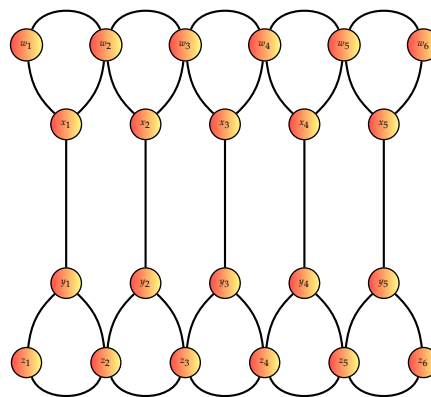


Figure 3. Double Balloon Graph DBl_5

Theorem 3. Given DBl_η , $\eta \geq 1$ with DBl_ζ -covering. Then, for $1 \leq \zeta \leq \eta$,

$$tHs(DBl_\eta, DBl_\zeta) = \left\lceil \frac{10\zeta + \eta + 2}{11\zeta + 2} \right\rceil.$$

PROOF. The double balloon graph, DBl_η , $\eta \geq 1$ admits DBl_ζ -covering with exactly $\eta - \zeta + 1$ double balloon DBl_ζ , $1 \leq \zeta \leq \eta$ and η is a positive integer. From Theorem 1, we obtain

$$tHs(DBl_\eta, DBl_\zeta) \geq \left\lceil 1 + \frac{(\eta - \zeta + 1) - 1}{11\zeta + 2} \right\rceil = \left\lceil \frac{10\zeta + \eta + 2}{11\zeta + 2} \right\rceil.$$

Next, put $p = \left\lceil \frac{10\zeta + \eta + 2}{11\zeta + 2} \right\rceil$, $tHs(DBl_\eta, DBl_\zeta) \leq p$ will be proven. On double balloon graph DBl_η with DBl_ζ -covering, $1 \leq \zeta \leq \eta$, we define

$$\Psi_{\eta, \zeta} : V(DBl_\eta) \cup E(DBl_\eta) \rightarrow \{1, 2, \dots, p\},$$

for $\zeta = 1, 2, \dots, \eta$, as follows:

for $q = 1, 2, \dots, \eta$,

$$\Psi_{\eta, \zeta}(x_q) = \left\lceil \frac{q + 7\zeta}{11\zeta + 2} \right\rceil, \Psi_{\eta, \zeta}(y_q) = \left\lceil \frac{q + 4\zeta}{11\zeta + 2} \right\rceil, \Psi_{\eta, \zeta}(x_q y_q) = \left\lceil \frac{q + 5\zeta}{11\zeta + 2} \right\rceil,$$

$$\Psi_{\eta, \zeta}(y_q z_q) = \left\lceil \frac{q + 2\zeta}{11\zeta + 2} \right\rceil, \Psi_{\eta, \zeta}(y_q z_{q+1}) = \left\lceil \frac{q + 3\zeta}{11\zeta + 2} \right\rceil,$$

$$\Psi_{\eta, \zeta}(z_q z_{q+1}) = \left\lceil \frac{q}{11\zeta + 2} \right\rceil, \Psi_{\eta, \zeta}(w_q w_{q+1}) = \left\lceil \frac{q + \zeta}{11\zeta + 2} \right\rceil,$$

$$\Psi_{\eta, \zeta}(x_q w_q) = \left\lceil \frac{q + 6\zeta}{11\zeta + 2} \right\rceil, \Psi_{\eta, \zeta}(x_q w_{q+1}) = \left\lceil \frac{q + 8\zeta}{11\zeta + 2} \right\rceil.$$

for $q = 1, 2, \dots, \eta + 1$,

$$\Psi_{\eta, \zeta}(w_q) = \left\lceil \frac{q + 10\zeta + 1}{11\zeta + 2} \right\rceil, \Psi_{\eta, \zeta}(z_q) = \left\lceil \frac{q + 9\zeta}{11\zeta + 2} \right\rceil.$$

The largest label is reached when $q = \eta + 1$, that is $\Psi_{\eta, \zeta}(w_q) = \left\lceil \frac{10\zeta + \eta + 2}{11\zeta + 2} \right\rceil$.

For every $r = 1, 2, 3, \dots, \eta - \zeta + 1$, the vertex set is

$$V(DBl_\zeta) = \{x_{r+q}, y_{r+q} : q = 0, 1, 2, \dots, \zeta - 1\} \cup \{w_{r+q}, z_{r+q} : q = 0, 1, 2, \dots, \zeta\}$$

and the edge set is

$$E(DBl_\zeta) = \{w_{r+q} w_{r+q+1}, x_{r+q} w_{r+q}, x_{r+q} w_{r+q+1}, x_{r+q} y_{r+q}, y_{r+q} z_{r+q}, \\ y_{r+q} z_{r+q+1}, z_{r+q} z_{r+q+1} : q = 0, 1, 2, \dots, \zeta - 1\}.$$

Furthermore, for every subgraph DBl_ζ^r in DBl_ζ , the weight of DBl_ζ^r , $r = 1, 2, \dots, \eta - \zeta + 1$, with respect to $\Psi_{\eta, \zeta} : V(DBl_\zeta) \cup E(DBl_\zeta) \rightarrow \{1, 2, \dots, p\}$ is

$$wt_{\Psi_{\eta, \zeta}}(DBl_\zeta^r) = \sum_{v \in V(DBl_\zeta^r)} \Psi_{\eta, \zeta}(v) + \sum_{e \in E(DBl_\zeta^r)} \Psi_{\eta, \zeta}(e).$$

Then, for $r = 1, 2, \dots, \eta - \zeta$, we have

$$\begin{aligned}
wt_{\Psi_{\eta,\zeta}} DBL_{\zeta}^{r+1} - wt_{\Psi_{\eta,\zeta}} DBL_{\zeta}^r &= \Psi_{\eta,\zeta}(w_{r+\zeta+1}) + \Psi_{\eta,\zeta}(w_{r+\zeta}w_{r+\zeta+1}) \\
&\quad + \Psi_{\eta,\zeta}(x_{r+\zeta}w_{r+\zeta}) + \Psi_{\eta,\zeta}(x_{r+\zeta}w_{r+\zeta+1}) \\
&\quad + \Psi_{\eta,\zeta}(x_{r+\zeta}) + \Psi_{\eta,\zeta}(y_{r+\zeta}) + \Psi_{\eta,\zeta}(z_{r+\zeta+1}) \\
&\quad + \Psi_{\eta,\zeta}(x_{r+\zeta}y_{r+\zeta}) + \Psi_{\eta,\zeta}(y_{r+\zeta}z_{r+\zeta}) \\
&\quad + \Psi_{\eta,\zeta}(y_{r+\zeta}z_{r+\zeta+1}) + \Psi_{\eta,\zeta}(z_{r+\zeta}z_{r+\zeta+1}) \\
&\quad - \Psi_{\eta,\zeta}(x_r) - \Psi_{\eta,\zeta}(y_r) - \Psi_{\eta,\zeta}(z_r) - \Psi_{\eta,\zeta}(w_r) \\
&\quad - \Psi_{\eta,\zeta}(w_rw_{r+1}) - \Psi_{\eta,\zeta}(x_rw_r) - \Psi_{\eta,\zeta}(x_rw_{r+1}) \\
&\quad - \Psi_{\eta,\zeta}(x_ry_r) - \Psi_{\eta,\zeta}(y_rz_r) - \Psi_{\eta,\zeta}(y_rz_{r+1}) \\
&\quad - \Psi_{\eta,\zeta}(z_rz_{r+1}) \\
&= \left\lceil \frac{r+11\zeta+2}{11\zeta+2} \right\rceil + \left\lceil \frac{r+2\zeta}{11\zeta+2} \right\rceil + \left\lceil \frac{r+7\zeta}{11\zeta+2} \right\rceil \\
&\quad + \left\lceil \frac{r+9\zeta}{11\zeta+2} \right\rceil + \left\lceil \frac{r+8\zeta}{11\zeta+2} \right\rceil + \left\lceil \frac{r+5\zeta}{11\zeta+2} \right\rceil \\
&\quad + \left\lceil \frac{r+10\zeta+1}{11\zeta+2} \right\rceil + \left\lceil \frac{r+6\zeta}{11\zeta+2} \right\rceil + \left\lceil \frac{r+3\zeta}{11\zeta+2} \right\rceil \\
&\quad + \left\lceil \frac{r+4\zeta}{11\zeta+2} \right\rceil + \left\lceil \frac{r+\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+7\zeta}{11\zeta+2} \right\rceil \\
&\quad - \left\lceil \frac{r+4\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+9\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+10\zeta+1}{11\zeta+2} \right\rceil \\
&\quad - \left\lceil \frac{r+\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+6\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+8\zeta}{11\zeta+2} \right\rceil \\
&\quad - \left\lceil \frac{r+5\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+2\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+3\zeta}{11\zeta+2} \right\rceil \\
&\quad - \left\lceil \frac{r}{11\zeta+2} \right\rceil \\
&= 1.
\end{aligned}$$

For every $r = 1, 2, \dots, \eta - \zeta$ we get $wt_{\Psi_{\eta,\zeta}}(DBL_{\zeta}^r) < wt_{\Psi_{\eta,\zeta}}(DBL_{\zeta}^{r+1})$. Hence, $wt_{\Psi_{\eta,\zeta}}(DBL_{\zeta}^{r+1}) = 1 + wt_{\Psi_{\eta,\zeta}}(DBL_{\zeta}^r)$. Furthermore, all DBL_{ζ} -weights are distinct. Thus, $\Psi_{\eta,\zeta}$ is total DBL_{ζ} -irregular labeling of DBL_{η} .

Since $tHs(DBL_{\eta}, DBL_{\zeta}) \geq p$ and $tHs(DBL_{\eta}, DBL_{\zeta}) \leq p$, we have

$$tHs(DBL_{\eta}, DBL_{\zeta}) = p,$$

which can be expressed as

$$tHs(DBL_{\eta}, DBL_{\zeta}) = \left\lceil \frac{10\zeta + \eta + 2}{11\zeta + 2} \right\rceil.$$

□

Figure 4 is given as an example of DBL_{19} with DBL_2 -covering, and with $tHs(DBL_{19}, DBL_2) = 2$.

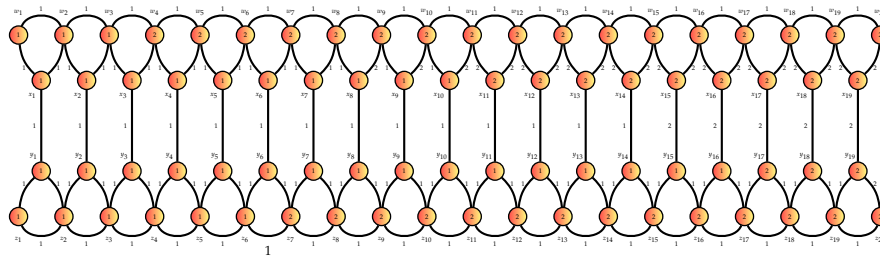


Figure 4. Double Balloon Graph DBL_{19} with DBL_2 -covering

Subsequently, Definition 3 furnishes the definition of a double balloon ladder graph.

Definition 3. A double balloon ladder graph DBL_{η} , $\eta \geq 1$, is a graph with

$$V(DBL_{\eta}) = \{x_q, y_q : q = 1, 2, \dots, \eta\} \cup \{w_q, z_q : q = 1, 2, \dots, \eta + 1\}$$

and

$$E(DBL_{\eta}) = \{w_q w_{q+1}, x_q w_q, x_q w_{q+1}, x_q y_q, y_q z_q, y_q z_{q+1}, z_q z_{q+1} : q = 1, 2, \dots, \eta\} \\ \cup \{w_q z_q : q = 1, 2, \dots, \eta + 1\}.$$

The following is provided as an example of double balloon ladder graph.

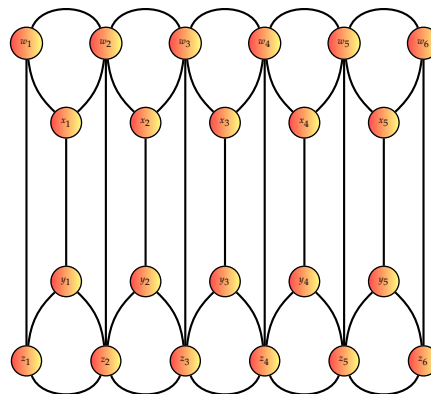


Figure 5. Double Balloon Ladder Graph DBL_5

Theorem 4. Given DBL_{η} , $\eta \geq 1$ with DBL_{ζ} -covering. Then, for $1 \leq \zeta \leq \eta$,

$$tHs(DBL_{\eta}, DBL_{\zeta}) = \left\lceil \frac{11\zeta + \eta + 3}{12\zeta + 3} \right\rceil.$$

Proof. The balloon DBL_{η} , $\eta \geq 1$ admits DBL_{ζ} -covering with exactly $\eta - \zeta + 1$ double balloon ladder DBL_{ζ} , $1 \leq \zeta \leq \eta$ and η is a positive integer. By Theorem 1,

$$tHs(DBL_{\eta}, DBL_{\zeta}) \geq \left\lceil 1 + \frac{(\eta - \zeta + 1) - 1}{12\zeta + 3} \right\rceil = \left\lceil \frac{11\zeta + \eta + 3}{12\zeta + 3} \right\rceil.$$

Next, put $p = \left\lceil \frac{11\zeta + \eta + 3}{12\zeta + 3} \right\rceil$, $tHs(DBL_{\eta}, DBL_{\zeta}) \leq p$ will be proven. On double balloon ladder DBL_{η} with DBL_{ζ} -covering, $1 \leq \zeta \leq \eta$, we define

$$\Psi_{\eta, \zeta} : V(DBL_{\eta}) \cup E(DBL_{\eta}) \rightarrow \{1, 2, \dots, p\},$$

for $\zeta = 1, 2, \dots, \eta$, in this way:

for $q = 1, 2, \dots, \eta$,

$$\Psi_{\eta, \zeta}(x_q) = \left\lfloor \frac{q + 6\zeta}{12\zeta + 3} \right\rfloor, \Psi_{\eta, \zeta}(y_q) = \left\lfloor \frac{q + 3\zeta}{12\zeta + 3} \right\rfloor, \Psi_{\eta, \zeta}(x_q y_q) = \left\lfloor \frac{q + 4\zeta}{12\zeta + 3} \right\rfloor,$$

$$\Psi_{\eta, \zeta}(y_q z_q) = \left\lfloor \frac{q + \zeta}{11\zeta + 3} \right\rfloor, \Psi_{\eta, \zeta}(y_q z_{q+1}) = \left\lfloor \frac{q + 2\zeta}{12\zeta + 3} \right\rfloor,$$

$$\Psi_{\eta, \zeta}(z_q z_{q+1}) = \left\lfloor \frac{q}{12\zeta + 3} \right\rfloor, \Psi_{\eta, \zeta}(w_q w_{q+1}) = \left\lfloor \frac{q}{12\zeta + 3} \right\rfloor,$$

$$\Psi_{\eta, \zeta}(x_q w_q) = \left\lfloor \frac{q + 5\zeta}{12\zeta + 3} \right\rfloor, \Psi_{\eta, \zeta}(x_q w_{q+1}) = \left\lfloor \frac{q + 7\zeta}{12\zeta + 3} \right\rfloor.$$

for $q = 1, 2, \dots, \eta + 1$,

$$\Psi_{\eta, \zeta}(w_q) = \left\lfloor \frac{q + 11\zeta + 2}{12\zeta + 3} \right\rfloor, \Psi_{\eta, \zeta}(z_q) = \left\lfloor \frac{q + 9\zeta}{12\zeta + 3} \right\rfloor,$$

$$\Psi_{\eta, \zeta}(w_q z_q) = \left\lfloor \frac{q + 10\zeta + 1}{12\zeta + 3} \right\rfloor.$$

The largest label is obtained when $q = \eta + 1$, that is $\Psi_{\eta, \zeta}(w_q) = \left\lfloor \frac{11\zeta + \eta + 3}{12\zeta + 3} \right\rfloor$.

For every $r = 1, 2, 3, \dots, \eta - \zeta + 1$, the vertex set is

$$V(DBIL_{\zeta}) = \{x_{r+q}, y_{r+q} : q = 0, 1, 2, \dots, \zeta - 1\} \cup \{w_{r+q}, z_{r+q} : q = 0, 1, 2, \dots, \zeta\}$$

and the edge set

$$E(DBIL_{\zeta}) = \{w_{r+q} w_{r+q+1}, x_{r+q} w_{r+q}, x_{r+q} w_{r+q+1}, x_{r+q} y_{r+q}, y_{r+q} z_{r+q}, \\ y_{r+q} z_{r+q+1}, z_{r+q} z_{r+q+1}, w_q z_q : q = 0, 1, 2, \dots, \zeta - 1\}.$$

Furthermore, for every subgraph $DBIL_{\zeta}^r$ in $DBIL_{\zeta}$ we have weight $DBIL_{\zeta}^r, r = 1, 2, \dots, \eta - \zeta + 1$, with respect to $\Psi_{\eta, \zeta} : V(DBIL_{\zeta}) \cup E(DBIL_{\zeta}) \rightarrow \{1, 2, \dots, p\}$ is

$$wt_{\Psi_{\eta, \zeta}}(DBIL_{\zeta}^r) = \sum_{v \in V(DBIL_{\zeta}^r)} \Psi_{\eta, \zeta}(v) + \sum_{e \in E(DBIL_{\zeta}^r)} \Psi_{\eta, \zeta}(e).$$

Then, for $r = 1, 2, \dots, \eta - \zeta$, we have

$$\begin{aligned} wt_{\Psi_{\eta, \zeta}} DBIL_{\zeta}^{r+1} - wt_{\Psi_{\eta, \zeta}} DBIL_{\zeta}^r &= \Psi_{\eta, \zeta}(w_{r+\zeta+1}) + \Psi_{\eta, \zeta}(w_{r+\zeta} w_{r+\zeta+1}) \\ &\quad + \Psi_{\eta, \zeta}(x_{r+\zeta} w_{r+\zeta}) + \Psi_{\eta, \zeta}(x_{r+\zeta} w_{r+\zeta+1}) \\ &\quad + \Psi_{\eta, \zeta}(x_{r+\zeta}) + \Psi_{\eta, \zeta}(y_{r+\zeta}) + \Psi_{\eta, \zeta}(z_{r+\zeta+1}) \\ &\quad + \Psi_{\eta, \zeta}(x_{r+\zeta} y_{r+\zeta}) + \Psi_{\eta, \zeta}(y_{r+\zeta} z_{r+\zeta}) \\ &\quad + \Psi_{\eta, \zeta}(y_{r+\zeta} z_{r+\zeta+1}) + \Psi_{\eta, \zeta}(z_{r+\zeta} z_{r+\zeta+1}) \\ &\quad + \Psi_{\eta, \zeta}(w_{r+\zeta+1} z_{r+\zeta+1}) - \Psi_{\eta, \zeta}(x_r) - \Psi_{\eta, \zeta}(y_r) \\ &\quad - \Psi_{\eta, \zeta}(z_r) - \Psi_{\eta, \zeta}(w_r) - \Psi_{\eta, \zeta}(w_r w_{r+1}) \\ &\quad - \Psi_{\eta, \zeta}(x_r w_r) - \Psi_{\eta, \zeta}(x_r w_{r+1}) - \Psi_{\eta, \zeta}(x_r y_r) \\ &\quad - \Psi_{\eta, \zeta}(y_r z_r) - \Psi_{\eta, \zeta}(y_r z_{r+1}) - \Psi_{\eta, \zeta}(z_r z_{r+1}) \\ &\quad - \Psi_{\eta, \zeta}(w_r z_r) \end{aligned}$$

$$\begin{aligned}
&= \left\lceil \frac{r+12\zeta+3}{12\zeta+3} \right\rceil + \left\lceil \frac{r+9\zeta}{12\zeta+3} \right\rceil + \left\lceil \frac{r+6\zeta}{12\zeta+3} \right\rceil \\
&\quad + \left\lceil \frac{r+8\zeta}{12\zeta+3} \right\rceil + \left\lceil \frac{r+7\zeta}{12\zeta+3} \right\rceil + \left\lceil \frac{r+4\zeta}{12\zeta+3} \right\rceil \\
&\quad + \left\lceil \frac{r+10\zeta+1}{12\zeta+3} \right\rceil + \left\lceil \frac{r+5\zeta}{12\zeta+3} \right\rceil + \left\lceil \frac{r+2\zeta}{12\zeta+3} \right\rceil \\
&\quad + \left\lceil \frac{r+3\zeta}{12\zeta+3} \right\rceil + \left\lceil \frac{r+\zeta}{13\zeta+2} \right\rceil + \left\lceil \frac{r+11\zeta+2}{12\zeta+3} \right\rceil \\
&\quad - \left\lceil \frac{r+6\zeta}{12\zeta+3} \right\rceil - \left\lceil \frac{r+3\zeta}{12\zeta+3} \right\rceil - \left\lceil \frac{r+9\zeta}{12\zeta+3} \right\rceil \\
&\quad - \left\lceil \frac{r+11\zeta+2}{12\zeta+3} \right\rceil - \left\lceil \frac{r+8\zeta}{12\zeta+3} \right\rceil - \left\lceil \frac{r+5\zeta}{12\zeta+3} \right\rceil \\
&\quad - \left\lceil \frac{r+7\zeta}{12\zeta+3} \right\rceil - \left\lceil \frac{r+4\zeta}{12\zeta+3} \right\rceil - \left\lceil \frac{r+\zeta}{12\zeta+3} \right\rceil \\
&\quad - \left\lceil \frac{r+2\zeta}{12\zeta+3} \right\rceil - \left\lceil \frac{r}{12\zeta+3} \right\rceil - \left\lceil \frac{r+10\zeta+1}{12\zeta+3} \right\rceil \\
&= 1.
\end{aligned}$$

For every $r = 1, 2, \dots, \eta - \zeta$ we get $wt_{\Psi_{\eta, \zeta}}(DBIL_{\zeta}^r) < wt_{\Psi_{\eta, \zeta}}(DBIL_{\zeta}^{r+1})$. Hence, $wt_{\Psi_{\eta, \zeta}}(DBIL_{\zeta}^{r+1}) = 1 + wt_{\Psi_{\eta, \zeta}}(DBIL_{\zeta}^r)$. Furthermore, all $DBIL_{\zeta}$ -weights are distinct. Thus, $\Psi_{\eta, \zeta}$ is total $DBIL_{\zeta}$ -irregular labeling of $DBIL_{\eta}$.

Since $tHs(DBIL_{\eta}, DBIL_{\zeta}) \geq p$ and $tHs(DBIL_{\eta}, DBIL_{\zeta}) \leq p$ we have

$$tHs(DBIL_{\eta}, DBIL_{\zeta}) = p,$$

which can be expressed as

$$tHs(DBIL_{\eta}, DBIL_{\zeta}) = \left\lceil \frac{11\zeta + \eta + 3}{12\zeta + 3} \right\rceil.$$

□

The following on Figure 6 is an example of $DBIL_{28}$ with $DBIL_2$ -covering, and we get $tHs(DBIL_{28}, DBIL_2) = 2$.

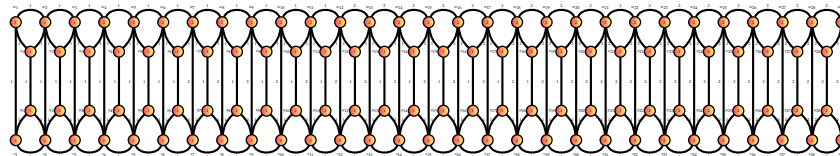


Figure 6. Total $DBIL$ -Irregularity Labeling of $DBIL_{28}$

3. Concluding Remarks

This research focuses on the total H -irregularity strength of balloon graph Bl_{η} with Bl_{ζ} -covering, double balloon graph DBl_{η} , and double balloon ladder graph $DBIL_{\eta}$ with DBL_{ζ} -covering. We found that the lower bound given in Theorem 1 is sharp for all these graphs.

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