Article

Not peer-reviewed version

Information-Complete and Paradox-Free Finite Ring Calculus

Yosef Akhtman *

Posted Date: 30 June 2025

doi: 10.20944/preprints202506.2454.v1

Keywords: Finite Ring Calculus; Finite Ring Continuum; Information-complete arithmetic; Paradox-free foundations; Godel diagonal collapse; Decidable first-order theory; Ultrafinitist logi; Finite fields; Model theory; Self-reference elimination; Logical completeness

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 June 2025 doi:10.20944/preprints202506.2454.v1

Article

Information-Complete and Paradox-Free Finite Ring Calculus

Yosef Akhtman

AGH University of Science and Technology, Switzerland; ya@gamma.earth

Abstract

Finite Ring Calculus (FRC) is a first-order formal system whose sole intended model is a fixed finite ring $\mathbb{Z}_q$. By eliminating unbounded induction and restricting all syntactic numerals to residues modulo q, FRC blocks standard diagonal constructions, thereby evading the classical Gödel-Turing-Russell paradox pattern. The resulting theory is shown to be recursive, consistent, complete and decidable, hence *information-complete* in the precise sense that every well-formed sentence admits an effective truth-value within $\mathbb{Z}_q$. We sketch how this finite arithmetic can act as a foundational layer for physics and computation without re-introducing classical paradoxes. We furthermore stress how these properties dovetail with the smooth-geometry and harmonic-analysis layers already constructed in the broader FRC framework.

Keywords: Finite Ring Continuum; information-complete arithmetic; paradox-free foundations; gödel incompleteness; decidable first-order theory; ultrafinitist logic; self-reference elimination; logical completeness

---

1. Introduction

The classical limits of formal logic, exemplified by the seminal arguments of Russell's set paradox [1], Gödel's incompleteness [2], and Turing's halting problem [3], all rely on an unbounded supply of natural-number codes. If however arithmetic is embedded into a *finite* algebra, the usual diagonal fixed-point method fails to escape the code-space, and no undecidable sentence is produced. In other words, if the universe of discourse is collapsed to a *finite* algebraic structure, the fixed-point machinery that fuels those paradoxes can no longer produce objects "outside the list".

This idea dates back to early ultrafinitist proposals [4–6], but is now developed into a coherent multi-layered framework: the *Finite Ring Continuum* (FRC) of [7–9]. The aim of this article is to make this intuition formal, yielding a fully decidable calculus that remains expressive enough for algebraic-numeric reasoning and, prospectively, for physics. The present paper is intended as a self-contained logical core for the FRC programme.

Remark 1. *Throughout the extent of the FRC programme, we keep the "FRC" acronym intentionally polyvalent, referring to any of the notions of* Finite Relativistic/Relational/Ring Continuum/Cosmology/Calculus *interchangeably depending on the context. Our intent is to emphasise how these deeply interlocked notions are all just different aspects of the exact same conceptual framework.*

Goal. Formulate a minimal first-order theory

$$\mathsf{FRC}_q \;=\; \mathsf{Ring}_q \;+\; \text{“there are exactly } q \text{ elements”}$$

whose sole model is a fixed finite ring $\mathbb{Z}_q$, of some composite cardinality $q = \prod_n p_n$, with p_n odd primes, and prove:

Consistency by explicit model;

 doi:10.20944/preprints202506.2454.v1

Completeness because every sentence can be brute-forced on F_q;

Decidability of the theory and its proof relation;

Collapse of diagonalisation: any Gödel sentence is already in the bounded code space, so no paradox arises.

2. Relation to the Finite Ring Continuum Framework

The FRC programme proceeds in three layers (Algebra → Geometry → Composition) as summarised in Fig. 1 of [8]. We briefly recall its main notation.

Definition 1 (FRC notation summary [9])**.** *Let* $p \equiv 1 \pmod 4$.

(a) $\mathbb{Z}_q$ *- framed finite ring;* $\mathbb{R}_q$ *- ultrapower pseudo-reals;* $\mathbb{C}_q = \mathbb{R}_q[i_q]$.

(b) *Canonical invarients:* $1,\ \pi_q,\ e_q,\ i_q$.

Our logical calculus *lives entirely inside* the algebraic layer $\mathbb{Z}_q$ but is consistent with the smooth-geometry layer carried by $\mathcal{S}_q^2 \subset \mathbb{R}^4$ [8,9]. Completeness of FRC_q therefore *does not* contradict the geometric richness exploited later in the continuum programme; it merely shows that paradox-free arithmetic is possible at the foundational stratum.

3. Language and Axioms of FRC_q

3.1. Signature

$$\mathcal{L} = \langle 0, 1, \pi_q, e_q, i_q,\ +, -, \cdot,\ \{c_k\}_{k=0}^{q-1} \rangle$$

extends the usual ring language by a constant for each field element, matching the FRC convention that *all numerals are concrete tokens* ([7], §2.1).

3.2. Finite Axiom Set

(A1) Ring axioms (associativity, distributivity, …).
(A2) Characteristic p (or q for non-prime moduli).
(A3) Frobenius clause $x^q = x$ (*cf.* ([7], Eq. (2.3))).
(A4) At least q distinct constants c_k.
(A5) No more than q elements (completed by a finite scheme).

Exactly as in ([8], Thm. 2.3), these axioms single out the intended finite ring.

4. Gödel Coding and Its Modular Collapse

Let g : Sentences $\twoheadrightarrow \mathbb{N}$ be any primitive-recursive Gödel numbering.

$$\overline{g}\ :\ \mathbb{Z}_q \longleftarrow \text{Sentences}, \qquad \overline{g}(\varphi) = g(\varphi) \bmod q.$$

Because range$(\overline{g}) \subseteq \{0, \ldots, q-1\}$, the fixed-point lemma yields a sentence G with $\overline{g}(G) = \overline{g}(\psi)$ for some earlier ψ; hence G is *not* new. This implements the "wrap-around defeats diagonal" slogan of ([7], §4) inside the present proof system.

Lemma 1 (Incompleteness blocked)**.** *For every recursive Gödel numbering g there is* no *sentence that is provably undecidable in* FRC_q.

Proof. Any candidate Gödel sentence duplicates a prior code and is therefore syntactically decidable in the finite proof search. □

 doi:10.20944/preprints202506.2454.v1

Compare with the geometric reading where the code space sits as a hyperfinite lattice in C_q ([8], §2.2); Lemma 1 is the algebraic counterpart of constant curvature $K = 1$ preventing "run-away" self-reference on $\mathcal{S}_q$.

5. Consistency, Completeness and Decidability

Lemma 2. FRC_q *is consistent, complete and decidable.*

Proof. *Consistency*: $\mathbb{Z}_q$ is a model. *Completeness*: brute-force truth-table on $\mathbb{Z}_q$. *Decidability*: enumerate proofs or evaluate truth; the two coincide. □

This result supplies the logical keystone required for the higher-level geometry, harmonic analysis and Seifert topology developed in [8,9].

6. Discussion

The Finite Ring Calculus provides the "paradox-free arithmetic engine" hypothesised in ([7], §1). Its compatibility with the canonical constants $(1, i_p, \pi_p, e_p)$ ensures smooth interoperability with the analytic structures (Fourier kernel, discrete S^2 of constant curvature) that power subsequent layers of the FRC stack.

References

1. Russell, B. *The Principles of Mathematics*; Cambridge University Press: Cambridge, 1903. See Chapter 10 for the first published statement of Russell's paradox.
2. Gödel, K. Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme I. *Monatshefte für Mathematik und Physik* **1931**, *38*, 173–198.
3. Turing, A.M. On Computable Numbers, with an Application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society* **1936**, *42*, 230–265. https://doi.org/10.1112/plms/s2-42.1.230.
4. Yessenin-Volpin, A.S. The Ultra-Intuitionistic Criticism and the Antitraditional Program for Foundations of Mathematics. Samizdat manuscript, Moscow; English translation circulated privately in 1962, 1961. Reprinted in: A. Gruszczyński et al. (eds.), *Non-Classical Logic*, Ossolineum, 1982, pp. 121–146.
5. Parikh, R.J. Existence and Feasibility in Arithmetic. *Journal of Symbolic Logic* **1971**, *36*, 494–508. https://doi.org/10.2307/2272491.
6. Nelson, E. *Predicative Arithmetic*; Princeton University Press: Princeton, 1986. Introduces a finitistic framework that strongly influenced later ultrafinitist research.
7. Akhtman, Y. Relativistic Algebra over Finite Ring Continuum. *Preprints* **2025**. https://doi.org/10.20944/preprints202505.2118.v5.
8. Akhtman, Y. Geometry and Constants in Finite Ring Continuum. *Preprints* **2025**. https://doi.org/10.20944/preprints202505.2112.v4.
9. Akhtman, Y. Prime Composition in Finite Ring Continuum. *Preprints* **2025**. https://doi.org/10.20944/preprints202506.0697.v1.