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Noncommutative Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle

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Abstract: Let $\{\tau_n\}_{n=1}^\infty$ and $\{\omega_m\}_{m=1}^\infty$ be two modular Parseval frames for a Hilbert C*-module $\mathcal{E}$. Then for every $x \in \mathcal{E} \setminus \{0\}$, we show that $\|\theta_\tau x\|_0 \|\theta_\omega x\|_0 \geq \frac{1}{\sup_{n,m\in\mathbb{N}} \|\langle \tau_n, \omega_m \rangle\|^2}$. We call Inequality (**??**) as **Noncommutative Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle**. Inequality (**??**) is the noncommutative analogue of breakthrough Ricaud-Torrésani uncertainty principle *[IEEE Trans. Inform. Theory, 2013]*. In particular, Inequality (**??**) extends Elad-Bruckstein uncertainty principle *[IEEE Trans. Inform. Theory, 2002]* and Donoho-Stark uncertainty principle *[SIAM J. Appl. Math., 1989]*.

Keywords: uncertainty principle; parseval frame; Hilbert C*-module

MSC: 42C15; 46L08

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1. Introduction

In 1989, Donoho and Stark derived following uncertainty principle which is one of the greatest inequality of all time in both pure and applied Mathematics [1]. For $h \in \mathbb{C}^d$, let $\|h\|_0$ be the number of nonzero entries in h. Let $\widehat{\cdot} : \mathbb{C}^d \to \mathbb{C}^d$ be the Fourier transform defined by

$$\widehat{(a_j)_{j=0}^{d-1}} := \frac{1}{\sqrt{d}} \left(\sum_{j=0}^{d-1} a_j e^{\frac{-2\pi ijk}{d}} \right)_{k=0}^{d-1}, \quad \forall (a_j)_{j=0}^{d-1} \in \mathbb{C}^d.$$

Theorem 1. *(**Donoho-Stark Uncertainty Principle**) [1,2] For every $d \in \mathbb{N}$,*

$$\left(\frac{\|h\|_0 + \|\widehat{h}\|_0}{2} \right)^2 \geq \|h\|_0 \|\widehat{h}\|_0 \geq d, \quad \forall h \in \mathbb{C}^d \setminus \{0\}. \tag{1}$$

By noting that Fourier transform is unitary and unitary operators are in one to one correspondence with orthonormal bases, in 2002, Elad and Bruckstein generalized Inequality (1) to arbitrary orthonormal bases [3]. To state the result we need some notations. Given a collection $\{\tau_j\}_{j=1}^n$ in a finite dimensional Hilbert space $\mathcal{H}$ over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), we define

$$\theta_\tau : \mathcal{H} \ni h \mapsto \theta_\tau h := (\langle h, \tau_j \rangle)_{j=1}^n \in \mathbb{K}^n.$$

Theorem 2. *(**Elad-Bruckstein Uncertainty Principle**) [3,4] Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$\left(\frac{\|\theta_\tau h\|_0 + \|\theta_\omega h\|_0}{2} \right)^2 \geq \|\theta_\tau h\|_0 \|\theta_\omega h\|_0 \geq \frac{1}{\max_{1\leq j,k \leq n} |\langle \tau_j, \omega_k \rangle|^2}, \quad \forall h \in \mathcal{H} \setminus \{0\}.$$

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In 2013, Ricaud and Torrésani showed that orthonormal bases in Theorem 2 can be improved to Parseval frames [5]. Recall that a collection $\{\tau_j\}_{j=1}^n$ in a finite dimensional Hilbert space $\mathcal{H}$ is said to be a Parseval frame for $\mathcal{H}$ [6] if

$$\|h\|^2=\sum_{j=1}^{n}|\langle h,\tau_j\rangle|^2,\quad \forall h\in\mathcal{H}.$$

Theorem 3. *(**Ricaud-Torrésani Uncertainty Principle**) [5] Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two Parseval frames for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$\left(\frac{\|\theta_\tau h\|_0+\|\theta_\omega h\|_0}{2}\right)^2\geq\|\theta_\tau h\|_0\|\theta_\omega h\|_0\geq\frac{1}{\max_{1\leq j,k\leq n}|\langle\tau_j,\omega_k\rangle|^2},\quad \forall h\in\mathcal{H}\setminus\{0\}.$$

The main purpose of this paper is to generalize and derive a noncommutative version of Theorem 3. For this we want generalization of Hilbert spaces known as Hilbert C*-modules. Hilbert C*-modules are first introduced by Kaplansky [7] for modules over commutative C*-algebras and later developed for modules over arbitrary C*-algebras by Paschke [8] and Rieffel [9].

Definition 1. *[7–9] Let $\mathcal{A}$ be a unital C*-algebra. A left module $\mathcal{E}$ over $\mathcal{A}$ is said to be a (left) Hilbert C*-module if there exists a map $\langle\cdot,\cdot\rangle:\mathcal{E}\times\mathcal{E}\to\mathcal{A}$ such that the following hold.*
(i) $\langle x,x\rangle\geq 0$, $\forall x\in\mathcal{E}$. If $x\in\mathcal{E}$ satisfies $\langle x,x\rangle=0$, then $x=0$.
(ii) $\langle x+y,z\rangle=\langle x,z\rangle+\langle y,z\rangle$, $\forall x,y,z\in\mathcal{E}$.
(iii) $\langle ax,y\rangle=a\langle x,y\rangle$, $\forall x,y\in\mathcal{E}$. $\forall a\in\mathcal{A}$.
(iv) $\langle x,y\rangle=\langle y,x\rangle^*$, $\forall x,y\in\mathcal{E}$.
(v) $\mathcal{E}$ is complete w.r.t. the norm $\|x\|:=\sqrt{\|\langle x,x\rangle\|}$, $\forall x\in\mathcal{E}$.

We are going to use the following inequality.

Lemma 1. *[8] (Noncommutative Cauchy-Schwarz inequality) If $\mathcal{E}$ is a Hilbert C*-module over $\mathcal{A}$, then*

$$\langle x,y\rangle\langle y,x\rangle\leq\|\langle y,y\rangle\|\langle x,x\rangle,\quad \forall x,y\in\mathcal{E}.$$

Given a unital C*-algebra $\mathcal{A}$, define

$$\ell^2(\mathbb{N},\mathcal{A}):=\left\{\{a_n\}_{n=1}^\infty:a_n\in\mathcal{A},\forall n\in\mathbb{N},\sum_{n=1}^{\infty}a_na_n^*\text{ converges in }\mathcal{A}\right\}.$$

Modular $\mathcal{A}$-inner product on $\ell^2(\mathbb{N},\mathcal{A})$ is defined as

$$\langle\{a_n\}_{n=1}^\infty,\{b_n\}_{n=1}^\infty\rangle:=\sum_{n=1}^{\infty}a_nb_n^*,\quad \forall\{a_n\}_{n=1}^\infty,\{b_n\}_{n=1}^\infty\in\ell^2(\mathbb{N},\mathcal{A}).$$

Hence the norm on $\ell^2(\mathbb{N},\mathcal{A})$ becomes

$$\|\{a_n\}_{n=1}^\infty\|:=\left\|\sum_{n=1}^{\infty}a_na_n^*\right\|^{\frac{1}{2}},\quad \forall\{a_n\}_{n=1}^\infty\in\ell^2(\mathbb{N},\mathcal{A}).$$

2. Noncommutative Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle

We start by recalling the definition of Parseval frames for Hilbert C*-modules by Frank and Larson [10].

Definition 2. *[10] Let $\mathcal{E}$ be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$. A collection $\{\tau_n\}_{n=1}^\infty$ in $\mathcal{E}$ is said to be a* ***modular Parseval frame*** *for $\mathcal{E}$ if*

$$\langle x, x\rangle = \sum_{n=1}^{\infty} \langle x, \tau_n\rangle \langle \tau_n, x\rangle, \quad \forall x \in \mathcal{E}.$$

As shown in [10] a modular Parseval frame $\{\tau_n\}_{n=1}^\infty$ for $\mathcal{E}$ gives an adjointable isometry

$$\theta_\tau : \mathcal{E} \ni x \mapsto \theta_\tau x := \{\langle x, \tau_n\rangle\}_{n=1}^\infty \in \ell^2(\mathbb{N}, \mathcal{A})$$

with adjoint

$$\theta_\tau^* : \ell^2(\mathbb{N}, \mathcal{A}) \ni \{a_n\}_{n=1}^\infty \mapsto \theta_\tau^*\{a_n\}_{n=1}^\infty := \sum_{n=1}^{\infty} a_n\tau_n \in \mathcal{E}.$$

We this preliminaries we can derive noncommutative analogue of Theorem 3. In the following theorem, given a subset $\Lambda \subseteq \mathbb{N}$, we set the notation

$$o(\Lambda) := \text{Number of elements in } \Lambda.$$

Theorem 4. *(**Noncommutative Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle**) For any two modular Parseval frames $\{\tau_n\}_{n=1}^\infty$ and $\{\omega_m\}_{m=1}^\infty$ for a Hilbert C*-module $\mathcal{E}$, we have*

$$\left(\frac{\|\theta_\tau x\|_0 + \|\theta_\omega x\|_0}{2}\right)^2 \geq \|\theta_\tau x\|_0 \|\theta_\omega x\|_0 \geq \frac{1}{\sup_{n,m\in\mathbb{N}} \|\langle \tau_n, \omega_m\rangle\|^2}, \quad \forall x \in \mathcal{E}, x \neq 0.$$

Proof. Let $x \in \mathcal{E}$ be nonzero. Using Lemma 1 and the well-known fact in C*-algebra that 'norm respects ordering of positive elements', we get

$$\begin{aligned}
\|x\|^2 &= \|\langle x, x\rangle\| = \left\|\sum_{n=1}^{\infty} \langle x, \tau_n\rangle\langle \tau_n, x\rangle\right\| = \left\|\sum_{n\in\operatorname{supp}(\theta_\tau x)} \langle x, \tau_n\rangle\langle \tau_n, x\rangle\right\| \\
&= \left\|\sum_{n\in\operatorname{supp}(\theta_\tau x)} \left\langle \sum_{m=1}^{\infty} \langle x, \omega_m\rangle\omega_m, \tau_n\right\rangle \left\langle \tau_n, \sum_{k=1}^{\infty} \langle x, \omega_k\rangle\omega_k\right\rangle\right\| \\
&= \left\|\sum_{n\in\operatorname{supp}(\theta_\tau x)} \left\langle \sum_{m\in\operatorname{supp}(\theta_\omega x)} \langle x, \omega_m\rangle\omega_m, \tau_n\right\rangle \left\langle \tau_n, \sum_{k\in\operatorname{supp}(\theta_\omega x)} \langle x, \omega_k\rangle\omega_k\right\rangle\right\| \\
&= \left\|\sum_{n\in\operatorname{supp}(\theta_\tau x)} \left(\sum_{m\in\operatorname{supp}(\theta_\omega x)} \langle x, \omega_m\rangle\langle \tau_n, \omega_m\rangle^*\right) \left(\sum_{k\in\operatorname{supp}(\theta_\omega x)} \langle x, \omega_k\rangle\langle \tau_n, \omega_k\rangle\right)^*\right\| \\
&\leq \left\|\sum_{n\in\operatorname{supp}(\theta_\tau x)} \left\|\sum_{m\in\operatorname{supp}(\theta_\omega x)} \langle \tau_n, \omega_m\rangle\langle \tau_n, \omega_m\rangle^*\right\| \left(\sum_{k\in\operatorname{supp}(\theta_\omega x)} \langle x, \omega_k\rangle\langle \omega_k, x\rangle\right)\right\| \\
&\leq \left\|\sum_{n\in\operatorname{supp}(\theta_\tau x)} \sum_{m\in\operatorname{supp}(\theta_\omega x)} \|\langle \tau_n, \omega_m\rangle\langle \tau_n, \omega_m\rangle^*\| \left(\sum_{k\in\operatorname{supp}(\theta_\omega x)} \langle x, \omega_k\rangle\langle \omega_k, x\rangle\right)\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\sup_{n,m\in\mathbb{N}} \|\langle \tau_n, \omega_m\rangle\|^2\right)\left\| \sum_{n\in\operatorname{supp}(\theta_\tau x)} \sum_{m\in\operatorname{supp}(\theta_\omega x)} 1\cdot\left(\sum_{k\in\operatorname{supp}(\theta_\omega x)} \langle x,\omega_k\rangle\langle\omega_k, x\rangle\right)\right\| \\
&\leq \left(\sup_{n,m\in\mathbb{N}} \|\langle \tau_n, \omega_m\rangle\|^2\right)\left\| o(\operatorname{supp}(\theta_\tau x)o(\operatorname{supp}(\theta_\omega x)\left(\sum_{k\in\operatorname{supp}(\theta_\omega x)} \langle x,\omega_k\rangle\langle\omega_k, x\rangle\right)\right\| \\
&= \left(\sup_{n,m\in\mathbb{N}} \|\langle \tau_n, \omega_m\rangle\|^2\right)\|\theta_\tau x\|_0\|\theta_\omega x\|_0\left\|\sum_{k\in\operatorname{supp}(\theta_\omega x)} \langle x,\omega_k\rangle\langle\omega_k, x\rangle\right\| \\
&= \left(\sup_{n,m\in\mathbb{N}} \|\langle \tau_n, \omega_m\rangle\|^2\right)\|\theta_\tau x\|_0\|\theta_\omega x\|_0\|\langle x, x\rangle\| \\
&= \left(\sup_{n,m\in\mathbb{N}} \|\langle \tau_n, \omega_m\rangle\|^2\right)\|\theta_\tau x\|_0\|\theta_\omega x\|_0\|x\|^2.
\end{aligned}
$$

By canceling $\|x\|$ we get the stated inequality. ☐

Using Chebotarev theorem, in 2005, Tao [11,12] improved Theorem 1 for prime dimensions d.

Theorem 5. *(**Tao Uncertainty Principle**) [11] For every prime p,*

$$\|h\|_0 + \|\widehat{h}\|_0 \geq p+1, \quad \forall h \in \mathbb{C}^p \setminus \{0\}.$$

In view of Theorem 5 we make the following conjecture.

Conjecture 6. *Let p be a prime and $\mathcal{A}$ be a unital C*-algebra with invariant basis number property. Let $\widehat{\cdot} : \mathcal{A}^p \to \mathcal{A}^p$ be the noncommutative Fourier transform defined by*

$$\widehat{(a_j)_{j=0}^{p-1}} := \frac{1}{\sqrt{p}}\left(\sum_{j=0}^{p-1} a_j e^{\frac{-2\pi ijk}{p}}\right)_{k=0}^{p-1}, \quad \forall (a_j)_{j=0}^{p-1} \in \mathcal{A}^p.$$

Then

$$\|x\|_0 + \|\widehat{x}\|_0 \geq p+1, \quad \forall x \in \mathcal{A}^p \setminus \{0\}.$$

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