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Definition of T-Norms and T-Conorms on Some Subfamilies of Type-2 Fuzzy Sets

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Abstract: In this paper, we obtain new t-norms and t-conorms on some important subfamilies of the set of functions from $[0,1]$ to $[0,1]$. In particular, we define these new operators on the subsets of the functions that are convex, normal, normal and convex, the functions taking only the values 0 or 1 and its subset of the functions whose support is a finite union of closed intervals. These t-norms and t-conorms are generalized to the type-2 fuzzy sets framework.

Keywords: Function from [0,1] to [0,1], normal function, convex function, type-2 fuzzy set, interval type-2 fuzzy set, t-norm, t-conorm.

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1. Introduction

When working with type-1 fuzzy sets (FSs), we may find that different agents assign different membership values to the same element. This disparity is inherent in the fact that different people may consider different meanings for the same words or different sensors may read the same data differently due to intrinsic errors in the measurements. More generally in fuzzy set theory every aspect is subject to the graduation of its membership including the degree of membership. To address this issue, L.A. Zadeh introduced type-2 fuzzy sets (T2FSs) as an extension of type-1 fuzzy sets (see [42,43]). A T2FS is determined by a membership function from the universe X to $\mathbf{M}$, where $\mathbf{M}$ is the set of functions from [0,1] to [0,1]. T2FSs are more general than FSs and more suitable for modeling uncertainty, vagueness and/or imprecision in specific situations. This is a consequence of the fact that, in the context of FSs, the degree of membership of an element to a set is given by a value in the interval $[0,1]$ while in the case of T2FSs this degree of membership is a fuzzy set in $[0,1]$ (see for instance [24,28,29,37]).

Many T2FSs families have been also developed to cope with the lack of knowledge or uncertainty of the experts valuations. The authors recommend the thorough overview [3]. Computationally efficient methods have been developed to transfer this reality into applications (see for example [7–9,22,23,25]). A large number of them are devoted to the feasibility of type-2 fuzzy logic systems (T2FLSs). As a result of these computational simplifications, the first applications are now being implemented (see [6,21,30,35]).

In this paper we consider T2FSs with membership degrees in some families of the set $\mathbf{M} = [0,1]^{[0,1]}$ of all functions from [0,1] to [0,1]. In particular, we will focus our attention in the next subsets of $\mathbf{M}$:

- $\mathbf{C}$: set of convex functions of $\mathbf{M}$.
- $\mathbf{N}$: set of normal functions of $\mathbf{M}$.
- $\mathbf{L}$: set of both convex and normal functions of $\mathbf{M}$.
- $\mathbf{K}$: functions of $\mathbf{N}$, whose images are 0 or 1 (but not all 0).
- $\mathbf{K}_c^F$: functions of $\mathbf{K}$ whose support is a finite union of closed intervals. In the notation $\mathbf{K}_c^F$, c stands for close and F for finite.

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Since the article of Bustince et al. [2], the interest in the set **K** has increased significantly (see for example [15,32]). In these works they show, among other things, theoretical and applied examples, about the advantage on the use of this set **K**. In particular, Ruiz-García et al. noted in [32] that it can be used to easily capture uncertainty without imposing unnecessary and unrealistic conditions on IVFSs, which can be extremely useful in intelligent systems.

The aim of this work is to define new triangular norms and triangular conorms in the aforementioned subsets of **M**. Triangular norms (t-norms) were first introduced by Menger in [27] in the context of metric probabilistic spaces. Later, Schweizer and Sklar reformulated the definition of t-norms in [33,34] establishing the axioms now used to define them. A thorough study about t-norms is given in [19]. Fuzzy set theory is strongly related to order theory (see for instance [12]). Hence the usefulness of defining t-norms on bounded partially ordered sets also known as bounded posets (see [4,5]). Specifically, it is interesting to define t-norms on bounded lattices as Ray did in [31].

The study of t-norms and t-conorms over more complex types of fuzzy sets started with Gehrke et al. in [11], where they extended the definitions of t-norm and t-conorm to interval-valued fuzzy sets (IVFSs). Walker and Walker extended these axioms to T2FSs (see [37,38]) and presented two new families of binary operations on **M**. They also determined that, under certain conditions, they are t-norms and t-conorms on **L**. In [18], Hernández et al. obtained t-norms and t-conorms on **L** which are extensions of those established in [37,38]. Furthermore, the same authors defined in [17] new t-norms and t-conorms on **L** that are not obtained with the formulas given in previous works. Later, Wu et al. carried out a similar study introducing different new t-norms on this same set (see [39,40]). Neither t-norms nor t-conorms on **C**, **N**, **K**, or $\mathbf{K}_c^F$ can be found in the literature. Even though **K** is not a lattice, in applications the operations on this set require less computational resources than those required on **M**.

The two main objectives of this paper are to analyze the operations presented in previous works (e.g. [17,18,37,38]) and to examine more general families of binary operations on **M**. More precisely, it is studied whether these operators satisfy the necessary axioms to be t-norms or t-conorms on **M**, **C**, **N**, **L**, **K** and $\mathbf{K}_c^F$.

The article is organized as follows. Section 2 establishes definitions, notations and properties required in the rest of this work. Subsection 2.1 is devoted to review some definitions and properties of FSs, IVFSs, T2FSs and IT2FSs. Subsection 2.2 provides some background on t-norms and t-conorms on such sets. Section 3 is the main part of the article. In Subsection 3.1 the operations considered in [16,18,37,38] are studied and we conclude that they are not t-norms or t-conorms on **M**, **C**, **N**, **K** and $\mathbf{K}_c^F$ in general. Subsection 3.2 introduces new families of operations on the aforementioned subsets of **M**. More precisely, the properties of these operations are analyzed in order to determine if they are t-norms or t-conorms on **M**, **C**, **N**, **L**, **K** and $\mathbf{K}_c^F$. Finally, Section 4 summarizes the main results and states some conclusions.

2. Preliminaries

Throughout the paper, X will denote a non-empty set which will represent the universe of discourse. Additionally, $\leq$ will denote the usual order relation in the lattice of real numbers, and $\vee$ and $\wedge$ the maximum and the minimum operators on the lattice $([0,1], \leq)$, respectively.

2.1. Some Types of Fuzzy Sets and Operations

In this subsection, we present the definition of fuzzy set, interval-valued fuzzy set, type-2 fuzzy set and interval type-2 fuzzy set. Moreover, we establish some important properties and operations related to them.

Definition 1. *([41]) A type-1 fuzzy set (FS) A is characterized by a membership function* μ_A,

$$\mu_A : X \to [0,1],$$

where $\mu_A(x)$ is the degree of membership of an element $x \in X$ to the set A.

Definition 2. *([1,36]) An interval-valued fuzzy set (IVFS)* A *is characterized by a membership function σ_A,*

$$\sigma_A : X \to I([0,1])$$

where $I([0,1])$ is the set of all closed intervals in $[0,1]$,

$$I([0,1]) = \{[a,b] : 0 \leq a \leq b \leq 1\}.$$

Accordingly, the degree of membership of an element $x \in X$ to the set A is a closed interval in $[0,1]$.

Definition 3. *([29]) A type-2 fuzzy set (T2FS)* A *is characterized by a membership function:*

$$\mu_A : X \to \mathbf{M}$$

where $\mathbf{M}$ is the set of all functions from the interval $[0,1]$ to itself,

$$\mathbf{M} = [0,1]^{[0,1]} = Map\ ([0,1],[0,1]).$$

That is, $\mu_A(x)$ is a fuzzy set on the interval $[0,1]$ and also the degree of membership of an element $x \in X$ to the set A. Therefore,

$$\mu_A(x) = f_x, \text{ where } f_x : [0,1] \to [0,1].$$

Next, let us present some subsets of **M** that we will consider in this work.

Definition 4. *A function $f \in \mathbf{M}$ is normal if,*

$$\sup\{f(x) : x \in [0,1]\} = 1$$

and it is convex if for any $x \leq y \leq z$, the inequality:

$$f(y) \geq f(x) \wedge f(z)$$

holds.

The set of all normal functions of **M** will be denoted by **N**, and the set of all convex functions of **M** will be denoted by **C**. Moreover, **L** will be the set of all normal and convex functions of **M**.

From now on, the notation for intervals between two slashes, $/a,b/$, will refer to any non-empty interval (closed, open or half-open interval) in $[0,1]$, and its characteristic function $\overline{/a,b/}$ is defined as follows.

Definition 5. *([18]) Let $/a,b/ \subseteq [0,1]$, with $0 \leq a \leq b \leq 1$, $/a,b/ \neq \emptyset$. The characteristic function of $/a,b/$ is $\overline{/a,b/} : [0,1] \to \{0,1\}$, where:*

$$\overline{/a,b/}(x) = \begin{cases} 1 & \text{if } x \in /a,b/, \\ 0 & \text{if } x \notin /a,b/. \end{cases}$$

Let us note that, the characteristic function of any interval in $[0,1]$ is an element of **L**.

Interval type-2 fuzzy sets are defined in [15] as follows:

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Definition 6. *([15]) A type-2 fuzzy set is said to be an interval type-2 fuzzy set (IT2FS) if for all $x \in X$,*

$$f_x \in Map\ ([0,1], \{0,1\}) \setminus \{\mathbf{0}\}$$

where $\mathbf{0}$ is a constant function such that $\mathbf{0}(y) = 0$ for all $y \in [0,1]$. That is, $f_x(y) \in \{0,1\}$ for all $y \in [0,1]$ and $f_x \neq \mathbf{0}$.

Note that the support of the function f_x, in Definition 6, can be any subset of the interval $[0,1]$ and therefore it does not necessarily have to be a convex subset. Moreover, let us note that in [2,20,26] the authors include the constant function $\mathbf{0}$ (with empty support), but in Definition 6 we exclude this function so as not to have two functions (the constant functions $\mathbf{0}$ and $\mathbf{1} = \overline{[0,1]}$) to represent the lack of information (see [15]).

Let $\mathbf{K} = Map\ ([0,1], \{0,1\}) \setminus \{\mathbf{0}\}$. Obviously, $\mathbf{K} \subset \mathbf{N} \subset \mathbf{M}$. Let us note that the support of any $f \in \mathbf{K}$ ($Supp(f)$) is not empty, and it is the finite or infinite union of closed, open or half-open intervals. In addition, we consider the subset of $\mathbf{K}$, denoted by $\mathbf{K}_c^F$, constituted by the functions whose support is the finite union of closed intervals. Consequently $\mathbf{K}_c^F \subset \mathbf{K}$.

The algebraic operations join, meet and complementation on $\mathbf{M}$, given in the next definition, were determined from Zadeh's Extension Principle ([41,42]).

Definition 7. *([10,14,37]) The operations $\sqcup$ (extended maximum or join), $\sqcap$ (extended minimum or meet) , $\neg$ (complementation) and the elements $\bar{0}$ and $\bar{1}$ are defined on $\mathbf{M}$ as follows (see Figure 1):*

$$(f \sqcup g)(x) = \sup\{f(y) \wedge g(z) : y \vee z = x\},$$

$$(f \sqcap g)(x) = \sup\{f(y) \wedge g(z) : y \wedge z = x\},$$

$$\neg f(x) = \sup\{f(y) : 1 - y = x\} = f(1 - x),$$

$$\bar{0}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases} \quad \bar{1}(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{if } x \neq 1. \end{cases}$$

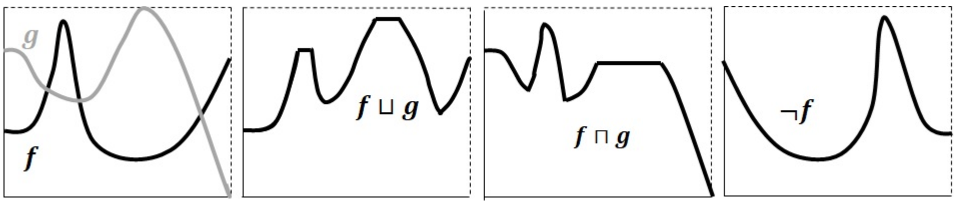

Figure 1. ([16], Fig. 5) Example for the operations $\sqcup$, $\sqcap$, and $\neg$.

Remark 1. *Note that $\sqcup$ and $\sqcap$ are idempotent, that is, $f \sqcap f = f$ and $f \sqcup f = f$, for all $f \in \mathbf{M}$. They also satisfy De Morgan's laws respect to the given operation $\neg$ (see [37] for more details). Additionally, when $\mathbf{M}$ is interpreted as the set of all linguistic labels of the "TRUTH" variable, then $\bar{0}$ and $\bar{1}$ (singletons of 0 and 1) represent the "completely false" and "completely true" labels, respectively.*

$\mathbb{M} = (\mathbf{M}, \sqcup, \sqcap, \neg, \bar{0}, \bar{1})$ does not have a lattice structure since it does not comply with the absorption law (see [14,37]). However, the operations $\sqcup$ and $\sqcap$ fulfill the properties required to each of them to define a partial order on $\mathbf{M}$.

Definition 8. *([29,37]) The partial orders defined on $\mathbf{M}$ are as follows:*

$$f \sqsubseteq g \quad \text{if } f \sqcap g = f; \quad f \preceq g \quad \text{if } f \sqcup g = g.$$

Remark 2. *As a consequence of [29,37] we can state that:*

- *The two partial orders* $\sqcap$ *and* $\sqcup$ *do not generally coincide.*
- $f \sqcap \bar{1} = f$*, and so* $f \sqsubseteq \bar{1}$*, for all* $f \in \mathbf{M}$*, that is,* $\bar{1}$ *is the largest element of the partial order* $\sqsubseteq$*.*
- $f \sqcup \bar{0} = f$*, and then* $\bar{0} \preceq f$*, for all* $f \in \mathbf{M}$*, that is,* $\bar{0}$ *is the smallest element of the partial order* $\preceq$*.*

The following definition and theorems were given in previous papers in order to facilitate the operations on $\mathbf{M}$:

Definition 9. *([10,14,37]) For each* $f \in \mathbf{M}$*, we define* $f^L, f^R \in \mathbf{M}$ *as follows:*

$$f^L(x) = \sup\{f(y) : y \leq x\}, \;\; f^R(x) = \sup\{f(y) : y \geq x\}.$$

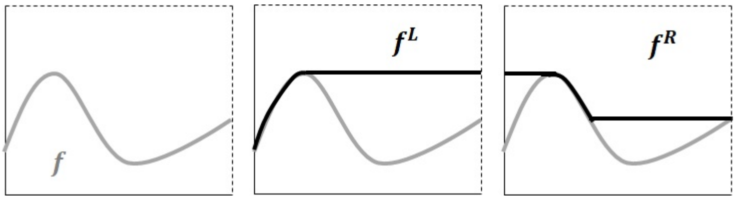

Figure 2. ([16], Fig. 7) Examples of f^L and f^R.

Remark 3. *In [37], some of the properties of these new functions are obtained:*

- f^L *and* f^R *are monotonically increasing and decreasing, respectively (see for example Figure 2).*
- $f \leq f^L$ *and* $f \leq f^R$ *where* $\leq$ *is the usual pointwise order in the set of functions (*$f \leq g$ *if and only if* $f(x) \leq g(x)$*, for all* $x \in [0,1]$*).*
- $(f^L)^L = f^L$ *and* $(f^R)^R = f^R$*.*
- *If we define* $f^{RL} = (f^R)^L$ *and* $f^{LR} = (f^L)^R$*, the next assertion holds:*

$$f^{RL} = f^{LR} = \sup f.$$

The following characterization was also shown in [37].

Theorem 1. *([37]) Let* $f, g \in \mathbf{M}$*. Then:*

$$f \sqsubseteq g \;\Leftrightarrow\; (f^R \wedge g) \leq f \leq g^R,$$

$$f \preceq g \;\Leftrightarrow\; (g^L \wedge f) \leq g \leq f^L.$$

Note that the operations $\vee$ and $\wedge$ have the usual meaning in the set of functions, that is, $(f \vee g)(x) = f(x) \vee g(x)$, and $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in [0,1]$.

The family $\mathbb{L} = (\mathbf{L}, \sqcup, \sqcap, \neg, \bar{0}, \bar{1})$ is a subalgebra of $\mathbb{M}$. In $\mathbf{L}$, the partial orders $\sqsubseteq$ and $\preceq$ coincide, and therefore $\mathbb{L}$ is a complete and bounded lattice where $\bar{0}$ and $\bar{1}$ are the minimum and the maximum, respectively (see [13,14,29,37] for more details). In $\mathbf{L}$, the following characterization holds.

Theorem 2. *([13,14]) Let* $f, g \in \mathbf{L}$*.* $f \sqsubseteq g$ *if and only if* $g^L \leq f^L$ *and* $f^R \leq g^R$*.*

2.2. T-norms and t-conorms on bounded posets

In this section we recall some definitions and results about t-norms and t-conorms which will be used throughout Section 3. Remember that a t-norm on $[0,1]$ is a binary operation $T : [0,1]^2 \rightarrow [0,1]$, which is commutative, associative, increasing on each argument, and with neutral element 1. Furthermore, a t-conorm on $[0,1]$ is a binary operation $S : [0,1]^2 \rightarrow [0,1]$, commutative, associative,

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increasing on each argument and with neutral element 0. Similar definitions are applied to bounded posets (see [4,5]).

Definition 10. *([4,5]) Let $(R, \leq_R, 0_R, 1_R)$ be a bounded poset. The binary operation $T : R^2 \to R$ is a t-norm on R if:*

1. $T(a,b) = T(b,a)$ *for all* $a, b \in R$ *(commutativity),*
2. $T(a, T(b,c)) = T(T(a,b), c)$ *for all* $a, b, c \in R$ *(associativity),*
3. $T(a, 1_R) = a$, *for all* $a \in R$ *(neutral element),*
4. *Let* $a, b, c \in R$ *such that* $b \leq_R c$, *then* $T(a,b) \leq_R T(a,c)$ *(monotony).*

Definition 11. *([4,5]) A binary operation $S : R^2 \to R$ is a t-conorm (triangular conorm) on the bounded poset $(R, \leq_R, 0_R, 1_R)$ if the axioms 1, 2 and 4 of the t-norm and the axiom:*

3′. $S(f, 0_R) = f$,

are satisfied.

Example 1. *Here we present some important examples of t-norms and t-conorms on $[0,1]$ which will be used in the following:*

1. *The minimum t-norm* $x \wedge y = \min\{x, y\}$ *and the maximum t-conorm* $x \vee y = \max\{x, y\}$.
2. *The product t-norm* $T_P(x,y) = xy$ *and the probabilistic sum* $S_P(x,y) = x + y - xy$.
3. *The drastic t-norm* $T_D(x,y) = \begin{cases} x \wedge y & \text{if } x \vee y = 1, \\ 0 & \text{otherwise} \end{cases}$ *and the drastic t-conorm:*

$$S_D(x,y) = \begin{cases} x \vee y & \text{if } x \wedge y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

In [36–38] it was shown that $\sqcap$ and $\sqcup$ are t-norm and t-conorm, respectively, on **L**, but no other study was done of these operations on other subsets of **M**. In [16,18] the two following families of binary operations on **M** were proposed. These operations are extensions of the ones given in [37,38].

Definition 12. *([16,18]) Let $\star$ and $\triangle$ be continuous t-norms on $[0,1]$, and $\triangledown$ a continuous t-conorm on $[0,1]$. For each $f, g \in \mathbf{M}$, we define the binary operations $\blacktriangle$ and $\blacktriangledown$ as:*

$$(f \blacktriangle g)(x) = \sup\{f(y) \star g(z) : y \triangle z = x\},$$
$$(f \blacktriangledown g)(x) = \sup\{f(y) \star g(z) : y \triangledown z = x\}.$$

In [18] it was shown that $\blacktriangle$ ($\blacktriangledown$) is a t-norm (t-conorm) on **L** given the order $\sqsubseteq$ (in this case $\sqsubseteq \equiv \preceq$). Furthermore, the axioms of definitions 10 and 11 were studied on **M**, **C** and **N**, except for the monotony. We will study if these two operators are, respectively, t-norm and t-conorm on these and other subfamilies of **M** with both orders $\sqsubseteq$ and $\preceq$. In addition, we will define new operators that are indeed t-norms or t-conorms on some of these subsets.

The following theorem presents some properties of $\blacktriangle$ and $\blacktriangledown$, that will allow us to prove some results in Section 3.

Theorem 3. *([18]) For the operations $\blacktriangle$ and $\blacktriangledown$ given in Definition 12, the following properties hold:*

1. $\blacktriangle$ *and* $\blacktriangledown$ *are commutative and associative in* $\mathbf{M}$.
2. $f \blacktriangle \bar{1} = f$, $f \blacktriangledown \bar{0} = f$, $f \blacktriangle \mathbf{0} = \mathbf{0}$ *and* $f \blacktriangledown \mathbf{0} = \mathbf{0}$ *for all* $f \in \mathbf{M}$.
3. $f \blacktriangle \mathbf{1} = f^R$, $f \blacktriangledown \mathbf{1} = f^L$ *for all* $f \in \mathbf{M}$ *where* $\mathbf{1} = \overline{[0,1]}$.
4. $f^R \blacktriangle g^R = f \blacktriangle g^R = f^R \blacktriangle g = (f \blacktriangle g)^R$ *for all* $f, g \in \mathbf{M}$.

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5. $f^L \blacktriangledown g^L = f \blacktriangledown g^L = f^L \blacktriangledown g = (f \blacktriangledown g)^L$ *for all* $f, g \in \mathbf{M}$.
6. $f^L \blacktriangle g^L = (f \blacktriangle g)^L$ *and* $f^R \blacktriangledown g^R = (f \blacktriangledown g)^R$ *for all* $f, g \in \mathbf{M}$.
7. *Given* $f, g, h \in \mathbf{M}$, *such that* $g \leq h$, *then:*

$$(f \blacktriangle g) \preceq (f \blacktriangle h) \text{ and } (f \blacktriangledown g) \preceq (f \blacktriangledown h).$$

8. $f \blacktriangle \bar{0} = \bar{0}$, $f \blacktriangledown \bar{1} = \bar{1}$ *for all* $f \in \mathbf{N}$.
9. *For all* $a, b, c, d \in [0,1]$ *such that* $a \leq b$ *and* $c \leq d$*:*

$$\overline{[a,b]} \blacktriangle \overline{[c,b]} = \overline{[a \triangle c, b \triangle d]} \text{ and}$$
$$\overline{[a,b]} \blacktriangledown \overline{[c,b]} = \overline{[a \triangledown c, b \triangledown d]}.$$

10. *If* $\overline{/a,b/}, \overline{/c,d/} \neq \mathbf{0}$, *then:*

$$\overline{/a,b/} \blacktriangle \overline{/c,b/}, \ \overline{/a,b/} \blacktriangledown \overline{/c,b/} \in \mathbf{K}.$$

11. $\blacktriangle$ *and* $\blacktriangledown$ *are closed on* $\mathbf{M}$, $\mathbf{C}$, $\mathbf{N}$, *and* $\mathbf{L}$.
12. $\blacktriangle$ *and* $\blacktriangledown$ *are t-norms and t-conorms, respectively, on the lattice* $(\mathbf{L}, \sqsubseteq, \bar{0}, \bar{1})$.

3. T-norms and t-conorms on M, C, N, L, K and $\mathbf{K}_c^F$.

In this section, we will prove that, in general, the operations $\blacktriangle$ and $\blacktriangledown$ are not t-norm and t-conorm on $\mathbf{C}$, $\mathbf{N}$, $\mathbf{K}$, and $\mathbf{K}_c^F$ respectively. Nevertheless, we will show that they are indeed t-norm and t-conorm in the particular case where $\blacktriangle = \sqcap$ and $\blacktriangledown = \sqcup$. Additionally we will perform a similar study, introducing new families of operators and analyzing for different orders if they are t-norms or t-conorms on $\mathbf{M}$, $\mathbf{C}$, $\mathbf{N}$, $\mathbf{L}$, $\mathbf{K}$ and $\mathbf{K}_c^F$.

3.1. The operations $\blacktriangle$ *and* $\blacktriangledown$ *on* $\mathbf{M}$, $\mathbf{C}$, $\mathbf{N}$, $\mathbf{K}$ *and* $\mathbf{K}_c^F$*.*

The main purpose of this subsection is to show that $\blacktriangle$ and $\blacktriangledown$ are not t-norms and t-conorms in general in any of the sets $\mathbf{C}$, $\mathbf{N}$, $\mathbf{K}$ and $\mathbf{K}_c^F$. In order to find the corresponding counterexamples, we need to go deeper into the structure of these families regarding the partial orders $\sqsubseteq$ and $\preceq$.

From the results in [37], it can be deduced that $\mathbf{0}$ is the minimum and $\bar{1}$ is the maximum element with respect to the partial order $\sqsubseteq$ on $\mathbf{M}$ and on $\mathbf{C}$. Moreover, $\bar{0}$ is the minimum and $\mathbf{0}$ is the maximum regarding $\preceq$ on these same sets. It is also well known (see [15]) that $\bar{0}$ and $\bar{1}$ are, respectively, the minimum and the maximum of each one of the posets $(\mathbf{K}, \sqsubseteq)$, $(\mathbf{K}, \preceq)$, $(\mathbf{K}_c^F, \sqsubseteq)$ and $(\mathbf{K}_c^F, \preceq)$. In the next result, we will show that these particular elements are the same in $\mathbf{N}$.

Proposition 1. *The functions* $\bar{0}$ *and* $\bar{1}$*, are, respectively, the minimum and the maximum of* $\mathbf{N}$*, respect to the partial orders* $\sqsubseteq$ *and* $\preceq$*.*

Proof. In [15] it was proved that $\bar{1}$ is the maximum of $(\mathbf{N}, \sqsubseteq)$ and $\bar{0}$ is the minimum of $(\mathbf{N}, \preceq)$. Let us prove that $\bar{0} \sqsubseteq f$, for all $f \in \mathbf{N}$. It is known that $\bar{0}^R = \bar{0}$, and that if $f \in \mathbf{N}$, then $\bar{0} \leq f^R$. Hence:

$$\bar{0}^R \wedge f = \bar{0} \wedge f \leq \bar{0} \leq f^R,$$

and, according to Theorem 1, $\bar{0} \sqsubseteq f$ for all $f \in \mathbf{N}$. The same procedure can be applied to show that $\bar{1}$ is the maximum element of $(\mathbf{N}, \preceq)$. □

In [18] it was proved that $\blacktriangle$ and $\blacktriangledown$ are closed in $\mathbf{C}$, $\mathbf{N}$, and $\mathbf{M}$. In the next result, we show that both operations are also closed in $\mathbf{K}$ and $\mathbf{K}_c^F$.

Proposition 2. $\blacktriangle$ *and* $\blacktriangledown$ *are binary operators in* $\mathbf{K}$ *and* $\mathbf{K}_c^F$*.*

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Proof. By definition, $\mathbf{K} = Map([0,1],\{0,1\}) \setminus \{\mathbf{0}\}$. We will only prove that $\blacktriangle$ is a closed operation on $\mathbf{K}$ since the proof is analogous for $\blacktriangledown$. If $f, g \in Map([0,1],\{0,1\})$, it is clear that $f \blacktriangle g \in Map([0,1],\{0,1\})$ by the way we defined this operation. Consequently, we only need to show that $f \blacktriangle g \neq \mathbf{0}$ whenever $f, g \neq \mathbf{0}$. In that case there exist $u \in Supp(f)$ and $v \in Supp(g)$ such that $f(u) = 1$ and $g(v) = 1$. Fixing $x = u \triangle v$ we have that:

$$(f \blacktriangle g)(x) = \sup\{f(y) \star g(z) : \ y \triangle z = x\} = f(u) \star g(v) = 1,$$

which concludes this part of the proof.

Let us now show that $\blacktriangle$ is closed on $\mathbf{K}_c^F$ (the proof for $\blacktriangledown$ is analogous). Given $f, g \in \mathbf{K}_c^F$, we only need to prove that $f \blacktriangle g \in \mathbf{K}_c^F$. Since $\blacktriangle$ is closed on $\mathbf{K}$, this is equivalent to state that $Supp(f \blacktriangle g)$ is a union of closed intervals. In fact, as $Supp(f) = \cup_{i=1}^{n}[a_i, b_i]$ and $Supp(g) = \cup_{j=1}^{m}[c_j, d_j]$ for some finite $n, m \in \mathbb{N}$, let us see that:

$$Supp(f \blacktriangle g) = \bigcup_{(i,j) \in \{1,\ldots,n\} \times \{1,\ldots,m\}} [a_i \triangle c_j, b_i \triangle d_j] \tag{1}$$

First, let us take an arbitrary $x \in Supp(f \blacktriangle g)$. In this case:

$$(f \blacktriangle g)(x) = \sup\{f(y) \star g(z) : y \triangle z = x\} = 1.$$

The only possibility here is the existance of $y \in Supp(f)$ and $z \in Supp(g)$ such that $y \triangle z = x$. That is, there exist $i_0 \in \{1, \ldots, n\}$ and $j_0 \in \{1, \ldots, m\}$ with $y \in [a_{i_0}, b_{i_0}]$ and $z \in [c_{j_0}, d_{j_0}]$. Consequently, and making use of the monotony of $\triangle$, we have that $x \in [a_{i_0} \triangle c_{j_0}, b_{i_0} \triangle d_{j_0}]$ and hence:

$$x \in \bigcup_{(i,j) \in \{1,\ldots,n\} \times \{1,\ldots,m\}} [a_i \triangle c_j, b_i \triangle d_j].$$

Finally, let us consider $x \in [a_{i_1} \triangle c_{j_1}, b_{i_1} \triangle d_{j_1}]$ for some $(i_1, j_1) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$. Since $\triangle$ is a continuous function, there exist $y \in [a_{i_1}, b_{i_1}] \subseteq Supp(f)$ and $z \in [c_{j_1}, d_{j_1}] \subseteq Supp(g)$ with $y \triangle z = x$. Taking this into account, $f(y) = g(z) = f(y) \star g(z) = 1$ and thus, $(f \blacktriangle g)(x) = 1$. Consequently, $x \in Supp(f \blacktriangle g)$ and equation (1) holds. Therefore, $Supp(f \blacktriangle g)$ is the union of closed intervals and $f \blacktriangle g \in \mathbf{K}_c^F$. □

Note that Theorem 3 establishes that the two operations satisfy the axioms 1 and 2 of t-norm and t-conorm in $\mathbf{M}$, and also establishes that the operation $\blacktriangle$ satisfies axiom 3 on the poset $(\mathbf{M}, \sqsubseteq, \mathbf{0}, \overline{1})$ and $\blacktriangledown$ satisfies axiom 3' on $(\mathbf{M}, \preceq, \overline{0}, \mathbf{0})$. Nevertheless, in general, they are not t-norms or t-conorms in $\mathbf{M}$, $\mathbf{N}$, $\mathbf{K}$ and $\mathbf{K}_c^F$ respect to each partial order. Corollary 1, established in [15], will help us to reach this result.

Corollary 1. *([15]) Let $f, g \in \mathbf{K}_c^F$. And let $v_i = \inf\{Supp(f)\}$, $w_i = \inf\{Supp(g)\}$, $v_s = \sup\{Supp(f)\}$, $w_s = \sup\{Supp(g)\}$. Then,*

- *$f \sqsubseteq g$ if and only if $v_i \leq w_i$, $v_s \leq w_s$, and $f(x) \geq g(x)$, for all $x \in [v_i, v_s]$.*
- *$f \preceq g$ if and only if $v_i \leq w_i$, $v_s \leq w_s$, and $g(x) \geq f(x)$, for all $x \in [w_i, w_s]$.*

Proposition 3. *$\blacktriangle$ and $\blacktriangledown$, in general, are neither t-norm nor t-conorm on $\mathbf{M}$, $\mathbf{N}$, $\mathbf{K}$ and $\mathbf{K}_c^F$.*

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Proof. It is enough to find the appropriate counterexamples where the operations are not increasing with respect to any of the two partial orders. In that case, t-norm (t-conorm) axiom 4 fails. Since $\mathbf{K}_c^F \subset \mathbf{K} \subset \mathbf{N} \subset \mathbf{M}$, we only need to find these counterexamples in $\mathbf{K}_c^F$. Let $f, g \in \mathbf{K}_c^F$, where

$$f(x) = \begin{cases} 1 & \text{if } x \in \{0.1, 0.25, 0.3\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$g(x) = \begin{cases} 1 & \text{if } x \in \{0.3, 0.4\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence of Corollary 1, $f \sqsubseteq g$ and $f \preceq g$. Let us consider $\blacktriangle$, where for each $x, y \in [0,1]$ we have $\triangle = T_P$, and $\star = \wedge$. In this case,

$$(f \blacktriangle f)(x) = \begin{cases} 1 & \text{if } x \in \{0.01, 0.025, 0.03, 0.0625, 0.075, 0.09\}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(f \blacktriangle g)(x) = \begin{cases} 1 & \text{if } x \in \{0.03, 0.04, 0.075, 0.09, 0.1, 0.12\}, \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 1 we conclude that $f \blacktriangle f \not\sqsubseteq f \blacktriangle g$ and $f \blacktriangle f \not\preceq f \blacktriangle g$. Therefore, $\blacktriangle$ is not always increasing on the bounded posets $(\mathbf{K}_c^F, \sqsubseteq, \overline{0}, \overline{1})$, $(\mathbf{K}, \sqsubseteq, \overline{0}, \overline{1})$, $(\mathbf{N}, \sqsubseteq, \overline{0}, \overline{1})$, $(\mathbf{K}_c^F, \preceq, \overline{0}, \overline{1})$, $(\mathbf{K}, \preceq, \overline{0}, \overline{1})$ and $(\mathbf{N}, \preceq, \overline{0}, \overline{1})$ and, consequently, on $(\mathbf{M}, \preceq, \overline{0}, \mathbf{0})$ and $(\mathbf{M}, \sqsubseteq, \mathbf{0}, \overline{1})$.

Analogously, it is easy to prove that the operator $\blacktriangledown$, where $\triangledown = S_P$ and $\star = \wedge$ for each $x, y \in [0,1]$, is not increasing with respect to any of the aforementioned partial orders. The same functions f and g defined above can be used. In this case:

$$(f \blacktriangledown f)(x) = \begin{cases} 1 & \text{if } x \in \{0.19, 0.325, 0.37, 0.4375, 0.475, 0.51\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$(f \blacktriangledown g)(x) = \begin{cases} 1 & \text{if } x \in \{0.37, 0.46, 0.475, 0.51, 0.55, 0.58\}, \\ 0 & \text{otherwise.} \end{cases}$$

Once again, using Corollary 1 we can check that $f \blacktriangledown f \not\sqsubseteq f \blacktriangledown g$ and $f \blacktriangledown f \not\preceq f \blacktriangledown g$ so the monotony of this operator on the posets of interest does not hold. □

A similar result to the previous one can be obtained for the set $\mathbf{C}$.

Proposition 4. *$\blacktriangle$ and $\blacktriangledown$, in general, are neither t-norm nor t-conorm on $(\mathbf{C}, \sqsubseteq)$ or $(\mathbf{C}, \preceq)$.*

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Proof. First, we define:

$$f(x) = \tfrac{1}{4}x, \qquad g(x) = \begin{cases} x & \text{if } x \in \left[0, \tfrac{1}{2}\right), \\ 0 & \text{otherwise,} \end{cases}$$

$$h(x) = \begin{cases} 0 & \text{if } x \in \left[0, \tfrac{1}{8}\right), \\ \tfrac{4}{3}x - \tfrac{1}{6} & \text{if } x \in \left[\tfrac{1}{8}, \tfrac{7}{8}\right), \\ 1 & \text{if } x \in \left[\tfrac{7}{8}, 1\right], \end{cases} \qquad p(x) = \tfrac{3}{4}(1-x),$$

$$q(x) = \begin{cases} 1 - x & \text{if } x \in \left[\tfrac{1}{2}, 1\right], \\ 0 & \text{otherwise,} \end{cases} \qquad s(x) = \frac{1}{2}$$

$$r(x) = \begin{cases} 1 & \text{if } x \in \left[0, \tfrac{1}{8}\right), \\ \tfrac{7}{6} - \tfrac{4}{3}x & \text{if } x \in \left[\tfrac{1}{8}, \tfrac{7}{8}\right), \\ 0 & \text{if } x \in \left[\tfrac{7}{8}, 1\right]. \end{cases} \quad \text{and} \quad w(x) = \overline{[0,1]}.$$

It is easy to check that $f, g, h, s, w \in \mathbf{C}$. Moreover, by Theorem 1 we have that $g \sqsubseteq h$ since $(g^R \wedge h) \leq g \leq h^R$. However, if we set $\star = \triangle = T_P$ to define $\blacktriangle$ as in Definition 12 we can show that $f \blacktriangle g \not\sqsubseteq f \blacktriangle h$. With this purpose, let us prove that:

$$(f \blacktriangle g)^R(x) \wedge (f \blacktriangle h)(x) > (f \blacktriangle g)(x) \tag{2}$$

for $x = \frac{1}{4}$. Since:

$$\begin{aligned} (f \blacktriangle g)\left(\tfrac{1}{4}\right) &= \sup\left\{f(y)g(z) : yz = \tfrac{1}{4}\right\} \\ &= \sup\left\{\tfrac{3}{4}yz : yz = \tfrac{1}{4} \leq z < \tfrac{1}{2}\right\} = \tfrac{3}{16}, \end{aligned}$$

$$\begin{aligned} (f \blacktriangle g)^R\left(\tfrac{1}{4}\right) &= (f^R \blacktriangle g^R)\left(\tfrac{1}{4}\right) \\ &= \sup\left\{f^R(y)g^R(z) : yz = \tfrac{1}{4}\right\} = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}, \end{aligned}$$

$$(f \blacktriangle h)\left(\tfrac{1}{4}\right) = \sup\left\{f(y)h(z) : yz = \tfrac{1}{4}\right\} \geq f\left(\tfrac{1}{3}\right)h\left(\tfrac{3}{4}\right) = \tfrac{5}{24},$$

then inequality (2) holds. As a consequence of this and by means of Theorem 1 we get to the result. With this discussion, we have proven that $\blacktriangle$ is not always monotonically increasing in $\mathbf{C}$ with respect to the order $\sqsubseteq$. Therefore, $\blacktriangle$ is neither t-norm nor t-conorm in $(\mathbf{C}, \sqsubseteq)$.

Let us now show that $\blacktriangledown$ is not monotonically increasing either, when we take $\star = T_P$ and $\triangledown = S_P$ in its definition. Note that $s \sqsubseteq w$ by Theorem 1 and that:

$$\begin{aligned} (f \blacktriangledown s)\left(\tfrac{1}{4}\right) &= \sup\left\{f(y)s(z) : y + z - yz = \tfrac{1}{4}\right\} \\ &= f\left(\tfrac{1}{4}\right)s(0) = \tfrac{3}{4}\tfrac{1}{4}\tfrac{1}{2} = \tfrac{3}{32}, \end{aligned}$$

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$$
\begin{aligned}
(f \blacktriangledown s)^R\left(\tfrac{1}{4}\right) &= (f^R \blacktriangledown s^R)\left(\tfrac{1}{4}\right) \\
&= \sup\left\{f^R(y)s^R(z) : y + z - yz = \tfrac{1}{4}\right\} \\
&= \tfrac{3}{4}\tfrac{1}{2} = \tfrac{3}{8},
\end{aligned}
$$

$$
\begin{aligned}
(f \blacktriangledown w)\left(\tfrac{1}{4}\right) &= \sup\left\{f(y)w(z) : y + z - yz = \tfrac{1}{4}\right\} \\
&= f\left(\tfrac{1}{4}\right)w(0) = \tfrac{3}{4}\tfrac{1}{4}1 = \tfrac{3}{16},
\end{aligned}
$$

Thus we can state that:

$$(f \blacktriangledown s)^R\left(\tfrac{1}{4}\right) \wedge (f \blacktriangledown w)\left(\tfrac{1}{4}\right) > (f \blacktriangledown s)\left(\tfrac{1}{4}\right).$$

By Theorem 1, $f \blacktriangledown s \not\sqsubseteq f \blacktriangledown w$ and $\blacktriangledown$ is neither t-norm nor t-conorm on $\mathbf{C}$ with respect to the order $\sqsubseteq$.

Let us now consider the order $\preceq$. Since $s^L \wedge w \leq s \leq w^L$, we have that $w \preceq s$ and that:

$$(p \blacktriangle s)\left(\tfrac{3}{4}\right) = p\left(\tfrac{3}{4}\right)s(1) = \tfrac{3}{4}\left(1 - \tfrac{3}{4}\right)\tfrac{1}{2} = \tfrac{3}{32},$$

$$(p \blacktriangle s)^L\left(\tfrac{3}{4}\right) = (p^L \blacktriangle s^L)\left(\tfrac{3}{4}\right) = \tfrac{3}{4}\tfrac{1}{2} = \tfrac{3}{8},$$

$$(p \blacktriangle w)\left(\tfrac{3}{4}\right) = p\left(\tfrac{3}{4}\right)w(1) = \tfrac{3}{4}\left(1 - \tfrac{3}{4}\right)1 = \tfrac{3}{16},$$

Hence:

$$(p \blacktriangle s)^L\left(\tfrac{3}{4}\right) \wedge (p \blacktriangle w)\left(\tfrac{3}{4}\right) > (p \blacktriangle s)\left(\tfrac{3}{4}\right).$$

As a consequence, $p \blacktriangle w \not\preceq p \blacktriangle s$ and $\blacktriangle$ is neither t-norm nor t-conorm on $\mathbf{C}$ with respect to the order $\preceq$. Moreover, it is clear that $r \preceq q$ because the inequality $q^L \wedge r \leq q \leq r^L$ holds. However:

$$
\begin{aligned}
(p \blacktriangledown q)\left(\tfrac{3}{4}\right) &= \sup\left\{p(y)q(z) : y + z - yz = \tfrac{3}{4}\right\} \\
&= \sup\left\{\tfrac{3}{4}(1-y)(1-z) : \tfrac{1}{2} \leq z \leq y + z - yz = \tfrac{3}{4}\right\} \\
&= \sup\left\{\tfrac{3}{4}(1 - y - z + yz) : y + z - yz = \tfrac{3}{4}\right\} \\
&= \tfrac{3}{4}\left(1 - \tfrac{3}{4}\right) = \tfrac{3}{16},
\end{aligned}
$$

$$(p \blacktriangledown q)^L\left(\tfrac{3}{4}\right) = (p^L \blacktriangledown q^L)\left(\tfrac{3}{4}\right) = p^L(0)q^L\left(\tfrac{3}{4}\right) = \tfrac{3}{4}\tfrac{1}{2} = \tfrac{3}{8},$$

$$
\begin{aligned}
(p \blacktriangledown r)\left(\tfrac{3}{4}\right) &= \sup\left\{p(y)r(z) : y + z - yz = \tfrac{3}{4}\right\} \\
&\geq p\left(\tfrac{2}{3}\right)r\left(\tfrac{1}{4}\right) = \tfrac{5}{24},
\end{aligned}
$$

Once again, we can make use of Theorem 1 and the inequality:

$$(p \blacktriangledown q)^L\left(\tfrac{3}{4}\right) \wedge (p \blacktriangledown r)\left(\tfrac{3}{4}\right) > (p \blacktriangledown q)\left(\tfrac{3}{4}\right),$$

to show that $p \blacktriangledown r \not\preceq p \blacktriangledown q$. Therefore, $\blacktriangledown$ is neither t-norm nor t-conorm on $\mathbf{C}$ with respect to the order $\preceq$. □

Remark 4. *It should be noted that in the particular cases of $\blacktriangle$ and $\blacktriangledown$, with $\star = \wedge$, these operators are t-norm and t-conorm on **C**, respectively, respect to both partial orders $\sqsubseteq$ and $\preceq$. See [37] and [18] for more details.*

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*However, in Proposition 4, we have shown that, generally, neither ▲ nor ▼ are monotonically increasing on **C** with respect to any of the partial orders.*

In spite of the previous results, there are particular cases in which ▲ and ▼ are t-norm and t-conorm, respectively, on the particular subsets of **M** that we are studying. We will show one of these cases.

Proposition 5. *$\sqcap$ ($\sqcup$) is a t-norm (t-conorm) on $\mathbf{M}$, $\mathbf{C}$, $\mathbf{N}$, $\mathbf{K}$ and $\mathbf{K}_c^F$ respect to the partial order $\sqsubseteq$ ($\preceq$).*

Proof. In [37] it was established that $\sqcap$ and $\sqcup$ are commutative and associative. Moreover, due to Theorem 3 and Proposition 2, we know that these functions are closed on $\mathbf{M}$, $\mathbf{C}$, $\mathbf{N}$, $\mathbf{K}$ and $\mathbf{K}_c^F$. Again, by Theorem 3 we have that the neutral element for $\sqcap$ is $\bar{1}$ and for $\sqcup$ is $\bar{0}$.

Let us check the monotony of $\sqcap$ ($\sqcup$) respect to the partial order $\sqsubseteq$ ($\preceq$). Let $f, g, h \in \mathbf{M}$, with $g \sqsubseteq h$. Let us recall that $g \sqsubseteq h$ if and only if $g \sqcap h = g$. As $\sqcap$ is commutative, associative and idempotent:

$$(f \sqcap g) \sqcap (f \sqcap h) = (f \sqcap f) \sqcap (g \sqcap h) = (f \sqcap g).$$

Thus, $\sqcap$ is increasing in each argument on $(\mathbf{M}, \sqsubseteq)$. Similarly, we can prove that $\sqcup$ is increasing on $(\mathbf{M}, \preceq)$.

As a consequence, since $\mathbf{K}_c^F \subset \mathbf{K} \subset \mathbf{N} \subset \mathbf{M}$ and $\mathbf{C} \subset \mathbf{M}$, we have that $\sqcap$ ($\sqcup$) is a t-norm (t-conorm) on $\mathbf{M}$, $\mathbf{C}$, $\mathbf{N}$, $\mathbf{K}$ and $\mathbf{K}_c^F$, respect to the partial order $\sqsubseteq$ ($\preceq$). □

Remark 5. *Note that $\bar{0}$ ($\bar{1}$) is the absorbent element of $\sqcap$ ($\sqcup$) on $\mathbf{N}$ (see Theorem 3). Nevertheless, when working on $\mathbf{C}$ (or $\mathbf{M}$) the constant function $\mathbf{0}$ is the absorbent element for both operators.*

3.2. The operations $\perp$ and $\top$ on $\mathbf{M}$, $\mathbf{C}$, $\mathbf{N}$, $\mathbf{L}$, $\mathbf{K}$ and $\mathbf{K}_c^F$.

In this subsection, two new operations, $\perp$ and $\top$, will be introduced. It will be proven that $\perp$ is a t-norm respect to the partial order $\sqsubseteq$ and $\top$ is a t-conorm respect to $\preceq$ on $\mathbf{N}$, $\mathbf{K}$ and $\mathbf{K}_c^F$. In addition, we will show that $\perp$ is a t-norm and $\top$ is a t-conorm on the lattice $(\mathbf{L}, \sqsubseteq)$. Nevertheless, we will prove that there exist counterexamples where $\perp$ ($\top$) is not t-norm (t-conorm) on $\mathbf{C}$ (and consequently, on $\mathbf{M}$) since, in this case, $\perp$ is equivalent to ▲ and $\top$ is equivalent to ▼.

Definition 13. *Let $f, g \in \mathbf{M}$, and ▲, ▼ the operations given in Definition 12. We define the following operations:*

$$f \perp g = \begin{cases} f & \text{if } g = \bar{1}, \\ g & \text{if } f = \bar{1}, \\ (f^L \wedge f^R) \blacktriangle (g^L \wedge g^R) & \text{otherwise,} \end{cases}$$

$$f \top g = \begin{cases} f & \text{if } g = \bar{0}, \\ g & \text{if } f = \bar{0}, \\ (f^L \wedge f^R) \blacktriangledown (g^L \wedge g^R) & \text{otherwise.} \end{cases}$$

Remark 6.
- *In Definition 13 the minimum t-norm $\wedge$ is used. However, when we work on $\mathbf{N}$, all the results obtained are also fulfilled when we employ any other t-norm $\barwedge$ on $[0,1]$. This fact is easy to check. When $f \in \mathbf{N}$, then for all $x \in [0,1]$ either $f^L(x) = 1$ or $f^R(x) = 1$. Since all t-norms are equivalent when one of the arguments takes the value 1, then $f^L \barwedge f^R = f^L \wedge f^R$ and it does not matter which t-norm we use to define $\perp$ or $\top$.*
- *$\perp$ and $\top$ are equivalent to ▲ and ▼, respectively, on $\mathbf{C}$. If $f, g \in \mathbf{C}$, then $f = f^L \wedge f^R$ and $g = g^L \wedge g^R$ (see [37]). Moreover since $f \blacktriangle \bar{1} = f$ and $f \blacktriangledown \bar{0} = f$ for all $f \in \mathbf{M}$, we can state that $\perp \equiv$ ▲, $\top \equiv$ ▼ on $\mathbf{C}$.*

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Consequently, Proposition 4 provides counterexamples where $\perp$ and $\top$ are neither t-norm nor t-conorm with respect to either order $\sqsubseteq$ or $\preceq$ on $\mathbf{C}$, and therefore on $\mathbf{M}$.

- *Since $\mathbf{L} \subset \mathbf{C}$ and as a consequence of the previous point, $\perp$ and $\top$ are also equivalent to $\blacktriangle$ and $\blacktriangledown$, respectively, on $\mathbf{L}$. It was proven in [18] that $\blacktriangle$ ($\blacktriangledown$) is t-norm (t-conorm) on $(\mathbf{L}, \sqsubseteq, \bar{0}, \bar{1})$ so $\perp$ ($\top$) is also t-norm (t-conorm).*
- *If $f \notin \mathbf{C}$ or $g \notin \mathbf{C}$ we can find examples where $\perp \not\equiv \blacktriangle$ and $\top \not\equiv \blacktriangledown$. Let us consider the function:*

$$f(x) = \begin{cases} 1 & \text{if } x \in \{0,1\}, \\ 0 & \text{otherwise.} \end{cases}$$

We have that $f \blacktriangle f = f$, $f \blacktriangledown f = f$, but $f \perp f = \overline{[0,1]} \neq f$, and $f \top f = \overline{[0,1]} \neq f$. Consequently, $\perp$ and $\top$ are not equivalent in general to $\blacktriangle$ and $\blacktriangledown$ on $\mathbf{N}$, $\mathbf{K}$ or $\mathbf{K}_c^F$.

The following proposition establishes that $\perp$ ($\top$) satisfy the axioms 1, 2 and 3 (1,2 and 3´) of t-norm (t-conorm) on $\mathbf{M}$.

Proposition 6. *The operations $\perp$ and $\top$ are commutative and associative on $\mathbf{M}$. Moreover, $f \perp \bar{1} = f$ and $f \top \bar{0} = f$, for all $f \in \mathbf{M}$.*

Proof. The operations $\perp$ and $\top$ are commutative and associative since $\blacktriangle$ and $\blacktriangledown$ are commutative and associative (see Theorem 3). In addition, $f \perp \bar{1} = f$ and $f \top \bar{0} = f$ by definition. □

Remark 7. *The boundary conditions of $\perp$ and $\top$ in Definition 13, guarantee the fulfillment of the axioms 3 and 3′, respectively. In fact, if they had not been added, these axioms would not always have to be fulfilled.*

To prove this fact, let us suppose that we do not include the boundary conditions. If $f \notin \mathbf{C}$, we have that $\bar{1}$ would not be the neutral element of the operation $\perp$, since:

$$f \perp \bar{1} = (f^L \wedge f^R) \blacktriangle \bar{1} = (f^L \wedge f^R) \neq f.$$

Moreover, $\bar{0}$ would not be the neutral element of $\top$, since:

$$f \top \bar{0} = (f^L \wedge f^R) \blacktriangledown \bar{0} = (f^L \wedge f^R) \neq f.$$

In order to analyze if these new operations are closed on $\mathbf{N}$, $\mathbf{K}$ and $\mathbf{K}_c^F$, we previously present some properties.

Proposition 7. *i) If $f \in \mathbf{N}$, then $(f^L \wedge f^R) \in \mathbf{L}$.*
ii) $(f^L \wedge f^R)^L = f^L$ for all $f \in \mathbf{M}$.
iii) $(f^L \wedge f^R)^R = f^R$ for all $f \in \mathbf{M}$.

Proof. i) It is known (see [37]) that a function is convex if and only if it is the minimum of two functions, one of them increasing and the other one decreasing. Since f^L is increasing and f^R is decreasing, $(f^L \wedge f^R) \in \mathbf{C}$ for all $f \in \mathbf{M}$. Moreover, $f \leq (f^L \wedge f^R)$ and:

$$1 = \sup\{f\} \leq \sup\{f^L \wedge f^R\} \leq 1$$

since $f \in \mathbf{N}$. Therefore, $\sup\{f^L \wedge f^R\} = 1$ and $(f^L \wedge f^R) \in \mathbf{N}$. Consequently, $(f^L \wedge f^R) \in \mathbf{L}$.

ii) For all $f \in \mathbf{M}$, we have that $f \leq (f^L \wedge f^R) \leq f^L$. Hence:

$$f^L \leq (f^L \wedge f^R)^L \leq (f^L)^L = f^L$$

and the desired property is proven.

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iii) The proof is analogous to the previous one.
□

Proposition 8. *The following properties hold:*

i) $(f^L \wedge f^R)\blacktriangle(g^L \wedge g^R) \in \mathbf{C}$ *and* $(f^L \wedge f^R)\blacktriangledown(g^L \wedge g^R) \in \mathbf{C}$ *for all* $f, g \in \mathbf{M}$*,*
ii) $(f^L \wedge f^R)\blacktriangle(g^L \wedge g^R) \in \mathbf{L}$ *and* $(f^L \wedge f^R)\blacktriangledown(g^L \wedge g^R) \in \mathbf{L}$ *for all* $f, g \in \mathbf{N}$*.*

Consequently, operations $\perp$ *and* $\top$ *are closed on* $\mathbf{N}$*,* $\mathbf{C}$ *and* $\mathbf{L}$*.*

Proof. i) Given that both $f^L \wedge f^R$ and $g^L \wedge g^R$ are in $\mathbf{C}$ for all $f, g \in \mathbf{M}$, and that the operations $\blacktriangle$ y $\blacktriangledown$ are closed on $\mathbf{C}$ (see Theorem 3) the property is directly deduced.
ii) By Proposition 7 i), if $f, g \in \mathbf{N}$, then $(f^L \wedge f^R) \in \mathbf{L}$ and $(g^L \wedge g^R) \in \mathbf{L}$. Once again, by Theorem 3, we know that $\blacktriangle$ and $\blacktriangledown$ are closed operations on $\mathbf{L}$ so the result is verified.
The fact that $\perp$ and $\top$ are closed on $\mathbf{N}$, $\mathbf{C}$ and $\mathbf{L}$ is a direct consequence of the previous properties. □

In the following proposition we state that the defined operators are binary operations on $\mathbf{K}$ and $\mathbf{K}_c^F$.

Proposition 9. $\perp$ *and* $\top$ *are closed on* $\mathbf{K}$ *and on* $\mathbf{K}_c^F$*.*

Proof. Let us first note that if $f \in \mathbf{K}$ with $v_i = \inf\{Supp(f)\}$ and $v_s = \sup\{Supp(f)\}$, then:

$$f^L = \overline{/v_i, 1]}, \quad f^R = \overline{[0, v_s/},$$

Given $f \in \mathbf{K}$, for all $x \in Supp(f)$ we have $f(x) = 1$ and for all $x \notin Supp(f)$ we have $f(x) = 0$. Consequently, $/v_i, 1]$ will be closed if $v_i \in Supp(f)$, and half-open otherwise. Similarly, $[0, v_s/$ will be closed if $v_s \in Supp(f)$, and half-open otherwise. In particular, if $f \in \mathbf{K}_c^F$, we have that $v_i, v_s \in Supp(f)$ and then:

$$f^L = \overline{[v_i, 1]}, \quad f^R = \overline{[0, v_s]}.$$

That is, they have closed supports.

The next step is to see that if $f \in \mathbf{K}$, then:

$$(f^L \wedge f^R) = \overline{/v_i, v_s/} \in \mathbf{K}. \tag{3}$$

Moreover, if $v_i \in Supp(f)$ or $v_s \in Supp(f)$ the interval will be closed in such endpoint. Otherwise, it will be open. In particular, if $f \in \mathbf{K}_c^F$, we have will have:

$$(f^L \wedge f^R) = \overline{[v_i, v_s]} \in \mathbf{K}_c^F.$$

Since $f \in \mathbf{K}$, we know that $f \neq \mathbf{0}$, $f^L = \overline{/v_i, 1]} \neq \mathbf{0}$ and $f^R = \overline{[0, v_s/} \neq \mathbf{0}$. Therefore:

$$(f^L \wedge f^R) = \overline{/v_i, 1]} \wedge \overline{[0, v_s/} = \overline{/v_i, v_s/} \neq \mathbf{0}.$$

We can be sure that $\overline{/v_i, v_s/} \neq \mathbf{0}$ because, for each $f \neq \mathbf{0}$, there exists $x_0 \in [0, 1]$ such that $f(x_0) = 1$. Consequently:

$$1 = f(x_0) \leq (f^L \wedge f^R)(x_0) = \overline{/v_i, v_s/}(x_0) \leq 1,$$

and $\overline{/v_i, v_s/}(x_0) = 1$ so it is clear that $\overline{/v_i, v_s/} \neq \mathbf{0}$. This proves the assertion (3).

In addition, if $v_i \in Supp(f)$, clearly $f(v_i) = 1$ and $(f^L(v_i) \wedge f^R(v_i)) = 1$. Thus, $/v_i, v_s/$ is closed in v_i. Otherwise, if $v_i \notin Supp(f)$, then $f(v_i) = 0$, and $(f^L(v_i) \wedge f^R(v_i)) = 0 \wedge 1 = 0$. Therefore, $/v_i, v_s/$ is open in v_i. A similar analysis can be done in the other endpoint v_s. Hence, if $v_i \in Supp(f)$ or $v_s \in Supp(f)$, then $/v_i, v_s/$ will be closed in the corresponding endpoint.

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Now we can check that the operations $\perp$ and $\top$ are closed on $\mathbf{K}_c^F$ and on $\mathbf{K}$. Let $f, g \in \mathbf{K}_c^F$. If $f, g \neq \bar{1}$, then $(f^L \wedge f^R) = \overline{[v_i, v_s]}$ and $(g^L \wedge g^R) = \overline{[w_i, w_s]}$, as shown above. Moreover, by Theorem 3, we have:

$$\overline{[v_i, v_s]} \blacktriangle \overline{[w_i, w_s]} = \overline{[v_i \vartriangle w_i, v_s \vartriangle w_s]} \in \mathbf{K}_c^F.$$

Otherwise, if $f = \bar{1}$ or $g = \bar{1}$, the operation is trivially closed on $\mathbf{K}_c^F$ since $(f \perp \bar{1}) = f \in \mathbf{K}_c^F$ and $(\bar{1} \perp g) = g \in \mathbf{K}_c^F$.

We can prove that $\top$ is a binary operation on $\mathbf{K}_c^F$ in a similar way.

Finally, let $f, g \in \mathbf{K}$. If $f, g \neq \bar{1}$, then $(f^L \wedge f^R) = \overline{/v_i, v_s/} \neq \mathbf{0}$ and $(g^L \wedge g^R) = \overline{/w_i, w_s/} \neq \mathbf{0}$. Using Theorem 3 again, we can state that $\overline{/v_i, v_s/} \blacktriangle \overline{/w_i, w_s/} \in \mathbf{K}$. Otherwise, $(f \perp \bar{1}) = f \in \mathbf{K}$ and $(\bar{1} \perp g) = g \in \mathbf{K}$.

The fact that $\top$ is closed on $\mathbf{K}$ can be shown similarly. □

The following proposition presents the absorbent elements of $\perp$ and $\top$ in $\mathbf{N}$. Since these elements belong to the subsets $\mathbf{L}$, $\mathbf{K}$ and $\mathbf{K}_c^F$, they will also be absorbent elements of these sets.

Proposition 10. *If $f \in \mathbf{N}$, then $f \perp \bar{0} = \bar{0}$ and $f \top \bar{1} = \bar{1}$.*

Proof. According to Definition 13, we have $\bar{0} \perp \bar{1} = \bar{1} \perp \bar{0} = \bar{0}$ and $\bar{0} \top \bar{1} = \bar{1} \top \bar{0} = \bar{1}$. Moreover, by Theorem 3:

$$f \perp \bar{0} = (f^L \wedge f^R) \blacktriangle (\bar{0}^L \wedge \bar{0}^R) = (f^L \wedge f^R) \blacktriangle \bar{0} = \bar{0},$$
$$f \top \bar{1} = (f^L \wedge f^R) \blacktriangledown (\bar{1}^L \wedge \bar{1}^R) = (f^L \wedge f^R) \blacktriangledown \bar{1} = \bar{1}.$$

□

Our next goal is to study the monotony of $\perp$ and $\top$ on $\mathbf{N}$ and hence on $\mathbf{K}$ and $\mathbf{K}_c^F$. However, let us previously analyze the monotony of these operations without considering the boundary conditions.

Proposition 11. *Let $f, g, h \in \mathbf{N}$, with $g \sqsubseteq h$. Then:*

$$(f^L \wedge f^R) \blacktriangle (g^L \wedge g^R) \sqsubseteq (f^L \wedge f^R) \blacktriangle (h^L \wedge h^R).$$

Let $f, g, h \in \mathbf{N}$, with $g \preceq h$. Then:

$$(f^L \wedge f^R) \blacktriangledown (g^L \wedge g^R) \preceq (f^L \wedge f^R) \blacktriangledown (h^L \wedge h^R).$$

Proof. We will only prove the first inequality since the second one can be shown analogously. Let $f, g, h \in \mathbf{N}$, with $g \sqsubseteq h$. First note that, according to Theorem 3 and and Theorem 7 ii):

$$((f^L \wedge f^R) \blacktriangle (g^L \wedge g^R))^R = (f^L \wedge f^R)^R \blacktriangle (g^L \wedge g^R)^R = (f^R \blacktriangle g^R).$$

In a similar way:

$$((f^L \wedge f^R) \blacktriangle (h^L \wedge h^R))^R = (f^R \blacktriangle h^R),$$
$$((f^L \wedge f^R) \blacktriangle (g^L \wedge g^R))^L = (f^L \blacktriangle g^L),$$
$$((f^L \wedge f^R) \blacktriangle (h^L \wedge h^R))^L = (f^L \blacktriangle h^L).$$

Since $g \sqsubseteq h$, by Lemma 1 in [15] we have that $g^R \leq h^R$ and $h^L \leq g^L$. With this fact and Theorem 3:

$$(f^R \blacktriangle g^R) \leq (f^R \blacktriangle h^R)$$

and

$$(f^L \blacktriangle h^L) \leq (f^L \blacktriangle g^L).$$

Moreover, if $f, g, h \in \mathbf{N}$, then $(f^L \wedge f^R)\blacktriangle(g^L \wedge g^R) \in \mathbf{L}$, $(f^L \wedge f^R)\blacktriangle(h^L \wedge h^R) \in \mathbf{L}$ (see Proposition 8). Hence, from Theorem 2:

$$(f^L \wedge f^R)\blacktriangle(g^L \wedge g^R) \sqsubseteq (f^L \wedge f^R)\blacktriangle(h^L \wedge h^R)$$

and the monotony on $(\mathbf{N}, \sqsubseteq)$ is proved. □

Proposition 12. *The following properties hold:*

i) The operation $\perp$ is increasing in each argument on $(\mathbf{N}, \sqsubseteq)$, $(\mathbf{K}, \sqsubseteq)$ and $(\mathbf{K}_c^F, \sqsubseteq)$.
ii) The operation $\top$ is increasing in each argument, on $(\mathbf{N}, \preceq)$, $(\mathbf{K}, \preceq)$ and $(\mathbf{K}_c^F, \preceq)$.

Proof. Let us prove that the operation $\perp$ is increasing respect to the partial order $\sqsubseteq$. Let $f, g, h \in \mathbf{N}$, such that $g \sqsubseteq h$. We have to distinguish four cases:

1. If all functions f, g and h are different from $\bar{1}$, by Proposition 11 we have:

$$\begin{aligned}(f \perp g) &= (f^L \wedge f^R)\blacktriangle(g^L \wedge g^R) \\ &\sqsubseteq (f^L \wedge f^R)\blacktriangle(h^L \wedge h^R) = (f \perp h).\end{aligned}$$

2. If $g = \bar{1}$, since $g \sqsubseteq h$ and g is the maximum, then $h = \bar{1}$. Thus, $(f \perp g) = f = (f \perp h)$.
3. If $f = \bar{1}$, then $(f \perp g) = g \sqsubseteq h = (f \perp h)$.
4. Finally, let us see the case in which $f \neq \bar{1}$ and $\bar{1} \neq g \sqsubseteq h = \bar{1}$. Here, $(f \perp g) = (f^L \wedge f^R)\blacktriangle(g^L \wedge g^R)$ and $(f \perp h) = f$. As a consequence, it is sufficient to prove that:

$$(f^L \wedge f^R)\blacktriangle(g^L \wedge g^R) \sqsubseteq f.$$

As $g \sqsubseteq \bar{1}$, from Proposition 11 and Theorem 3:

$$\begin{aligned}(f^L \wedge f^R)\blacktriangle(g^L \wedge g^R) &\sqsubseteq (f^L \wedge f^R)\blacktriangle(\bar{1}^L \wedge \bar{1}^R) \\ &= (f^L \wedge f^R)\blacktriangle\bar{1} = (f^L \wedge f^R).\end{aligned}$$

Let us check that $(f^L \wedge f^R) \sqsubseteq f$. By Theorem 1 the inequality:

$$(f^L \wedge f^R)^R \wedge f \leq f^L \wedge f^R \leq f^R$$

must hold. According to Proposition 7, this inequality is equivalent to

$$f = f^R \wedge f \leq f^L \wedge f^R \leq f^R,$$

which trivially holds. Then, $\perp$ is increasing on each argument on $(\mathbf{N}, \sqsubseteq)$. Consequently, it is also increasing in each argument on $(\mathbf{K}, \sqsubseteq)$ and $(\mathbf{K}_c^F, \sqsubseteq)$.

The proof is similar when it comes to showing that $\top$ is increasing on $(\mathbf{N}, \preceq)$, $(\mathbf{K}, \preceq)$ and $(\mathbf{K}_c^F, \preceq)$. □

Corollary 2. *The following statements hold:*

i) $\perp$ is a t-norm on $(\mathbf{N}, \sqsubseteq)$, $(\mathbf{K}, \sqsubseteq)$ and $(\mathbf{K}_c^F, \sqsubseteq)$, with neutral element $\bar{1}$ and absorbent element $\bar{0}$.
ii) $\top$ is a t-conorm on $(\mathbf{N}, \preceq)$, $(\mathbf{K}, \preceq)$ and $(\mathbf{K}_c^F, \preceq)$, with neutral element $\bar{0}$ and absorbent element $\bar{1}$.

Proof. All the necessary properties for $\perp$ to be a t-norm and for $\top$ to be a t-conorm in the above mentioned posets have been proved in the previous results of this subsection. □

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The following example shows that $\perp$ does not satisfy the monotony respect to the partial order $\preceq$ on **M**, **N**, **K** and $\mathbf{K}_c^F$. In the same way, it shows that $\top$ does not satisfy the axiom of monotony respect to $\sqsubseteq$ on these sets.

Example 2. *Let us consider the following functions on* $\mathbf{K}_c^F$*:*

$$f(x) = \begin{cases} 1 & si\ x \in \{0,1\}, \\ 0 & otherwise, \end{cases} \quad and\ g = \overline{0.5},$$

and let us take $\blacktriangle = \sqcap$. *In this case,* $g \preceq \bar{1}$. *Since:*

$$(f \perp g) = (f^L \wedge f^R) \blacktriangle (g^R \wedge g^L) = \overline{[0,1]} \sqcap \overline{0.5} = \overline{[0,0.5]},$$

we have that $(f \perp g) = \overline{[0,0.5]} \npreceq f = (f \perp \bar{1})$ *(see Corollary 1).*

Furthermore, $\bar{0} \sqsubseteq g$. *If we fix* $\blacktriangledown = \sqcup$, *the next identity holds:*

$$(f \top g) = (f^L \wedge f^R) \blacktriangledown (g^R \wedge g^L) = \overline{[0,1]} \sqcup \overline{0.5} = \overline{[0.5,1]}$$

and, by Corollary 1:

$$(f \top 0) = f \not\sqsubseteq \overline{[0.5,1]} = (f \top g).$$

Therefore, neither $\perp$ *nor* $\top$ *are increasing in each argument respect to the corresponding orders.*

4. Concluding remarks

In this paper, we have introduced some new binary operations on **M**: $\perp$ and $\top$. We have analyzed when the considered operations satisfy the required axioms for them to be t-norms or t-conorms on **M**, **C**, **N**, **L**, **K** and $\mathbf{K}_c^F$ with respect to the two most commonly used partial orders on **M**. Let us present a list that contains the main results obtained:

1. The operator $\blacktriangle$, with $\triangle = T_P$ and $\star = \wedge$, is neither increasing with respect to $\sqsubseteq$ nor with respect to $\preceq$ on **M**, **N**, **K** and $\mathbf{K}_c^F$.
2. The operator $\blacktriangledown$, with $\triangledown = S_D$ and $\star = \wedge$ is neither increasing with respect to $\sqsubseteq$ nor with respect to $\preceq$ on **M**, **N**, **K** and $\mathbf{K}_c^F$.
3. The operator $\blacktriangle = \sqcap$ is t-norm on **M**, **C**, **N**, **K** and $\mathbf{K}_c^F$ with respect to $\sqsubseteq$.
4. The operator $\blacktriangledown = \sqcup$ is t-conorm on **M**, **C**, **N**, **K** and $\mathbf{K}_c^F$ with respect to $\preceq$.
5. In general, the operators $\blacktriangle$, $\blacktriangledown$, $\perp$ and $\top$ are neither t-norm nor t-conorm, on **C** with respect to either $\sqsubseteq$ and $\preceq$.
6. The operator $\perp$ is t-norm on **N**, **L**, **K** and $\mathbf{K}_c^F$ with respect to the order $\sqsubseteq$. Moreover, $\perp = \blacktriangle$ on **C**.
7. The operator $\top$ is t-conorm on **N**, **L**, **K** and $\mathbf{K}_c^F$ with respect to the order $\preceq$. Moreover, $\top = \blacktriangledown$ on **C**.

We are currently conducting a study of different new operations, which could be t-norms or t-conorms on some of the families that we are considering. We hope to present these results soon. Moreover, we will study other structures in type-2 fuzzy sets. In particular, aggregations, contradictions or similarities on **M**, **C**, **N**, **K** and $\mathbf{K}_c^F$.

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Abbreviations

The following abbreviations are used in this manuscript:

FS	Fuzzy set
IVFS	Interval-Valued Fuzzy Set
T2FS	Type-2 Fuzzy Set
IT2FS	Interval Type-2 Fuzzy Set
T2FLS	Type-2 Fuzzy Logic System

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