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Not peer-reviewed version

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Posted Date: 1 April 2025

doi: 10.20944/preprints202404.1009.v16

Keywords: foundations of physics



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Article

Constructing Physics From Measurements

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Abstract: We present a reformulation of fundamental physics from an enumeration of independent axioms into the solution of a single optimization problem. Any experiment begins with an initial state preparation, involves some physical operation, and ends with a final measurement. Working from this structure, we maximize the entropy of a final measurement relative to its initial preparation subject to a measurement constraint. Solving this optimization problem for the *natural constraint* –the most permissive measurement constraint compatible with said problem– identifies a unified physical theory which contains quantum mechanics, general relativity (acting on spacetime) and Yang-Mills (acting on internal spaces). Notably, mathematical consistency further restricts valid solutions to 3+1 dimensions only, automatically yielding spacetime. This reformulation reveals that the apparent complexity of modern physics, with its various forces, symmetries, and dimensional constraints, emerges as the solution to an optimization problem constructed over all experiments realizable within the constraint of nature.

Keywords: foundations of physics

1. Introduction

Statistical mechanics (SM), in the formulation developed by E.T. Jaynes [1,2], is founded on an entropy optimization principle. Specifically, the Boltzmann entropy is maximized under the constraint of a fixed average energy $\bar{E}$:

$$\bar{E} := \sum_i \rho_i E_i \tag{1}$$

The Lagrange multiplier equation defining the optimization problem is:

$$\mathcal{L} := -k_B \sum_i \rho_i(\beta) \ln \rho_i(\beta) + \lambda \left(1 - \sum_i \rho_i(\beta) \right) + \beta \left(\bar{E} - \sum_i \rho_i(\beta) E_i \right) \tag{2}$$

where λ and β are Lagrange multipliers enforcing the normalization and average energy constraints. Solving this optimization problem yields the Gibbs measure:

$$\rho_i(\beta) = \frac{1}{Z(\beta)} \exp(-\beta E_i), \tag{3}$$

where $Z(\beta) := \sum_i \exp(-\beta E_i)$ is the partition function.

For comparison, quantum mechanics (QM) is not formulated as the solution to an optimization problem, but rather consists of a collection of axioms[3,4]:

- | | |
|-----------------|--|
| QM Axiom 1 of 5 | State Space: Every physical system is associated with a complex Hilbert space, and its state is represented by a ray (an equivalence class of vectors differing by a non-zero scalar multiple) in this space. |
| QM Axiom 2 of 5 | Observables: Physical observables correspond to Hermitian (self-adjoint) operators acting on the Hilbert space. |

- QM Axiom 3 of 5 **Dynamics:** The time evolution of a quantum system is governed by the Schrödinger equation, where the Hamiltonian operator represents the system's total energy.
- QM Axiom 4 of 5 **Measurement:** Measuring an observable projects the system into an eigenstate of the corresponding operator, yielding one of its eigenvalues as the measurement result.
- QM Axiom 5 of 5 **Probability Interpretation:** The probability of obtaining a specific measurement outcome is given by the squared magnitude of the projection of the state vector onto the relevant eigenstate (Born rule).

Physical theories have traditionally been constructed in two distinct ways. Some, like QM, are defined through a set of mathematical axioms that are first postulated and then verified against experiments. Others, like SM, emerge as solutions to optimization problems with experimentally-verified constraints.

We propose to generalize the optimization methodology of E.T. Jaynes to encompass all of physics, aiming to derive a unified theory from a single optimization problem.

To that end, we introduce the following constraint:

Axiom 1 (Nature).

$$\overline{\mathbf{M}} := \sum_i \rho_i \mathbf{M}_i$$

where $\mathbf{M}_i$ are $n \times n$ matrices, and $\overline{\mathbf{M}}$ is their average.

This constraint, as it replaces the scalar E_i with the matrix $\mathbf{M}_i$, extends E.T. Jaynes' optimization method to encompass non-commutative observables and symmetry group generators required for fundamental physics.

We then construct an optimization problem:

Definition 1 (Physics). *Physics is the solution to:*

$$\underbrace{\mathcal{L}}_{\text{an optimization problem}} := \underbrace{-\sum_i \rho_i(\tau) \ln \frac{\rho_i(\tau)}{\rho_i(0)}}_{\text{on the entropy of a measurement relative to its preparation over all}} + \underbrace{\lambda \left(1 - \sum_i \rho_i(\tau)\right)}_{\text{predictive theories}} + \underbrace{\tau \operatorname{tr} \left(\overline{\mathbf{M}} - \sum_i \rho_i(\tau) \mathbf{M}_i\right)}_{\text{of nature}}$$

where λ and τ are Lagrange multipliers enforcing the normalization and natural constraints, respectively.

This definition constitutes our complete proposal for reformulating fundamental physics—no additional principles will be introduced. By replacing the Boltzmann entropy with the relative Shannon entropy, the optimization problem extends beyond thermodynamic variables to encompass any type of experiment. This generalization occurs because relative entropy captures the essence of any experiment: the relationship between a final measurement state and its initial preparation state.

Two key constraints shape our framework. The normalization constraint ensures we are working with a proper predictive theory, while the natural constraint spawns the domain of applicability of the theory. The crucial insight is that because our formulation maintains complete generality in the structure of experiments while optimizing over all possible predictive theories, the resulting solution holds true, by construction, for all realizable experiments within its domain.

This approach reduces our reliance on postulating axioms through trial and error, and simplifies the foundations of physics. Specifically, when we employ the *natural constraint*—the most permissive constraint for this problem (see Discussion for proof)—, the solution spawns its largest domain,

pointing towards a unified physics where fundamental theories emerge naturally—e.g. SM when $\mathbf{M} \cong \mathbb{R}$, QM when $\mathbf{M} \cong \mathfrak{u}(1)$, and general relativity (acting on spacetime) + Yang-Mills (acting on internal spaces) when $\mathbf{M} \cong \mathbb{R} \oplus \mathfrak{spin}(3,1)$. As we found, these three solutions are the only possible ones, as those entailed by other algebras encounter obstructions which violates the axioms of probability theory.

Theorem 1. *The general solution of the optimization problem is:*

$$\rho_i(\tau) = \frac{1}{\sum_j \det \exp(-\tau \mathbf{M}_j) \det \psi_j(0)} \det \exp(-\tau \mathbf{M}_i) \det \psi_i(0)$$

where $\det \psi_i(\tau) := \rho_i(\tau)$.

Proof. We solve the maximization problem by setting the derivative of the Lagrange multiplier equation with respect to $\rho_i(\tau)$ to zero:

$$\frac{\partial \mathcal{L}[\rho_1(\tau), \dots, \rho_i(\tau), \dots, \rho_n(\tau)]}{\partial \rho_i(\tau)} = -\ln \frac{\rho_i(\tau)}{\rho_i(0)} - 1 - \lambda - \tau \operatorname{tr} \mathbf{M}_i = 0. \quad (4)$$

$$\implies \ln \frac{\rho_i(\tau)}{\rho_i(0)} = -1 - \lambda - \tau \operatorname{tr} \mathbf{M}_i. \quad (5)$$

$$\implies \rho_i(\tau) = \rho_i(0) \exp(-1 - \lambda) \exp(-\tau \operatorname{tr} \mathbf{M}_i). \quad (6)$$

Normalizing the probabilities using $\sum_j \rho_j(\tau) = 1$, we find:

$$1 = \sum_j \rho_j(\tau) = \exp(-1 - \lambda) \sum_j \rho_j(0) \exp(-\tau \operatorname{tr} \mathbf{M}_j), \quad (7)$$

$$\implies \exp(1 + \lambda) = \sum_j \rho_j(0) \exp(-\tau \operatorname{tr} \mathbf{M}_j). \quad (8)$$

Substituting back, we obtain:

$$\rho_i(\tau) = \frac{1}{\sum_j \rho_j(0) \exp(-\tau \operatorname{tr} \mathbf{M}_j)} \exp(-\tau \operatorname{tr} \mathbf{M}_i) \rho_i(0). \quad (9)$$

Finally, using the identity $\det \exp(\mathbf{M}) \equiv \exp \operatorname{tr} \mathbf{M}$ for square matrices $\mathbf{M}$, we get:

$$\rho_i(\tau) = \frac{1}{\sum_j \det \exp(-\tau \mathbf{M}_j) \det \psi_j(0)} \det \exp(-\tau \mathbf{M}_i) \det \psi_i(0). \quad (10)$$

where $\det \psi_i(\tau) := \rho_i(\tau)$. $\square$

As we will see in the results section, this solution encapsulates three distinct special cases:

1. Statistical Mechanics:

To recover SM from Equation 10, we consider the case where the matrices $\mathbf{M}_i$ are 1×1 , i.e., real scalars. Specifically, we set:

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i, \quad \text{with } \mathbf{M}_i = E_i, \quad (11)$$

and take $\rho_i(0)$ to be a uniform distribution. Then, Equation 10 reduces to the Gibbs distribution:

$$\rho_i(\tau) = \frac{1}{Z} \exp(-\tau E_i), \quad (12)$$

where τ corresponds to the β of SM. This demonstrates that our solution generalizes SM, as it recovers it when $\mathbf{M}_i$ are scalars.

2. Quantum Mechanics:

By choosing $\mathbf{M}_i$ to generate the $U(1)$ group, we derive the axioms of QM from entropy maximization. Specifically, we set:

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i, \quad \text{with} \quad \mathbf{M}_i = \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}, \quad (13)$$

where E_i are energy levels. In the results section, we will detail how this choice leads to the Born rule in lieu of the Gibbs measure, and that the partition function is unitary invariant—the solution is shown to satisfy all five axioms of QM.

3. Fundamental Physics:

Extending our approach, we choose $\mathbf{M}_i$ to be 4×4 matrices representing the $\mathbb{R} \oplus \text{spinc}(3,1)$ algebra. Specifically, we consider multivectors of the form $\mathbf{u} = a + \mathbf{f} + \mathbf{b}$, where a is a scalar, where $\mathbf{f}$ is a bivector and $\mathbf{b}$ is a pseudoscalar of the 3+1D geometric algebra $GA(3,1)$. This constitute its even sub-algebra. The matrix representation of $\mathbf{M}_i$ is:

$$\mathbf{M}_i = \begin{bmatrix} a + f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & a + f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & a - f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & a - f_{02} \end{bmatrix}, \quad (14)$$

where $f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23}$, and b correspond to the generators of the $\text{Spinc}(3,1)$ group, which includes both Lorentz boosts/rotations and the four-volume orientation, and where a is the generator of the group $\mathbb{R}^+$. Solving the optimization problem with this choice leads to a relativistic quantum probability measure extending the Born rule from $\mathbb{C}$ to $\mathbb{R}^+ \times \text{Spinc}(3,1)$. The solution is shown to uniquely satisfy both general relativity (acting on spacetime) and Yang-Mills (acting on its internal spaces).

4. Dimensional Obstructions:

Definition 1 yields valid probability measures only in specific cases of Axiom 1. Beyond the instances of statistical mechanics and quantum mechanics, Axiom 1 produces a consistent solution only in 3+1 dimensions. In other dimensional configurations, various obstructions arises violating the axioms of probability theory. The following table summarizes the geometric cases and their obstructions:

<i>Dimensions</i>	<i>Optimal Predictive Theory of Nature</i>	
GA(0)	Statistical Mechanics	(15)
GA(0,1)	Quantum Mechanics	(16)
GA(1,0)	Obstructed (Negative probabilities)	(17)
GA(2,0)	Quantum Mechanics	(18)
GA(1,1)	Obstructed (Negative probabilities)	(19)
GA(0,2)	Obstructed (Non-real probabilities)	(20)
GA(3,0)	Obstructed (Non-real probabilities)	(21)
GA(2,1)	Obstructed (Non-real probabilities)	(22)
GA(1,2)	Obstructed (Non-real probabilities)	(23)
GA(0,3)	Obstructed (Non-real probabilities)	(24)
GA(4,0)	Obstructed (Non-real probabilities)	(25)
GA(3,1)	Gravity + Yang-Mills	(26)
GA(2,2)	Obstructed (Negative probabilities)	(27)
GA(1,3)	Obstructed (Non-real probabilities)	(28)
GA(0,4)	Obstructed (Non-real probabilities)	(29)
GA(5,0)	Obstructed (Non-real probabilities)	(30)
⋮	⋮	
GA(6,0)	Suspected Obstructed (No observables)	(31)
⋮	⋮	

where $GA(p, q)$ means the geometric algebra of $p + q$ dimensions, where p is the number of positive signature dimensions and q of negative signature dimensions. QM shows up twice because both $GA(0, 1)$ and the even-subalgebra of $GA(2, 0)$ are isomorphic to $\mathbb{C}$.

We will first investigate the unobstructed cases in Section 2.1, 2.2 and 2.3 and then demonstrate the obstructions in Section 2.4. These obstructions are desirable because they automatically limit the theory to 3+1D, thus providing a built-in mechanism for the observed dimensionality of our universe.

2. Results

2.1. $u(1)$ -constraint: Quantum Mechanics

In SM, the central observation is that energy measurements of a thermally equilibrated system tend to cluster around a fixed average value (Equation 1). In contrast, QM is characterized by the presence of interference effects in measurement outcomes. To capture these features, we introduce the following special case of Axiom 1:

Definition 2 ($u(1)$ constraint). *We reduce the generality of Axiom 1 to the generator of the $U(1)$ group. Specifically, we replace*

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i \quad \text{with} \quad \mathbf{M}_i = \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix} = IE_i$$

where E_i are scalar values (e.g., energy levels), ρ_i are the probabilities of outcomes, the matrices $\mathbf{M}_i$ generate the $U(1)$ group, and where $I := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $I^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cong -1$.

Let us also substitute $\tau := t/\hbar$ analogously to $\beta := 1/(k_B T)$. Then, the Lagrange multiplier equation (Definition 1) becomes:

$$\mathcal{L} = - \sum_i \rho_i(t) \ln \frac{\rho_i(t)}{\rho_i(0)} + \lambda \left(1 - \sum_i \rho_i(t) \right) + \frac{t}{\hbar} \text{tr} \left(I\bar{E} - \sum_i \rho_i(t) I E_i \right) \quad (32)$$

The general solution of the optimization problem (Theorem 1), with the above-mentioned replacements, reduces to:

$$\rho_i(t) = \frac{1}{\sum_j \det \exp(-ItE_j/\hbar) \det \psi_j(0)} \det \exp(-ItE_i/\hbar) \det \psi_i(0) \quad (33)$$

Though initially unfamiliar, this form effectively establishes a comprehensive formulation of QM, as we will demonstrate.

To align our results with conventional QM notation, we note the following equivalence with the square modulus:

$$\det \exp \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \equiv r^2 \det \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}, \text{ where } r := \exp a \quad (34)$$

$$\equiv r^2 (\cos^2(b) + \sin^2(b)) \quad (35)$$

$$\equiv |r \cos(b) + ri \sin(b)|^2 \quad (36)$$

$$\equiv |r \exp(ib)|^2 \quad (37)$$

Consequently, changing to the square modulus representation for both the numerator and to the denominator, consolidates a unitary invariant ensemble, an evolution operator, and an initial preparation into the Born rule:

$$\rho_i(t) = \underbrace{\frac{1}{\sum_j |\exp(-itE_j/\hbar) \psi_j(0)|^2}}_{\text{Unitary Invariant Ensemble}} \underbrace{|\exp(-itE_i/\hbar)|^2}_{\text{Evolution Operator}} \underbrace{|\psi_i(0)|^2}_{\text{Initial Preparation}} \quad (38)$$

where $|\psi_i(\tau)|^2 := \rho_i(\tau)$.

This equation describes what occurs at the instant of measurement—specifically, a part of the state space structure (e.g. the complex phase) is erased under measurement via action of the square modulus, thus yielding a classical probability measure.

By inspection, we can recover the definition of the state space, which is a wavefunction. To obtain it, we decompose the square modulus into a complex number and its conjugate. The state space is then identified as a vector within a complex n -dimensional Hilbert space. The partition function acts as the inner product. This relationship is articulated as follows:

$$Z(t) := \sum_j |\exp(-itE_j/\hbar) \psi_j(0)|^2 = \langle \psi | \psi \rangle \quad (39)$$

where

$$\begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix} := \begin{bmatrix} \exp(-itE_1/\hbar) & & \\ & \ddots & \\ & & \exp(-itE_n/\hbar) \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{bmatrix} \quad (40)$$

Let us now investigate how the axioms of QM are recovered from this result:

- The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1/Z = 1/\langle \psi | \psi \rangle$. This normalization links $|\psi\rangle$ to a unit vector in Hilbert space. Furthermore, as physical states

associate to the probability measure, and the probability is defined up to a phase, we conclude that physical states map to Rays within Hilbert space. This demonstrates [QM Axiom 1 of 5](#).

- An observable of the ensemble must satisfy:

$$\overline{O} := \sum_j O_j |\exp(-itE_j/\hbar)\psi_j(0)|^2 \quad (41)$$

Since $Z = \langle \psi | \psi \rangle$, then any self-adjoint operator satisfying the condition $\langle \mathbf{O} \psi | \phi \rangle = \langle \psi | \mathbf{O} \phi \rangle$ will equate the above equation, simply because $\langle \mathbf{O} \rangle := \langle \psi | \mathbf{O} | \psi \rangle$. This demonstrates [QM Axiom 2 of 5](#).

- Upon transforming Equation 40 out of its eigenbasis through unitary operations, we find that the energy E_i transforms in the manner of a Hamiltonian operator:

$$|\psi(t)\rangle = \exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle \quad (42)$$

The system's dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

$$\frac{\partial}{\partial t} |\psi(t)\rangle = \frac{\partial}{\partial t} (\exp(-it\mathbf{H}/\hbar) |\psi(0)\rangle) \quad (43)$$

$$= -i\mathbf{H}/\hbar \exp(-it\mathbf{H}/\hbar) |\psi(0)\rangle \quad (44)$$

$$= -i\mathbf{H}/\hbar |\psi(t)\rangle \quad (45)$$

$$\implies \mathbf{H} |\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle \quad (46)$$

which is the Schrödinger equation. This demonstrates [QM Axiom 3 of 5](#).

- From Equation 40 it follows that the possible microstates E_i of the system correspond to specific eigenvalues of $\mathbf{H}$. An observation can thus be conceptualized as sampling from ρ , with the measured state being the occupied microstate i . Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, which directly corresponds to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability measure is:

$$\rho_i(t) = \frac{1}{\langle \psi | \psi \rangle} |\psi_i(t)|^2. \quad (47)$$

In scenarios where the probability measure $\rho_i(\tau)$ is expressed in a basis other than its eigenbasis, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on a eigenstate:

$$P(\lambda_i) = |\langle \lambda_i | \psi \rangle|^2 \quad (48)$$

Here, $|\langle \lambda_i | \psi \rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. As this argument hold for any observable, this demonstrates [QM Axiom 4 of 5](#).

- Finally, since the probability measure (Equation 38) replicates the Born rule, [QM Axiom 5 of 5](#) is also demonstrated.

Revisiting QM with this perspective offers a coherent and unified narrative. Specifically, the U(1) generating constraint is sufficient to entail the foundations of QM through the principle of entropy maximization—in this formulation, QM Axioms 1, 2, 3, 4, and 5 are not fundamental, but the solution to an optimization problem.

2.2. $\mathbb{R} \oplus \mathfrak{spin}(2)$ -constraint: Euclidean QM in 2D

In this section, we investigate a model, isomorphic to QM, that lives in 2D—it provides a valuable starting point before addressing the more complex 3+1D case. Before we solve the optimization

problem, we will first introduce tools to express a bilinear form over $\mathbf{M}$ using the multivectors of $\text{GA}(2,0)$.

2.2.1. Bilinear Form

In general a multivector $\mathbf{u} := a + \mathbf{x} + \mathbf{b}$ of $\text{GA}(2,0)$, where a is a scalar, $\mathbf{x} := x\mathbf{e}_1 + y\mathbf{e}_2$ is a vector and $\mathbf{b} := b\mathbf{e}_1\mathbf{e}_2$ a pseudo-scalar, can be represented as a real 2×2 matrix via an isomorphism:

Definition 3 (Pauli Algebra Isomorphism). *The map $\varphi : \text{GA}(2,0) \rightarrow \text{Mat}(2, \mathbb{R})$ defined by:*

$$\varphi(1) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \varphi(\mathbf{e}_x) := \sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \varphi(\mathbf{e}_y) := \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (49)$$

extends linearly and multiplicatively to an isomorphism between $\text{GA}(2,0)$ and the algebra of real 2×2 matrices. In particular, the basis bivector maps to:

$$\varphi(\mathbf{e}_x\mathbf{e}_y) = \sigma_x\sigma_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (50)$$

Definition 4 (Matrix Representation). *For a multivector $\mathbf{u} := a + \mathbf{x} + \mathbf{b}$, its matrix representation under φ is:*

$$\varphi(\mathbf{u}) = a\varphi(1) + x\varphi(\mathbf{e}_x) + y\varphi(\mathbf{e}_y) + b\varphi(\mathbf{e}_x\mathbf{e}_y) = \begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix},$$

We now introduce the multivector conjugate, also known as the Clifford conjugate, which generalizes the concept of complex conjugation to multivectors.

Definition 5 (Multivector Conjugate—in $\text{GA}(2,0)$). *Let $\mathbf{u} := a + \mathbf{x} + \mathbf{b}$ be in $\text{GA}(2,0)$. Then its multivector conjugate is defined as:*

$$\mathbf{u}^\dagger := a - \mathbf{x} - \mathbf{b}$$

The determinant of the matrix representation of a multivector can be expressed as a multivector self-product:

Theorem 2 (Multivector Determinant—in $\text{GA}(2,0)$). *Let $\mathbf{u} := a + \mathbf{x} + \mathbf{b}$ be in $\text{GA}(2,0)$, then:*

$$\mathbf{u}^\dagger \mathbf{u} \equiv \det \varphi(\mathbf{u}) \quad (51)$$

Proof. Let $\mathbf{u} := a + \mathbf{x} + \mathbf{b}$ thus $\varphi(\mathbf{u}) = \begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix}$. Then:

$$1: \quad \mathbf{u}^\dagger \mathbf{u} = (a + \mathbf{x} + \mathbf{b})^\dagger (a + \mathbf{x} + \mathbf{b}) \quad (52)$$

$$= (a - \mathbf{x} - \mathbf{b})(a + \mathbf{x} + \mathbf{b}) \quad (53)$$

$$= a^2 + a\mathbf{x} + a\mathbf{b} - \mathbf{x}a - \mathbf{x}^2 - \mathbf{x}\mathbf{b} - \mathbf{b}a - \mathbf{b}\mathbf{x} - \mathbf{b}^2 \quad (54)$$

$$= a^2 - \mathbf{x}^2 + b^2 \quad (55)$$

$$= a^2 - x^2 - y^2 + b^2, \text{ since } \mathbf{x}^2 = x^2 + y^2 \quad (56)$$

$$2: \quad \det \varphi(\mathbf{u}) = \det \begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \quad (57)$$

$$= (a+x)(a-x) - (y-b)(y+b) \quad (58)$$

$$= a^2 - x^2 - y^2 + b^2 \quad (59)$$

□

Building upon the concept of the multivector conjugate, we introduce the multivector conjugate transpose, which serves as an extension of the Hermitian conjugate to the domain of multivectors.

Definition 6 (Multivector Conjugate Transpose). Let $|V\rangle\rangle \in (\text{GA}(2,0))^n$:

$$|V\rangle\rangle := \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{b}_n \end{bmatrix} \quad (60)$$

The multivector conjugate transpose of $|V\rangle\rangle$ is defined as first taking the transpose and then the element-wise multivector conjugate:

$$\langle\langle V| := \begin{bmatrix} a_1 - \mathbf{x}_1 - \mathbf{b}_1 & \dots & a_n - \mathbf{x}_n - \mathbf{b}_n \end{bmatrix} \quad (61)$$

Definition 7 (Bilinear Form). Let $|V\rangle\rangle$ and $|W\rangle\rangle$ be two vectors valued in $\text{GA}(2,0)$:

$$|V\rangle\rangle := \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{b}_n \end{bmatrix} \quad |W\rangle\rangle := \begin{bmatrix} a'_1 + \mathbf{x}'_1 + \mathbf{b}'_1 \\ \vdots \\ a'_n + \mathbf{x}'_n + \mathbf{b}'_n \end{bmatrix} \quad (62)$$

We introduce the following bilinear form:

$$\langle\langle V|W\rangle\rangle = (a_1 - \mathbf{x}_1 - \mathbf{b}_1)(a'_1 + \mathbf{x}'_1 + \mathbf{b}'_1) + \dots + (a_n - \mathbf{x}_n - \mathbf{b}_n)(a'_n + \mathbf{x}'_n + \mathbf{b}'_n) \quad (63)$$

Theorem 3 (Inner Product). Restricted to the even sub-algebra of $\text{GA}(2,0)$, the bilinear form is an inner product.

Proof.

$$\langle\langle V|W\rangle\rangle_{\mathbf{x} \rightarrow 0} = (a_1 - \mathbf{b}_1)(a_1 + \mathbf{b}_1) + \dots + (a_n - \mathbf{b}_n)(a_n + \mathbf{b}_n) \quad (64)$$

This is isomorphic to the inner product of a complex Hilbert space, with the identification $(\sigma_x \sigma_y)^2 \cong -1$ $\square$

2.2.2. 1+1D Obstruction

The reader may wonder why we are using 2D instead of the more physically relevant 1+1D for the lower dimensional example. As stated in the introduction the 1+1D is obstructed. Specifically, the 1+1D theory results in a split-complex quantum theory due to the bilinear form $(a - b\mathbf{e}_0 \wedge \mathbf{e}_1)(a + b\mathbf{e}_0 \wedge \mathbf{e}_1)$, which yields negative probabilities: $a^2 - b^2 \in \mathbb{R}$ for certain wavefunction states, in contrast to the non-negative probabilities $a^2 + b^2 \in \mathbb{R}^{\geq 0}$ obtained in the Euclidean 2D case. As such, neither 1+1D nor any of its sub-algebras satisfy all axioms of probability theory, hence it is obstructed. In contrast, the even sub-algebra of 2D is unobstructed simply because it is isomorphic to the complex numbers.

2.2.3. $\mathfrak{spin}(2)$ -constraint: $\cong$ Quantum Mechanics

Let us first investigate the $\mathfrak{spin}(2)$ -constraint, then we will investigate the more general $\mathbb{R} \oplus \mathfrak{spin}(2)$ -constraint. This constraint is recovered by posing $a \rightarrow 0$ and $\mathbf{x} \rightarrow 0$ then $\varphi(\mathbf{u})$ reduces as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b}|_{a \rightarrow 0, \mathbf{x} \rightarrow 0} = \mathbf{b} \implies \varphi(\mathbf{u}) = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \quad (65)$$

The fundamental Lagrange Multiplier Equation:

$$\mathcal{L} := - \sum_i \rho_i(\theta) \ln \frac{\rho_i(\theta)}{\rho_i(0)} + \lambda \left(1 - \sum_i \rho_i(\theta) \right) + \frac{1}{2} \theta \text{tr} \left(\bar{\mathbf{b}} - \sum_i \rho_i(\theta) \mathbf{b}_i \right) \quad (66)$$

where

1. λ and θ are the Lagrange multipliers
2. $\mathbf{b}_i$ are the multivectors of $\text{GA}(2,0)$, reduced by $a \rightarrow 0$ and $\mathbf{x} \rightarrow 0$
3. the factor $(1/2)$ is there to regularize the adjoint action on a vector $e^{-(1/2)\mathbf{b}_i} \mathbf{v} e^{(1/2)\mathbf{b}_i} = \mathbf{v}'$

It yields the following solution:

$$\rho_i = \frac{1}{\underbrace{\sum_j \det \exp \left(-\frac{1}{2} \theta \mathbf{b}_j \right) \det \psi_j(\theta)}_{\text{Spin(2) Invariant Ensemble}}} \underbrace{\det \exp \left(-\frac{1}{2} \theta \mathbf{b}_i \right)}_{\text{Evolution Operator}} \underbrace{\det \psi_i(0)}_{\text{Initial Preparation}} \quad (67)$$

As with the $\mathfrak{u}(1)$ -constraint case, this equation describes what must be erased from the state space structure at the instant of measurement, to yield a classical probability measure—in this case a $\text{Spin}(2)$ phase. The wavefunction with this structure is:

Definition 8 ($\text{Spin}(2)$ -valued Wavefunction).

$$\begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} := \begin{bmatrix} \exp \left(-\frac{1}{2} \theta \mathbf{b}_1 \right) & & \\ & \ddots & \\ & & \exp \left(-\frac{1}{2} \theta \mathbf{b}_n \right) \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{bmatrix}$$

The dynamics are described by a variant of the Schrödinger equation, which is derived by taking the derivative of the wavefunction with respect to the Lagrange multiplier, θ :

Definition 9 ($\text{Spin}(2)$ -valued Schrödinger Equation).

$$\frac{d}{d\theta} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} \mathbf{b}_1 & & \\ & \ddots & \\ & & \mathbf{b}_n \end{bmatrix} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix}$$

where

1. θ represents a global one-parameter evolution parameter akin to time
2. $\mathbf{b}_i$ is the generator of $\text{Spin}(2)$ transformations.

Since $\text{Spin}(2) \cong \text{U}(1)$, then it should come to no surprise that the theory resulting from the $\mathfrak{spin}(2)$ -constraint is of the same mathematical form as QM, obtained from the $\mathfrak{u}(1)$ -constraint.

2.2.4. $\mathbb{R} \oplus \mathfrak{spin}(2)$ -constraint: Euclidean QM in 2D

Now, we solve the optimization problem for the curvilinear case where $\frac{1}{2}(\mathbf{e}_\mu \mathbf{e}_\nu + \mathbf{e}_\nu \mathbf{e}_\mu) = g_{\mu\nu}$. We will also move to the continuum $\Sigma \rightarrow \int$. The optimization problem is described as follows:

$$\mathcal{L} := - \int_L \rho(\theta, r) \ln \frac{\rho(\theta, r)}{\rho(0, r)} \sqrt{h_\theta} dr + \lambda \left(1 - \int_L \rho(\theta, r) \sqrt{h_\theta} dr \right) + \frac{1}{2} \theta \left(\bar{\mathbf{M}} - \int_L \rho(\theta, r) \mathbf{M}(r) \sqrt{h_\theta} dr \right) \quad (68)$$

where

1. λ and θ are the Lagrange multipliers
2. $\sqrt{h_\theta}$ describes the metric of the 2D space foliated in slices of constant θ . This foliation results in radial lines from the origin to infinity with angle θ . A less sophisticated way of saying this is that we use polar coordinates in curved space (specifically, for flat space $\sqrt{h_\theta} = r$).
3. L is the integration length for the slices.
4. $\mathbf{M}$ is the lie algebra $\mathbb{R} \oplus \mathfrak{spin}(2)$.

We note that the lie algebra $\mathbb{R} \oplus \mathfrak{spin}(2)$ is isomorphic to the even subalgebra of $\text{GA}(2, 0)$:

$$\mathbb{R} \oplus \mathfrak{spin}(2) \cong a + \mathbf{b} \quad (69)$$

where a generates dilation (corresponding to translations on the θ -foliated slices) and $\mathbf{b}$ generates rotations (corresponding to circular motion on the θ -foliated slices). However, this isomorphism is only valid in the absence of curvature. In the presence of curvature, these generators are replaced with:

$$\sigma_1 \mathbf{e}^\mu \left(\partial_\mu + \frac{1}{2} \omega_\mu^{12} \mathbf{e}_1 \mathbf{e}_2 \right) \quad (70)$$

where $D_\mu := \partial_\mu + \frac{1}{2} \omega_\mu^{12} \mathbf{e}_1 \mathbf{e}_2$ is the covariant derivate. The expression includes ∂_μ , the partial derivative, and $\frac{1}{2} \omega_\mu^{12} \mathbf{e}_1 \mathbf{e}_2$, the $\text{Spin}(2)$ connection—acting on the wavefunction to transform its components in curved spacetime. The term $\mathbf{e}^\mu$ is required to contract the indices of ∂_μ and ω_μ , producing an odd multivector—and the term σ_1 converts it back to an even multivector. The term σ_1 also selects a preferred frame—the laboratory frame. Its role is similar to γ_0 in the Dirac Lagrangian.

Solving the optimization problem for this value of $\mathbf{M}$, yields:

$$\rho(\theta, r) = \frac{1}{\int_L \det \exp \left(-\frac{1}{2} \theta \sigma_1 \mathbf{e}^\mu D_\mu \right) \det \psi(0, r) \sqrt{h_\theta} dr} \det \exp \left(-\frac{1}{2} \theta \sigma_1 \mathbf{e}^\mu D_\mu \right) \det \psi(0, r) \quad (71)$$

Probabilities, normalized for a given initial slice, are conserved as the system evolves along θ .

As before, this equation identifies the structure of the state space that a measurement must erase to yield a classical probability measure. This structure yields the wavefunction, given as follows:

$$\psi(\theta, r) := \exp \left(-\frac{1}{2} \theta \sigma_1 \mathbf{e}^\mu D_\mu \right) \psi(0, r) \quad (72)$$

Then, taking the derivative with respect to θ yields the Schrödinger equation:

$$\frac{\partial}{\partial \theta} \psi(\theta, r) = -\frac{1}{2} \sigma_1 \mathbf{e}^\mu D_\mu \psi(\theta, r) \quad (73)$$

Revealing that the Hamiltonian, curiously expressed as a multivector, is $H := \frac{1}{2} \sigma_1 \mathbf{e}^\mu D_\mu$ (more on that in Theorem 4).

Definition 10 (David Hestenes' Formulation). *In 3+1D, the David Hestenes' formulation [5] of the wavefunction is $\psi = \sqrt{\rho} R e^{ib/2}$, where $R = e^{\mathbf{f}/2}$ is a Lorentz boost or rotation and where $e^{ib/2}$ is a phase. In 2D, as the algebra only admits a bivector, his formulation would reduce to $\psi = \sqrt{\rho} R$, where ρ is a probability density and R is a rotor—this is the form we have recovered.*

$$\sqrt{\rho} R \equiv \exp(a/2) \exp(\mathbf{f}/2)$$

The definition of the Dirac current applicable to our wavefunction follows the formulation of David Hestenes:

Definition 11 (Dirac Current). *The Dirac current for the 2D theory is defined as:*

$$J := \psi^\dagger \mathbf{e}_\mu \psi = \rho \underbrace{R^\dagger \mathbf{e}_\mu R}_{\text{SO}(2)} = \rho \mathbf{e}'_\mu$$

where $\mathbf{e}'_\mu$ is a $\text{SO}(2)$ -rotated basis vector.

We recall that in QM, the Hamiltonian relate to the Lagrangian as follows:

$$L = \text{tr} \left(\frac{\psi^\dagger H \psi}{\psi^\dagger \psi} \right) \quad (74)$$

With that, we can prove the following:

Theorem 4 (Dirac equation). *The equation of motion of the Schrödinger equation (Equation 73), is the Dirac equation. Since $H = \frac{1}{2} \sigma_1 \mathbf{e}^\mu D_\mu$ (Equation 73), we write:*

$$\frac{\delta}{\delta \psi^\dagger} \int_{\mathcal{M}} \text{tr} \left(\frac{\psi^\dagger \sigma_1 \mathbf{e}^\mu D_\mu \psi}{2 \psi^\dagger \psi} \right) \sqrt{g} d\theta dr = 0 \implies \underbrace{\mathbf{e}^\mu D_\mu \psi = 0}_{\text{Dirac Equation}} \quad (75)$$

Proof.

$$\frac{\delta}{\delta \psi^\dagger} \int_{\mathcal{M}} \text{tr} \left(\frac{\psi^\dagger \sigma_1 \mathbf{e}^\mu D_\mu \psi}{2 \psi^\dagger \psi} \right) \sqrt{g} d\theta dr = 0 \quad (76)$$

We can already recognize the numerator term $\psi^\dagger \sigma_1 \mathbf{e}^\mu D_\mu \psi$ as the Dirac Lagrangian (in 2D). Nonetheless, let us show explicitly:

$$\implies \frac{\delta}{\delta \psi^\dagger} \text{tr} \left(\frac{\psi^\dagger \sigma_1 \mathbf{e}^\mu D_\mu \psi}{2 \psi^\dagger \psi} \right) = 0 \quad (77)$$

$$\implies \frac{\sigma_1 \mathbf{e}^\mu D_\mu \psi 2 \psi^\dagger \psi}{2 \psi^\dagger \psi} \delta \psi^\dagger + \frac{\psi^\dagger \sigma_1 \mathbf{e}^\mu D_\mu \psi 2 \psi}{2 \psi^\dagger \psi} \delta \psi = 0 \quad (78)$$

$$\implies \sigma_1 \mathbf{e}^\mu D_\mu \psi \delta \psi^\dagger + \sigma_1 \mathbf{e}^\mu D_\mu \psi \delta \psi = 0 \quad (79)$$

$$\implies \mathbf{e}^\mu D_\mu \psi = 0 \quad (80)$$

which is the Dirac equation in 2D. $\square$

Observations: One might be tempted to simply extend $\mathbf{e}^\mu D_\mu \psi = 0$ from 2D to 3+1D by adding the basis vectors $\mathbf{e}_0$ and $\mathbf{e}_3$. This approach would produce the conventional Dirac equation in spacetime. However, when we solve the optimization problem directly in 3+1D, we discover that the Dirac equation represents only a subsolution of a more comprehensive probabilistic structure. This complete structure naturally incorporates both gravity (acting on spacetime) and Yang-Mills (acting on internal spaces). These results, presented in the following section, suggest that the conventional Dirac equation in 3+1D, in a sense, represents an extension of 2D quantum theory that captures only a part of the full probabilistic structure available in 3+1D spacetime.

2.3. $\mathbb{R} \oplus \mathfrak{spinc}(3,1)$ -constraint: Gravity + Yang-Mills

Extending the framework to relativistic quantum mechanics begins by considering a measurement constraint having the $\mathbb{R}^+ \times \text{Spinc}(3,1)$ symmetry. This allows for transformations that include dilations, boosts/rotations, and re-orientations (David Hestene describes "re-orientation" as representing the changing orientation of the spin plane due to Zitterbewegung).

Another necessary change regards the interpretation of ψ from a probability amplitude, to that of a field amplitude ϕ . As such, and consistently with usual quantum field theory (QFT) interpretation, the notion of charge conservation will replace that of probability conservation. The notation will be changed as follows:

$$\psi \rightarrow \phi \quad (81)$$

$$\sqrt{\rho}Re^{-ib/2} \rightarrow \sqrt{\chi}Re^{-ib/2} \quad (82)$$

$\mathbf{M}$ will represent the algebra of $\mathbb{R} \oplus \mathfrak{spinc}(3, 1)$:

$$\mathbf{M} = \begin{bmatrix} a + f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & a + f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & a - f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & a - f_{02} \end{bmatrix} \quad (83)$$

Using $\text{GA}(3, 1)$ notation, $\mathbf{M}$ can be equivalently represented as:

$$\mathbf{M} \cong a + \mathbf{f} + \mathbf{b} \quad (84)$$

where a is a scalar, $\mathbf{f}$ is a bivector and $\mathbf{b}$ is a pseudo-scalar.

As we did for the $\mathbb{R} \oplus \mathfrak{spin}(2)$ -constraint, we will develop the framework for the continuum case $\Sigma \rightarrow \int$ which can be obtained by solving the following Lagrange multiplier equation:

$$\mathcal{L} := - \int_V \chi(\zeta, x) \ln \frac{\chi(\zeta, x)}{\chi(0, x)} \sqrt{h_\zeta} d^3x + \frac{1}{2} \zeta \text{tr} \left(\overline{\mathbf{M}} - \int_V \chi(\zeta, x) \mathbf{M}(x) \sqrt{h_\zeta} d^3x \right) \quad (85)$$

where

1. ζ is the twisted-rapidity acting on $\mathbb{R} \oplus \mathfrak{spinc}(3, 1)$ to form a one-parameter group via the exponential map.
2. h_ζ is the determinant of the induced spatial metric on the hypersurface of constant twisted-rapidity ζ . Is is a general curvature version of Rindler's coordinates.
3. V represents the causally accessible region
4. the normalization constraint $\lambda(1 - \int_V \chi(x, \zeta) \sqrt{h_\zeta} d^3x)$ has been dropped, consistently with a conserved charge interpretation (which will come from the Lagrangian) replacing probability conservation (which comes from a constraint in the optimization problem).

The integration is performed over a foliation of spacetime by surfaces of constant ζ , where twisted-rapidity is measured relative to a specified reference frame. This ensures the solution exists only over events that are causally accessible to observers characterized by a specific rapidity, respecting both quantum principles and relativistic causal structure.

Using the same technique as Theorem 1, solving the optimization problem here yields:

$$\chi(\zeta, x) = \underbrace{\det \exp \left(-\frac{1}{2} \zeta \mathbf{M}(x) \right)}_{\substack{\text{Spinc}(3,1) \text{ Evolution} \\ \text{Operator}}} \underbrace{\det \phi(0, x)}_{\substack{\text{Initial} \\ \text{Preparation}}} \quad (86)$$

where $\det \phi(\zeta, x) := \chi(\zeta, x)$. The partition function is absent because we dropped the normalization constraint.

As before this equation describes the structure that is erased from the state space at the instant of measurement to yield a classical measure—in the present case a $\mathbb{R}^+ \times \text{Spinc}(3, 1)$ -valued phase is erased. In what follows, we will describe the wavefunction with this structure. But first, in the next section, we will express the determinant of 4×4 real matrices using the multivectors of $\text{GA}(3, 1)$.

2.3.1. The Multivector Determinant

As we did in the beginning of the 2D case, our goal here will be to express $\det \mathbf{M}$ as a multivector self-product. To achieve that, we begin by defining a general multivector in the geometric algebra $\text{GA}(3,1)$:

$$\mathbf{u} := a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b} \quad (87)$$

where a is a scalar, $\mathbf{x}$ a vector, $\mathbf{f}$ a bivector, $\mathbf{v}$ is pseudo-vector and $\mathbf{b}$ a pseudo-scalar. Explicitly,

$$\mathbf{u} := a \quad (88)$$

$$+ t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \quad (89)$$

$$+ f_{01}\gamma_0\gamma_1 + f_{02}\gamma_0\gamma_2 + f_{03}\gamma_0\gamma_3 + f_{12}\gamma_1\gamma_2 + f_{13}\gamma_1\gamma_3 + f_{23}\gamma_2\gamma_3 \quad (90)$$

$$+ p\gamma_1\gamma_2\gamma_3 + q\gamma_0\gamma_2\gamma_3 + v\gamma_0\gamma_1\gamma_3 + w\gamma_0\gamma_1\gamma_2 \quad (91)$$

$$+ b\gamma_0\gamma_1\gamma_2\gamma_3 \quad (92)$$

Definition 12 (Real-Majorana Algebra Isomorphism). *The map $\varphi : \text{GA}(3,1) \rightarrow \text{Mat}(4, \mathbb{R})$ defined by:*

$$\varphi(1) := \text{diag}(1, 1, 1, 1) \quad (93)$$

$$\varphi(\gamma_0) := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \varphi(\gamma_2) := \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (94)$$

$$\varphi(\gamma_0) := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \varphi(\gamma_2) := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (95)$$

$$\varphi(\gamma_\mu\gamma_\nu) := \varphi(\gamma_\mu)\varphi(\gamma_\nu) \quad (96)$$

$$\varphi(\gamma_\mu\gamma_\nu\gamma_\kappa) := \varphi(\gamma_\mu)\varphi(\gamma_\nu)\varphi(\gamma_\kappa) \quad (97)$$

$$\varphi(\gamma_0\gamma_1\gamma_2\gamma_3) := \varphi(\gamma_0)\varphi(\gamma_1)\varphi(\gamma_2)\varphi(\gamma_3) \quad (98)$$

extends linearly and multiplicatively to an isomorphism between $\text{GA}(3,1)$ and the algebra of real 4×4 matrices.

Definition 13 (Matrix Representation).

$$\varphi(\mathbf{u}) = \begin{bmatrix} a + f_{02} - q - z & b - f_{13} + w - x & -f_{01} + f_{12} - p + v & f_{03} + f_{23} + t + y \\ -b + f_{13} + w - x & a + f_{02} + q + z & f_{03} + f_{23} - t - y & f_{01} - f_{12} - p + v \\ -f_{01} - f_{12} + p + v & f_{03} - f_{23} + t - y & a - f_{02} + q - z & -b - f_{13} - w - x \\ f_{03} - f_{23} - t + y & f_{01} + f_{12} + p + v & b + f_{13} - w - x & a - f_{02} - q + z \end{bmatrix}$$

To manipulate and analyze multivectors in $\text{GA}(3,1)$, we introduce several important operations, such as the multivector conjugate, the pseudo-blade conjugate, and the multivector determinant.

Definition 14 (Multivector Conjugate—in $\text{GA}(3,1)$).

$$\mathbf{u}^\dagger := a - \mathbf{x} - \mathbf{f} + \mathbf{v} + \mathbf{b}$$

Definition 15 (Pseudo-Blade Conjugate—in $\text{GA}(3,1)$). *The pseudo-blade conjugate of $\mathbf{u}$ is*

$$\mathbf{u}^\dagger := a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b}$$

Lundholm[6] proposes a number the multivector norms, and shows that they are the *unique* forms which carries the properties of the determinants such as $N(\mathbf{uv}) = N(\mathbf{u})N(\mathbf{v})$ to the domain of multivectors:

Definition 16. The self-products associated with low-dimensional geometric algebras are:

$$\text{GA}(0,1) : \quad \mathbf{u}^* \mathbf{u} \quad (99)$$

$$\text{GA}(2,0) : \quad \mathbf{u}^\dagger \mathbf{u} \quad (100)$$

$$\text{GA}(3,0) : \quad (\mathbf{u}^\dagger \mathbf{u})^* \mathbf{u}^\dagger \mathbf{u} \quad (101)$$

$$\text{GA}(3,1) : \quad (\mathbf{u}^\dagger \mathbf{u})^\dagger \mathbf{u}^\dagger \mathbf{u} \quad (102)$$

$$\text{GA}(4,1) : \quad ((\mathbf{u}^\dagger \mathbf{u})^\dagger \mathbf{u}^\dagger \mathbf{u})^* ((\mathbf{u}^\dagger \mathbf{u})^\dagger \mathbf{u}^\dagger \mathbf{u}) \quad (103)$$

where $\mathbf{u}^*$ is a conjugate that reverses the sign of pseudo-scalar blade (i.e. the highest degree blade of the algebra).

We can now express the determinant of the matrix representation of a multivector via a self-product. This choice is unique:

Theorem 5 (The Multivector Determinant—in $\text{GA}(3,1)$).

$$(\mathbf{u}^\dagger \mathbf{u})^\dagger \mathbf{u}^\dagger \mathbf{u} \equiv \det \varphi(\mathbf{u})$$

Proof. Please find a computer assisted proof of this equality in Annex F. $\square$

As can be seen from this theorem, the relationship between determinants and multivector products becomes more sophisticated in 3+1D. Unlike the 2D case where the determinant could be expressed using a product of two terms, in $\text{GA}(3,1)$ the determinant requires two products involving four copies of the multivector. This is reflected in the structure $(\mathbf{u}^\dagger \mathbf{u})^\dagger \mathbf{u}^\dagger \mathbf{u}$, which cannot be reduced to a simpler self-product of two terms.

Theorem 6 (Positive-Definiteness over $\mathbb{R}^+ \times \text{Spin}(3,1)$). Let $\mathbf{u} = \exp(\frac{1}{2}(a + \mathbf{f} + \mathbf{b}))$ be a general invertible element of the even-subalgebra of $\text{GA}(3,1)$. As such, $\mathbf{u}$ is in $\mathbb{R}^+ \times \text{Spin}(3,1)$. Then the multivector determinant $(\mathbf{u}^\dagger \mathbf{u})^\dagger \mathbf{u}^\dagger \mathbf{u}$ is positive-definite.

Proof. Since scalars, bivectors and pseudoscalars commute, we have:

$$\exp(\frac{1}{2}(a + \mathbf{f} + \mathbf{b})) = e^{a/2} e^{\mathbf{f}/2} e^{\mathbf{b}/2} \quad (104)$$

Using this convenient form, the proof is as follows:

$$(\mathbf{u}^\dagger \mathbf{u})^\dagger \mathbf{u}^\dagger \mathbf{u} = e^{a/2} e^{-\mathbf{f}/2} e^{-\mathbf{b}/2} e^{a/2} e^{\mathbf{f}/2} e^{-\mathbf{b}/2} e^{a/2} e^{-\mathbf{f}/2} e^{\mathbf{b}/2} e^{a/2} e^{\mathbf{f}/2} e^{\mathbf{b}/2} \quad (105)$$

$$= e^{2a} \quad (106)$$

which is positive-definite—the exponential of a real number a is in $\mathbb{R}^+$. $\square$

2.3.2. The $\mathbb{R}^+ \times \text{Spin}^c(3,1)$ -valued Field

By inspection, the solution to the optimisation problem for the $\mathbb{R} \oplus \mathfrak{spin}(3,1)$ -constraint, identifies the following field:

Definition 17 ($\mathbb{R}^+ \times \text{Spin}^c(3,1)$ -valued Field).

$$\phi(\zeta) := \exp\left(\frac{1}{2}\zeta(a + \mathbf{f} + \mathbf{b})\right)\phi(0)$$

Theorem 7 (David Hestenes' Wavefunction). The $\mathbb{R}^+ \times \text{Spin}^c(3,1)$ -valued field is formulated using the same geometric structure as David Hestenes'[5] formulation of the wavefunction within $\text{GA}(3,1)$. Specifically, David Hestenes' wavefunction is a special case of our result, where the field magnitude sums to 1.

Proof.

$$\underbrace{e^{\frac{1}{2}(a(x)+\mathbf{f}(x)+\mathbf{b}(x))}}_{\text{ours}} \propto \underbrace{\sqrt{\rho(x)}R(x)e^{-ib(x)/2}}_{\text{Hestenes'}}$$

where $e^{\frac{1}{2}a(x)} \propto \sqrt{\rho(x)}$, $e^{\frac{1}{2}\mathbf{f}(x)} = R(x)$ and $e^{\frac{1}{2}\mathbf{b}(x)} = e^{-ib(x)/2}$. Here, $\rho(x)$ is a probability density (versus a field magnitude), $R(x)$ is a rotor and $e^{-ib(x)/2}$ describes the four-volume orientation. Adding the normalisation constraint to the optimisation problem forces the field magnitude to sum to 1, which recovers David Hestenes' wavefunction as a special case. $\square$

This field leads to a variant of the Schrödinger equation obtained by taking its derivative with respect to the Lagrange multiplier ζ :

Definition 18 ($\mathbb{R}^+ \times \text{Spin}^c(3,1)$ -valued Schrödinger equation).

$$\frac{d}{d\zeta}\phi(\zeta) = \frac{1}{2}(a + \mathbf{f} + \mathbf{b})\phi(\zeta) \quad (107)$$

This Schrödinger equation is able to act on the wavefunction to dilate its probability measure, rotate or boost its rotor and to jiggle its orientation in flat spacetime. In curved spacetime, it generalizes to:

$$\frac{d}{d\zeta}\phi(\zeta, x) = \frac{1}{2}\gamma_0 \mathbf{e}^\mu (\partial_\mu + \frac{1}{2}\omega_\mu^{ab}\gamma_{ab} + IV_\mu)\phi(\zeta, x) \quad (108)$$

where $D_\mu := \partial_\mu + \frac{1}{2}\omega_\mu^{ab}\gamma_{ab} + IV_\mu$ is the covariant derivative. The expression includes ∂_μ , the partial derivative, $\frac{1}{2}\omega_\mu^{ab}\gamma_{ab}$ the $\text{Spin}^c(3,1)$ connection and IV_μ a $U(1)$ connection acting on the four-volume orientation—all acting on the wavefunction to transform its components in curved spacetime. Likewise to the 2D case, $\mathbf{e}^\mu$ is used to contract with D_μ , leaving no free indices. But since it produces an odd-multivector in the process, the term γ_0 is added converting the result back into an even-multivector. It also picks a preferred frame—the laboratory frame. Its effect is similar to the presence of γ_0 in the Dirac Lagrangian.

2.3.3. Geometry

Definition 19 (Dirac Current). *Using a single-copy of the multivector determinant, the definition of the Dirac current is the same as Hestenes':*

$$J := \overbrace{\phi^\dagger \mathbf{e}_0 \phi}^{\text{one copy}} \quad (109)$$

$$= \chi R^\dagger e^{-ib/2} \mathbf{e}_0 e^{-ib/2} R \quad (110)$$

$$= \chi R^\dagger \mathbf{e}_0 e^{ib/2} e^{-ib/2} R \quad (111)$$

$$= \chi R^\dagger \mathbf{e}_0 R \quad (112)$$

$$= \chi \mathbf{e}'_0 \quad (113)$$

where $\mathbf{e}'_0$ is a $SO(3,1)$ rotated basis vector.

Theorem 8 (Metric Tensor). *Taking advantage of the multivector determinant formulation, we utilize both copies to obtain the metric tensor as a basis vectors measurement:*

$$\text{tr} \left(\frac{\overbrace{(\phi^\dagger \mathbf{e}_\mu \phi)^\dagger}^{\text{copy 1}} \overbrace{\phi^\dagger \mathbf{e}_\nu \phi}^{\text{copy 2}}}{\underbrace{(\phi^\dagger \phi)^\dagger \phi^\dagger \phi}_{\chi^2}} \right) = g_{\mu\nu}$$

Proof.

$$\text{tr} \left(\frac{(\phi^\dagger \mathbf{e}_\mu \phi)^\dagger \phi^\dagger \mathbf{e}_\nu \phi}{(\phi^\dagger \phi)^\dagger \phi^\dagger \phi} \right) = \text{tr} \left(R^\dagger e^{-ib/2} \mathbf{e}_\mu e^{-ib/2} R R^\dagger e^{-ib/2} \mathbf{e}_\nu e^{-ib/2} R \right) \quad (114)$$

$$= \text{tr} \left(R^\dagger \mathbf{e}_\mu e^{ib/2} e^{-ib/2} R R^\dagger \mathbf{e}_\nu e^{ib/2} e^{-ib/2} R \right) \quad (115)$$

$$= \text{tr} \left(R^\dagger e_\mu R R^\dagger e_\nu R \right) \quad (116)$$

$$= \text{tr} \left(\mathbf{e}'_\mu \mathbf{e}'_\nu \right) \quad (117)$$

$$= \text{tr} \left(\mathbf{e}'_\mu \cdot \mathbf{e}'_\nu + \mathbf{e}'_\nu \wedge \mathbf{e}'_\mu \right) \quad (118)$$

$$= \text{tr} \left(g_{\mu\nu} + \mathbf{e}'_\nu \wedge \mathbf{e}'_\mu \right) \quad (119)$$

$$= g_{\mu\nu} \quad (120)$$

□

2.3.4. Dynamics

In the $\mathbb{R} \oplus \mathfrak{spin}(2)$ -constraint section we utilized the correspondence between the Hamiltonian of the Schrodinger equation and the Lagrangian (Equation 74):

$$L = \text{tr} \left(\frac{\psi^\dagger H \psi}{\psi^\dagger \psi} \right) \quad (121)$$

This definition generalizes to the multivector determinant in 3+1D as follows:

$$L := \text{tr} \left(\frac{(\phi^\dagger H \phi)^\dagger \phi^\dagger H \phi}{(\phi^\dagger \phi)^\dagger \phi^\dagger \phi} \right) \quad (122)$$

which contains two copies.

Definition 20 (Kinetic Energy). *Applying the Hamiltonian $H = \frac{1}{2} \gamma_0 \mathbf{e}^\mu D_\mu$ to each bilinear copy, yields:*

$$T := \text{tr} \left(\frac{\overbrace{(\phi^\dagger \gamma_0 \mathbf{e}^\mu D_\mu \phi)^\dagger}^{\text{copy 1}} \overbrace{\phi^\dagger \gamma_0 \mathbf{e}^\nu D_\nu \phi}^{\text{copy 2}}}{4 \underbrace{(\phi^\dagger \phi)^\dagger \phi^\dagger \phi}_{\chi^2}} \right)$$

Theorem 9 (Dirac Equation). *Varying the action yields the Dirac equation as a sufficient (but not necessary) equation of motion:*

$$\delta \int_{\mathcal{M}} \text{tr} \left(\frac{(\phi^\dagger \gamma_0 \mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\dagger \gamma_0 \mathbf{e}^\nu D_\nu \phi}{4(\phi^\dagger \phi)^\dagger \phi^\dagger \phi} \right) \sqrt{-|g|} d^4x = 0 \implies \underbrace{\mathbf{e}^\mu D_\mu \phi = 0}_{\text{Dirac Equation}}$$

Proof.

$$\frac{\delta}{\delta\phi^{\dagger\dagger}} \text{tr} \left(\frac{(\phi^{\dagger}\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\dagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi}{4(\phi^{\dagger}\phi)^{\dagger}\phi^{\dagger}\phi} \right) = 0 \quad (123)$$

$$\Rightarrow \text{tr} \frac{\delta}{\delta\phi^{\dagger\dagger}} \left(\frac{(\phi^{\dagger}\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\dagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi}{4(\phi^{\dagger}\phi)^{\dagger}\phi^{\dagger}\phi} \right) = 0 \quad (124)$$

$$\Rightarrow \text{tr} \left(\frac{(\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\dagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi(\phi^{\dagger}\phi)^{\dagger}\phi^{\dagger}\phi}{((\phi^{\dagger}\phi)^{\dagger}\phi^{\dagger}\phi)^2} \delta\phi^{\dagger\dagger} + \frac{(\phi^{\dagger}\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\dagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi\phi^{\dagger}\phi^{\dagger}\phi}{((\phi^{\dagger}\phi)^{\dagger}\phi^{\dagger}\phi)^2} \delta\phi^{\dagger\dagger} \right) = 0 \quad (125)$$

$$\Rightarrow \text{tr} \left(\frac{(\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\dagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi}{(\phi^{\dagger}\phi)^{\dagger}\phi^{\dagger}\phi} \delta\phi^{\dagger\dagger} + \frac{(\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\dagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi}{(\phi^{\dagger}\phi)^{\dagger}\phi^{\dagger}\phi} \delta\phi^{\dagger\dagger} \right) = 0 \quad (126)$$

$$\Rightarrow (\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\dagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi = 0 \quad (127)$$

For the condition to be satisfied, it is sufficient but not necessary that $\mathbf{e}^{\mu}D_{\mu}\phi = 0$, which is the Dirac equation. A second condition $\phi^{\dagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi = 0$ also reduces to the Dirac equation because $\phi^{\dagger}$ is invertible by definition. $\square$

The multivector determinant formulation thus contains the solutions that satisfy the Dirac equation. However, broader solutions where the trace condition $\text{tr} \left((\phi^{\dagger}\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\dagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi \right) = 0$ holds without $\mathbf{e}^{\mu}D_{\mu}\phi = 0$, also exist. We will now investigate these broader solutions.

2.3.5. Gravity

Theorem 10 (Quantum Action). *Let us investigate a subspace of the field where $R = 1$ and $e^{-ib/2} = 1$, such that $\phi = \sqrt{\chi}$. Due to its non-linearity, the kinetic energy produces a quantum potential in addition to the usual kinetic energy term:*

$$\text{tr} \left(\frac{(\sqrt{\chi}\mathbf{e}^{\mu}D_{\mu}\chi)^{\dagger}\mathbf{e}^{\nu}D_{\nu}\sqrt{\chi}}{\chi^2} \right) = \underbrace{\frac{1}{2\chi^2}(\partial\chi)^2}_{\text{Quantum Kinetics}} - \underbrace{\left(\frac{1}{4\chi^2}(\partial\chi)^2 - \frac{\partial^2\chi}{2\chi} \right)}_{\text{Quantum Potential}} \quad (128)$$

The quantum potential herein described is the relativistic version of the quantum potential found in the Bohm-Broglie reformulation of QM, whereas the quantum kinetics can be understood as a scalar field kinetic term. When integrated, they define a quantity that we refer to as the quantum action:

$$S = \underbrace{\int \left(\frac{1}{4\chi^2}(\partial\chi)^2 + \frac{\partial^2\chi}{2\chi} \right) \sqrt{-|g|} d^4x}_{\text{Quantum Action}} \quad (129)$$

Proof.

$$\text{tr} \left(\left(\chi^{-2} \sqrt{\chi} \mathbf{e}^\mu D_\mu \chi \right)^\dagger \mathbf{e}^\nu D_\nu \sqrt{\chi} \right) \quad (130)$$

$$= -\text{tr} \left(\chi^{-2} \sqrt{\chi} (\mathbf{e}^\mu \partial_\mu \chi) \mathbf{e}^\nu \partial_\nu \sqrt{\chi} + \chi^{-2} \sqrt{\chi} \chi \mathbf{e}^\mu \partial_\mu \mathbf{e}^\nu \partial_\nu \sqrt{\chi} \right) \quad (131)$$

$$= -\text{tr} \left(\chi^{-2} 2^{-1} (\mathbf{e}^\mu \partial_\mu \chi) (\mathbf{e}^\nu \partial_\nu \chi) \right) + \text{tr} \left(\chi^{-1} \sqrt{\chi} 4^{-1} \chi^{-3/2} \mathbf{e}^\mu \partial_\mu \mathbf{e}^\nu \partial_\nu \chi \right) \\ - \text{tr} \left(\chi^{-1} \sqrt{\chi} 2^{-1} \chi^{-1/2} \mathbf{e}^\mu \partial_\mu \mathbf{e}^\nu \partial_\nu \chi \right) \quad (132)$$

$$= -\text{tr} \left(\frac{(\mathbf{e}^\mu \partial_\mu \chi)(\mathbf{e}^\nu \partial_\nu \chi)}{2\chi^2} - \frac{(\mathbf{e}^\mu \partial_\mu \chi)(\mathbf{e}^\nu \partial_\nu \chi)}{4\chi^2} + \frac{\mathbf{e}^\mu \partial_\mu \mathbf{e}^\nu \partial_\nu \chi}{2\chi} \right) \quad (133)$$

$$= \frac{1}{2\chi^2} (\partial\chi)^2 - \frac{1}{4\chi^2} (\partial\chi)^2 + \frac{\partial^2 \chi}{2\chi} \quad (134)$$

$$= \frac{1}{4\chi^2} (\partial\chi)^2 + \frac{\partial^2 \chi}{2\chi} \quad (135)$$

□

Theorem 11 (Equation of Motion). *Varying the quantum action:*

$$S = \int \left(\frac{1}{4\chi^2} (\partial\chi)^2 + \frac{\partial^2 \chi}{2\chi} \right) \sqrt{-|g|} d^4x \quad (136)$$

produces:

$$\partial^2 \chi = \chi \square \chi \quad (137)$$

as the equation of motion.

Proof.

$$\delta \left(\frac{1}{4\chi^2} (\partial\chi)^2 + \frac{\partial^2 \chi}{2\chi} \right) = 0 \quad (138)$$

$$\implies -\frac{(\partial\chi)^2}{2\chi^3} \delta\chi - \partial_\mu \left(\frac{\partial^\mu \chi}{2\chi^2} \right) \delta\chi + \frac{\partial^2(\delta\chi)}{2\chi} - \frac{\partial^2 \chi}{2\chi^2} \delta\chi = 0 \quad (139)$$

$$\implies -\frac{(\partial\chi)^2}{2\chi^3} \delta\chi + \frac{(\partial\chi)^2}{\chi^3} \delta\chi - \frac{\partial^2 \chi}{2\chi^2} \delta\chi + \frac{\partial^2(\delta\chi)}{2\chi} - \frac{\partial^2 \chi}{2\chi^2} \delta\chi = 0 \quad (140)$$

$$\implies \frac{(\partial\chi)^2}{2\chi^3} \delta\chi - \frac{\partial^2 \chi}{\chi^2} \delta\chi + \frac{\partial^2(\delta\chi)}{2\chi} = 0 \quad (141)$$

To proceed further, we are required to do integration by part for the last term:

$$\int \frac{\partial^2(\delta\chi)}{2\chi} d^4x = - \int \frac{(\partial\chi)\partial(\delta\chi)}{2\chi^2} d^4x \quad (142)$$

Then, a second integration by part, yields:

$$- \int \frac{(\partial\chi)\partial(\delta\chi)}{2\chi^2} d^4x = - \int \delta\chi \left(\frac{\partial^2 \chi}{2\chi^2} - \frac{(\partial\chi)^2}{\chi^3} \right) d^4x \quad (143)$$

$$\implies \frac{(\partial\chi)^2}{2\chi^3}\delta\chi - \frac{\partial^2\chi}{\chi^2}\delta\chi - \frac{\partial^2\chi}{2\chi^2}\delta\chi + \frac{(\partial\chi)^2}{\chi^3}\delta\chi = 0 \quad (144)$$

$$\implies \frac{3(\partial\chi)^2}{2\chi^3}\delta\chi - \frac{3\partial^2\chi}{2\chi^2}\delta\chi = 0 \quad (145)$$

$$\implies \partial^2\chi = \frac{(\partial\chi)^2}{\chi} \quad (146)$$

$$\implies \chi\Box\chi = (\partial\chi)^2 \quad (147)$$

□

To interpret this action and resulting equation of motion, let us now introduce the surprisal field and associated definitions.

Definition 21 (Surprisal Field). *We define a change of variable:*

$$\varphi := -\ln\chi$$

We call φ the surprisal field.

Definition 22 (Surprisal Equation of Motion). *We note that the change of variable $\varphi = -\ln\chi$, changes the equation of motion as follows:*

$$\chi\Box\chi = (\partial\chi)^2 \xrightarrow[\varphi = -\ln\chi]{} \Box\varphi = 0$$

which is the Klein-Gordon equation in curved spacetime, applied to the surprisal field.

Definition 23 (Surprisal Conservation). *The following current:*

$$\nabla_\mu(\partial^\mu\varphi) = 0$$

identifies the surprisal as the conserved charge of this action.

Definition 24 (Surprisal Expectation Value). *The surprisal expectation value is merely the entropy H of a region V of the manifold:*

$$\underbrace{\langle \ln\chi \rangle}_{\text{expectation value}} := - \underbrace{\int_V \chi(x) \underbrace{\ln\chi(x)}_{\text{observable}} \sqrt{h_\zeta} d^3x}_{\text{Definition of Entropy}}$$

Interpretation:

In information theory, the **surprisal** of an event x with probability density $\rho(x)$ is defined as $-\ln\rho(x)$, and the entropy $H = -\int \rho \ln\rho d^4x$ represents its expectation value. As the unit of surprisal is the bit, it represents the quantity of information associated to the event—and it is conserved by $\Box\varphi = 0$. In contrast, also in information theory, the units of entropy are the bits per symbol—this is not conserved.

In our framework, the field χ replaces ρ —it has most of its properties, but differs critically as follows:

- χ is **not a probability density**—it lacks a conserved current ($\nabla_\mu(\chi u^\mu) \neq 0$) and is not normalized—but it is positive-definite.

- Instead, χ is interpreted as an **information density**, encoding spacetime's local information content.

The **surprisal** is defined as $\varphi = -\ln \chi$, which in this theory satisfies the Klein-Gordon equation $\square \varphi = 0$. This ensures:

1. **Conservation:** The current $j^\mu = \partial^\mu \varphi$ is conserved ($\nabla_\mu j^\mu = 0$), making $Q = \int_V j^\mu \sqrt{h_\zeta} d^3x$ a conserved charge.
2. **Causal Propagation:** Surprisal propagates at light speed, enforcing that bits of information cannot spread superluminally—a core tenet of relativity.

Before we continue the interpretation of this theory, let us introduce a few more theorems.

Theorem 12 (Ricci Scalar). *Let us investigate another subspace of the field where $\sqrt{\chi} = 1$ and $e^{-ib/2} = 1$, such that $\phi = R$. Then the kinetic energy T reduces to the Ricci scalar $\mathcal{R}$.*

$$\text{tr}\left((\tilde{R}\mathbf{e}^\mu D_\mu R)^\dagger \tilde{R}\mathbf{e}^\nu D_\nu R\right) = \mathcal{R}$$

Proof.

$$\text{tr}\left((\tilde{R}\mathbf{e}^\mu D_\mu R)^\dagger \tilde{R}\mathbf{e}^\nu D_\nu R\right) \quad (148)$$

$$= -\text{tr}(\tilde{R}\mathbf{e}^\mu D_\mu \mathbf{e}^\nu D_\nu R) \quad \text{via } \tilde{R}R = 1 \quad (149)$$

$$= -\text{tr}(\tilde{R}D^2 R) \quad (150)$$

$$= -\text{tr}(R\tilde{R}D^2) \quad (151)$$

$$= \text{tr}(D^2) \quad (152)$$

$$= \mathcal{R} \quad \text{via Lichnerowicz-Weitzenböck identity} \quad (153)$$

which is the Ricci scalar. $\square$

Definition 25 (Gravity). *Let us now consider the full space of the wavefunction $\psi = \sqrt{\rho}Re^{-ib/2}$. We are automatically lead into a theory of gravity:*

$$S = \int_{\mathcal{M}} \text{tr}\left(\frac{(\phi^\dagger \mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\dagger \mathbf{e}^\nu D_\nu \phi}{(\phi^\dagger \phi)^\dagger \phi^\dagger \phi}\right) \sqrt{-|g|} d^4x \quad (154)$$

which expands, via Theorem 10 and 12, as follows:

$$S = \int_{\mathcal{M}} \left(\mathcal{R} + \text{cross-terms} + \frac{1}{4\chi^2} (\partial\chi)^2 + \frac{\partial^2 \chi}{2\chi} \right) \sqrt{-|g|} d^4x \quad (155)$$

We note the following equations of motion which must be simultaneously satisfied:

1. Varying with respect to $g_{\mu\nu}$ yields the EFE with the Einstein tensor from $\mathcal{R}$, and is sourced by the quantum action variation yielding the stress-energy tensor.
2. Varying with respect to χ gives equations of motion that define the flow of χ in spacetime.

Interpretation (cont'd):

Thus, while quantum mechanics relies on probabilistic amplitudes ψ , our formulation recasts general relativity as a **deterministic theory of information dynamics**, where spacetime geometry and surprisal flux are dual aspects of $\mathcal{R}$ and χ . The distribution of surprisal in spacetime dictates its geometric structure, which in turns dictates how it propagates. General relativity is to information, what quantum mechanics is to probability.

Revisiting General Relativity with this perspective shows that the natural constraint is sufficient to entail the theory through the principle of entropy maximization—in this formulation, the speed

of light as a limit on the propagation of the quantity of information (via the surprisal obeying the Klein-Gordon equation), and even the Einstein field equations are not fundamental, but the solution to an optimization problem on entropy.

2.3.6. Yang-Mills

In QFT, the standard method to identify a local gauge symmetry is to start with a global symmetry of the action or probability measure and then localize it by introducing gauge fields. For example, the $U(1)$ gauge symmetry arises naturally in electromagnetism as the group preserving the probability density (Born rule) under local phase transformations. However, the non-Abelian $SU(2)$ and $SU(3)$ gauge symmetries of the Standard Model are not derived from first principles in this way; their inclusion is empirically motivated by particle physics experiments.

Improvement via Multivector Determinant Formulation: Our framework demonstrates that Yang-Mills theories emerge naturally from constraints on the wavefunction's probability measure and Dirac current. Specifically:

1. **Probability Measure:** The quadratic form $(\phi^\dagger \phi)^\dagger \phi^\dagger \phi = \chi^2$ enforces rotor invariance $\phi \rightarrow R\phi$, restricting transformations to those satisfying $R^\dagger R = 1$, for some rotor R of a geometric algebra of n dimensions:

$$(\phi^\dagger R^\dagger R \phi)^\dagger \phi^\dagger R^\dagger R \phi = (\phi^\dagger \phi)^\dagger \phi^\dagger \phi \implies R^\dagger R = 1. \quad (156)$$

Solutions to $R^\dagger R = 1$ are rotor transformations generated by bivectors in the Clifford algebra. For a $2n$ -dimensional algebra, these generate $\text{Spin}(2n)$, whose subgroups include $SU(n)$.

2. **Dirac Current:** The spacetime current $\phi^\dagger \mathbf{e}_0 \phi = \mathbf{e}_0$ requires gauge generators to commute with $\mathbf{e}_0$, confining them to an internal space. This implies:

$$\phi^\dagger e^{-\theta^i f_i} \mathbf{e}_0 e^{\theta^i f_i} \phi = \phi^\dagger \mathbf{e}_0 \phi \implies [f_i, \mathbf{e}_0] = 0, \quad (157)$$

where f_i are bivector generators. Thus, f_i act only on **internal degrees of freedom**, orthogonal to spacetime.

3. **Spacetime:** The origin of the multivector determinant from STA, defines the resulting internal space against spacetime.

These constraints limit the allowable symmetry to groups generated by bivector exponentials (which are compact Lie groups), and acting on the internal spaces of spacetime. Since $SU(n) \subset \text{Spin}(2n)$, this framework inherently includes the Standard Model within its landscape but also generalizes to larger symmetries such as those found in condensed matter systems with emergent $SU(n)$ symmetries.

Wavefunction and Symmetry Structure:

The total wavefunction is a tensor product of spacetime (STA) and internal space components:

1. For $SU(n)$ Yang-Mills:

$$\phi_{\text{STA}} \otimes \phi_{\mathbb{C}^n}. \quad (158)$$

2. For the Standard Model $SU(3) \times SU(2) \times U(1)$:

$$\phi_{\text{STA}} \otimes \phi_{\mathbb{C}} \otimes \phi_{\mathbb{C}^2} \otimes \phi_{\mathbb{C}^3}. \quad (159)$$

Action:

Our previous gravitational action is reconstructed with a spectral function f :

$$S = \int_{\mathcal{M}} \text{tr} \left(f \left(\frac{1}{\Lambda^2} \frac{(\phi^\dagger \mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\dagger \mathbf{e}^\mu D_\mu \phi}{(\phi^\dagger \phi)^\dagger \phi^\dagger \phi} \right) \right) \sqrt{-|g|} d^4x. \quad (160)$$

A heat kernel expansion yields the invariants of the theory (more on that in a moment).

Covariant Derivative (Ex. Standard Model):

Taking the Standard Model as an example, the covariant derivative incorporates spacetime curvature (gravity) and gauge fields:

$$D_\mu := \begin{pmatrix} \partial_\mu + \frac{\omega_\mu^{ab}}{2} \gamma_{ab} + ig' Y B_\mu + ig \frac{\sigma^a}{2} W_\mu^a + ig_s \frac{\lambda^a}{2} G_\mu^a & \Phi \\ \Phi^\dagger & \partial_\mu + \frac{\omega_\mu^{ab}}{2} \gamma_{ab} + ig' Y B_\mu + ig_s \frac{\lambda^a}{2} G_\mu^a \end{pmatrix}, \quad (161)$$

where:

1. γ_{ab} : Generators of $\text{Spin}(3,1)$ (gravitational spin connection).
2. B_μ, W_μ^a, G_μ^a : $U(1)$, $SU(2)$, and $SU(3)$ gauge fields.
3. Φ : Higgs field ($SU(2)$ doublet).

It acts on the left/right split of the field.

Expanding f yield the field strength term $\text{tr}(f(D^2/\Lambda^2))$ which via the Heat kernel further yields the Standard Model + gravity (see A. H. Chamseddine and Alain Connes [?]] for method). The invariants recovered are:

1. Leading Terms:

- (a) Cosmological constant: $\propto \Lambda^4 \int \sqrt{-|g|} d^4x$.
- (b) Einstein-Hilbert term: $\propto \Lambda^2 \int R \sqrt{-|g|} d^4x$.

2. Yang-Mills and Higgs:

- (a) Gauge kinetic terms: $\propto \int \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \sqrt{-|g|} d^4x$.
- (b) Higgs kinetic and potential terms:

$$\propto \int \left(|D_\mu \Phi|^2 + \Lambda^2 |\Phi|^2 + \frac{1}{\Lambda^2} |\Phi|^4 \right) \sqrt{-|g|} d^4x. \quad (162)$$

3. Yukawa Couplings (from matter fields):

$$\propto \int y_{ij} \bar{\phi}_i \Phi \phi_j \sqrt{-|g|} d^4x. \quad (163)$$

Key Notes:

1. **Higher-Order Terms:** Higher order field strength terms appear but are suppressed by Λ^{-2} , making them negligible at low energies.
2. **Uniqueness:** The Standard Model is not uniquely selected by the optimization problem but resides within the landscape of allowed Yang-Mills theories.
3. **Experimental Consistency:** The framework resembles Connes' spectral action (see A. H. Chamseddine and Alain Connes [?]), recovering the Standard Model and general relativity while allowing for testable extensions (e.g., higher-curvature gravity).

This formulation unifies gauge symmetries and gravity within the multivector determinant structure.

2.3.7. Yang-Mills Axioms as Theorems

In Section 2.1, we demonstrated that all 5 axioms of quantum mechanics are derivable from the solution to the optimization problem in $GA(0, 1)$. Here, our aim is to do the same but for the axioms of Yang-Mills theory. First, let us list the axioms:

1. **Compact Gauge Group:** The symmetry group is a compact Lie group G .
2. **Local Gauge Invariance:** Fields transform under spacetime-dependent (local) group elements $T(x) \in G$.
3. **Gauge Connections:** Gauge fields A_μ are introduced as connections in the covariant derivative $D_\mu = \partial_\mu + A_\mu$.
4. **Field Strength:** The curvature $F_{\mu\nu} = [D_\mu, D_\nu]$ defines the dynamics.
5. **Yang-Mills Action:** The action depends on $F_{\mu\nu}$, e.g., $\int \text{tr}(F_{\mu\nu}F^{\mu\nu})$.

Now for the theorems.

Theorem 13 (Compact Gauge Group). *The allowed symmetries form a compact Lie group $G \subset \text{Spin}(2n)$.*

Proof. :

1. **Constraint:** $(\phi^\dagger\phi)^\dagger\phi^\dagger\phi = \chi^2$ implies invariance of arbitrary n-dimensional rotors: $R^\dagger R = 1$.
2. **Structure of Solutions:** Rotor transformations in finite-dimensional Clifford algebras are generated by bivectors. These generate $\text{Spin}(2n)$ and its subgroups, which are compact Lie groups.

Thus, the gauge group G is inherently compact and derived from the algebra structure. $\square$

Theorem 14 (Local Gauge Invariance). *The theory is invariant under spacetime-dependent $T(x) \in G$.*

Proof. Wavefunction Transformation: $\phi \rightarrow R(x)\phi$, where $R(x) = e^{\theta^i(x)f_i}$ (exponentials of spacetime-dependent bivectors).

Probability Measure: $(\phi^\dagger\phi)^\dagger\phi^\dagger\phi \rightarrow (\phi^\dagger R^\dagger R \phi)^\dagger\phi^\dagger R^\dagger R \phi = \chi^2$.

Dirac Current: $\phi^\dagger \mathbf{e}_0 \phi \rightarrow \phi^\dagger R^\dagger \mathbf{e}_0 R \phi = \phi^\dagger \mathbf{e}_0 \phi$, since $[f_i, \mathbf{e}_0] = 0$. $\square$

Theorem 15 (Gauge Connections). *The covariant derivative $D_\mu = \partial_\mu + A_\mu$ emerges to maintain invariance under local $R(x)$.*

Proof. :

1. **Minimal Coupling:** To preserve $D_\mu\phi \rightarrow R(x)D_\mu\phi$, the derivative must transform as $\partial_\mu \rightarrow \partial_\mu + A_\mu$, where $A_\mu = f_i A_\mu^i(x)$.
2. **Gauge Field Definition:** Let $\partial_\mu R(x) = A_\mu R(x)$, then: $D_\mu\phi = \partial_\mu\phi + A_\mu\phi \implies D_\mu(R\phi) = R D_\mu\phi$.
3. **Clifford Algebra Embedding:** The A_μ are bivector fields in $\mathcal{Cl}(2n)$, ensuring $A_\mu \in \mathfrak{g}$ (the Lie algebra of G).

$\square$

Theorem 16 (Field Strength). *The commutator $F_{\mu\nu} = [D_\mu, D_\nu]$ defines the field strength.*

Proof. :

1. **Kinetic Energy:** The kinetic energy expands to include the field strength tensor:

$$\frac{(\phi^\dagger \gamma_0 \mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\dagger \gamma_0 \mathbf{e}^\mu D_\mu \phi}{(\phi^\dagger \phi)^\dagger \phi^\dagger \phi} = \text{kinetic terms} + F_{\mu\nu} \quad (164)$$

where $F_{\mu\nu}$ is the field strength (Shown in Definition 25). $\square$

Theorem 17 (Yang-Mills Action). *The spectral action over the kinetic energy includes the kinetic term $\int \text{tr}(F_{\mu\nu}F^{\mu\nu})$.*

Proof. : Heat Kernel Expansion: As shown in Equation 160 (see A. H. Chamseddine and Alain Connes [?] for method), the field strength term of the spectral action $S = \text{tr}\left(f(D^2/\Lambda^2)\right)$ expands as:
$$S \sim \int \left(\cdots + F_{\mu\nu}^a F^{a\mu\nu} + \cdots \right) \sqrt{-|g|} \, d^4x. \quad \square$$

Revisiting Yang-Mills with this perspective shows that the natural constraint is sufficient to entail the theory through the principle of entropy maximization—in this formulation, Yang-Mills axioms 1, 2, 3, 4, and 5 are not fundamental, but the solution to this optimization problem.

2.4. Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to solve the entropy maximization problem for other dimensional configurations. We found that all geometric configurations except the previously explored cases are obstructed. By obstructed, we mean that the solution to the entropy maximization problem, ρ , does not satisfy all axioms of probability theory. These obstructions also holds for the less restrictive interpretation in 3+1D of χ as an information density, because this interpretation nonetheless requires positive-definiteness which is not satisfied in other dimensional configurations.

<i>Dimensions</i>	<i>Optimal Predictive Theory of Nature</i>	
GA(0)	Statistical Mechanics	(165)
GA(0,1)	Quantum Mechanics	(166)
GA(1,0)	Obstructed (Negative probabilities)	(167)
GA(2,0)	Quantum Mechanics	(168)
GA(1,1)	Obstructed (Negative probabilities)	(169)
GA(0,2)	Obstructed (Non-real probabilities)	(170)
GA(3,0)	Obstructed (Non-real probabilities)	(171)
GA(2,1)	Obstructed (Non-real probabilities)	(172)
GA(1,2)	Obstructed (Non-real probabilities)	(173)
GA(0,3)	Obstructed (Non-real probabilities)	(174)
GA(4,0)	Obstructed (Non-real probabilities)	(175)
GA(3,1)	Gravity + Yang-Mills	(176)
GA(2,2)	Obstructed (Negative probabilities)	(177)
GA(1,3)	Obstructed (Non-real probabilities)	(178)
GA(0,4)	Obstructed (Non-real probabilities)	(179)
GA(5,0)	Obstructed (Non-real probabilities)	(180)
⋮	⋮	
GA(6,0)	Suspected Obstructed (No observables)	(181)
⋮	⋮	

Let us now demonstrate the obstructions mentioned above.

Theorem 18 (Non-real probabilities). *The determinant of the matrix representation of the geometric algebras in this category is either complex-valued or quaternion-valued, making them unsuitable as a probability.*

Proof. These geometric algebras are classified as follows:

$$GA(0, 2) \cong \mathbb{H} \quad (182)$$

$$GA(3, 0) \cong M_2(\mathbb{C}) \quad (183)$$

$$GA(2, 1) \cong M_2^2(\mathbb{R}) \quad (184)$$

$$GA(1, 2) \cong M_2(\mathbb{C}) \quad (185)$$

$$GA(0, 3) \cong \mathbb{H}^2 \quad (186)$$

$$GA(4, 0) \cong M_2(\mathbb{H}) \quad (187)$$

$$GA(1, 3) \cong M_2(\mathbb{H}) \quad (188)$$

$$GA(0, 4) \cong M_2(\mathbb{H}) \quad (189)$$

$$GA(5, 0) \cong M_2^2(\mathbb{H}) \quad (190)$$

The determinant of these objects is valued in $\mathbb{C}$ or in $\mathbb{H}$, where $\mathbb{C}$ are the complex numbers, and where $\mathbb{H}$ are the quaternions. $\square$

Theorem 19 (Negative probabilities). *The even sub-algebra of these dimensional configurations allows for negative probabilities, making them unsuitable.*

Proof. This category contains three dimensional configurations:

$GA(1, 0)\psi = a + be_1$, then:

$$(a + be_1)^\dagger(a + be_1) = (a - be_1)(a + be_1) = a^2 - b^2e_1e_1 = a^2 - b^2 \quad (191)$$

which is valued in $\mathbb{R}$.

$GA(1, 1)\psi = a + be_0e_1$, then:

$$(a + be_0e_1)^\dagger(a + be_0e_1) = (a - be_0e_1)(a + be_0e_1) = a^2 - b^2e_0e_1e_0e_1 = a^2 - b^2 \quad (192)$$

which is valued in $\mathbb{R}$.

$GA(1, 2)\psi = a + be_0e_1e_2$, where $e_0^2 = -1, e_1^2 = -1, e_2^2 = 1, e_3^2 = 1$, then:

$$((a + \mathbf{b})^\dagger(a + \mathbf{b}))^\dagger(a + \mathbf{b})^\dagger(a + \mathbf{b}) \quad (193)$$

$$= (a^2 + 2a\mathbf{b} + \mathbf{b}^2)^\dagger(a^2 + 2a\mathbf{b} + \mathbf{b}^2) \quad (194)$$

We note that $\mathbf{b}^2 = b^2e_0e_1e_2e_0e_1e_2 = b^2$, therefore:

$$= (a^2 + b^2 - 2a\mathbf{b})(a^2 + b^2 + 2a\mathbf{b}) \quad (195)$$

$$= (a^2 + b^2)^2 - 4a^2\mathbf{b}^2 \quad (196)$$

$$= (a^2 + b^2)^2 - 4a^2b^2 \quad (197)$$

which is valued in $\mathbb{R}$.

In all of these cases the probability can be negative. $\square$

Conjecture 1 (No observables (6D)). *The multivector representation of the norm in 6D cannot satisfy any observables.*

Argument. In six dimensions and above, the self-product patterns found in Definition 16 collapse. The research by Acus et al.[7] in 6D geometric algebra concludes that the determinant, so far defined

through a self-products of the multivector, fails to extend into 6D. The crux of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \quad (198)$$

This equation is not a multivector self-product but a linear sum of two multivector self-products[7].

The full expression is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$a_0^4 - 2a_0^2 a_{47}^2 + b_2 a_0^2 a_{47}^2 p_{412} p_{422} + \langle 72 \text{ monomials} \rangle = 0 \quad (199)$$

$$b_1 a_0^3 a_{52} + 2b_2 a_0 a_{47}^2 a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials} \rangle = 0 \quad (200)$$

$$\langle 74 \text{ monomials} \rangle = 0 \quad (201)$$

$$\langle 74 \text{ monomials} \rangle = 0 \quad (202)$$

From Equation 198, it is possible to see that no observable $\mathbf{O}$ can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

$$b_1 \mathbf{O} B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 \mathbf{O} B g_5(g_4(B) g_3(g_2(B) g_1(B))) \quad (203)$$

Any equality of the above type between $b_1 \mathbf{O}$ and $b_2 \mathbf{O}$ is frustrated by the factors b_1 and b_2 , forcing $\mathbf{O} = 1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it is questionable that a satisfactory theory (with non-trivial observables) be constructible in 6D, using our method. $\square$

This conjecture proposes that the multivector representation of the determinant in 6D does not allow for the construction of non-trivial observables, which is a crucial requirement for a relevant quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

Conjecture 2 (No observables (above 6D)). *The norms beyond 6D are progressively more complex than the 6D case, which is already obstructed.*

These theorems and conjectures provide additional insights into the unique role of the unobstructed 3+1D signature in our proposal.

It is also interesting that our proposal is able to rule out $\text{GA}(1,3)$ even if in relativity, the signature of the metric $(+, -, -, -)$ versus $(-, -, -, +)$ does not influence the physics. However, in geometric algebra, $\text{GA}(1,3)$ represents 1 space dimension and 3 time dimensions. Therefore, it is not the signature itself that is ruled out but rather the specific arrangement of 3 time and 1 space dimensions, as this configuration yields quaternion-valued "probabilities" (i.e. $\text{GA}(1,3) \cong \mathbb{M}_2(\mathbb{H})$ and $\det \mathbb{M}_2(\mathbb{H}) \in \mathbb{H}$).

3. Discussion

When asked to define what a physical theory is, an informal answer may be that it is a measurements-constrained mathematical framework that applies to all possible experiments realizable within a domain, with nature as a whole being the most general domain. While physicists have expressed these theories through sets of axioms, we propose a more direct approach—mathematically realizing this fundamental definition itself. This definition is realized as an optimization problem (Definition 1) that can be solved directly (Theorem 1). The solution to this optimization problem yields

precisely those structures that realize the physical theory over said domain. Succinctly, physics is the solution to:

$$\underbrace{\mathcal{L}}_{\text{an optimization problem}} := \underbrace{-\sum_i \rho_i(\tau) \ln \frac{\rho_i(\tau)}{\rho_i(0)}}_{\text{on the entropy of a measurement relative to its preparation over all}} + \underbrace{\lambda \left(1 - \sum_i \rho_i(\tau)\right)}_{\text{predictive theories}} + \underbrace{\tau \operatorname{tr} \left(\bar{\mathbf{M}} - \sum_i \rho_i(\tau) \mathbf{M}_i\right)}_{\text{of nature}} \quad (204)$$

The relative Shannon entropy represents the basic structure of any experiment, quantifying the informational difference between its initial preparation and its final measurement.

The natural constraint is chosen to be the most general structure that admits a solution to this optimization problem. This generality follows from key mathematical requirements. The constraint must involve quantities that form an algebra, as the solution requires taking exponentials:

$$\exp X = 1 + X + \frac{1}{2} X^2 + \dots \quad (205)$$

which involves addition, powers, and scalar multiplication of X . The use of the trace operation further necessitates that X must be represented by square $n \times n$ matrices. Thus Axiom 1 involves $n \times n$ matrices:

$$\bar{\mathbf{M}} := \sum_i \rho_i \mathbf{M}_i \quad (206)$$

The trace operation is utilized because the constraint must be converted back to a scalar for use in the Lagrange multiplier equation; while any function that maps an algebra to a scalar would achieve that, picking the trace recovers QM in the $\text{GA}(0,1) \cong \mathbb{C}$ case and SM in the $\text{GA}(0) \cong \mathbb{R}$ case.

These mathematical requirements demonstrate that the natural constraint, as it admits the minimal mathematical structure required to solve an arbitrary entropy maximization problem, can be understood as the most general extension of the statistical mechanics average energy constraint which contains QM and SM (as induced by the trace) as specific solutions.

Thus, having established both the mathematical structure and its generality, we can understand how this minimal ontology operates. Since our formulation keeps the structure of experiments completely general, our optimization considers all possible predictive theories for that structure, and the constraint is the most general constraint possible for that structure, the resulting optimal physical theory applies, by construction, to all realizable experiments within its domain.

This ontology is both operational, being grounded in the basic structure of experiments rather than abstract entities, and constructive, showing how physical laws emerge from optimization over all possible predictive theories subject to the natural constraint. Physics is encapsulated not as a pre-defined collection of fundamental axioms but as the optimal solution to a well-defined optimization problem over all experiments realizable within the domain. This represents a significant philosophical shift from traditional physical ontologies where laws are typically taken as primitive.

The next step in our derivation is to represent the determinant of the $n \times n$ matrices through a self-product of multivectors involving various conjugate structures. By examining the various dimensional configurations of geometric algebras, we find that $\text{GA}(3,1)$, representing 4×4 real matrices, admits a sub-algebra whose determinant is positive-definite for its invertible members. All other dimensional configurations fail to admit such a positive-definite structure, with two exceptions: statistical mechanics (found in $\text{GA}(0)$) and quantum mechanics (found in $\text{GA}(0,1)$ and in a sub-algebra of $\text{GA}(2,0)$).

The solution reveals that the 3+1D case harbours a new type of field amplitude structure analogous to complex amplitudes, one that exhibits the characteristic elements of a quantum mechanical theory. Instead of complex-valued amplitudes, we have amplitudes valued in the invertible subset of the even sub-algebra of $\text{GA}(3,1)$. When normalized, this amplitude is identical to David Hestenes' wavefunction,

but comes with an extended Born rule represented by the determinant, and rather than a complex Hilbert space, it lives in a "double-product structure". This double-product structure automatically incorporates gravity via the $\text{Spin}(3,1)$ connection and local gauge theories as Yang-Mills theories. The square of the Dirac operator, automatically generated by the Lagrangian, then generates the invariants of gravity and of the Yang-Mills theory via a heat kernel expansion, along with the matter fields quantifying the system's information via surprisal and limiting its propagation speed.

Interpretation: At the foundation of any experiment lie two empirically irreducible elements: an **initial preparation** described by a classical probability distribution p_i , and a **final measurement** described by a classical probability distribution ρ_i . Between these endpoints, traditional physics posits dynamical laws governing the evolution of intermediate states (e.g., wavefunction). Our framework inverts this ontology: quantum mechanics, spacetime geometry, and gauge symmetries are not fundamental entities but *emergent tools* that optimally interpolate between p_i and ρ_i under the constraint of nature (Axiom 1).

Dissolving the Measurement Problem: Traditional interpretations reify intermediate quantum states, demanding ad hoc "collapse" rules to reconcile unitary evolution with definite outcomes. By contrast, our framework eliminates ontological commitment to the wavefunction entirely. The Schrödinger equation and Born rule describe not physical dynamics but *inferential relationships* between p_i and ρ_i . Measurement is not a physical process but a consistency condition: the optimized solution must align initial and final classical distributions. This negates the need for collapse, as intermediate states exist only as interpolational devices.

4. Conclusion

E.T. Jaynes fundamentally reoriented statistical mechanics by recasting it as a problem of inference rather than mechanics. His approach revealed that the equations of thermodynamics are not arbitrary physical laws but necessary consequences of maximizing entropy subject to measured constraints. This work extends Jaynes' inferential paradigm to address a more fundamental question: what is a physical theory itself?

A physical theory, at its essence, is a measurements-constrained mathematical framework that applies to all possible experiments realizable within a domain. While this definition is informal, our contribution lies in mathematizing this concept directly. By formulating it as an optimization problem—maximizing the relative entropy of measurement outcomes subject to the natural constraint—we transform an abstract definition into a precise, solvable mathematical structure.

This approach represents a profound methodological shift. Rather than constructing physical theories through trial and error enumerations of axioms, we derive them as necessary solutions to a well-defined optimization problem. Physics thus emerges not as a collection of independently discovered laws but as the unique optimal bridge between experimental preparation and measurement under the constraint of nature.

The power of this formulation lies in its generality: by varying only the algebraic structure of the constraint, we recover established physical theories as special cases of the same optimization principle. Jaynes showed that statistical inference with minimal assumptions yields thermodynamics; we suggest that this same principle, properly generalized, has the potential to yield the very foundation of all physics.

Statements and Declarations

- **Funding:** This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.
- **Competing Interests:** The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.
- **Data Availability Statement:** No datasets were generated or analyzed during the current study.

- During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.

Appendix E SM

Here, we solve the Lagrange multiplier equation of SM.

$$\mathcal{L} := \underbrace{-k_B \sum_i \rho_i \ln \rho_i}_{\text{Boltzmann Entropy}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\beta \left(\bar{E} - \sum_i \rho_i E_i\right)}_{\text{Average Energy Constraint}} \quad (\text{A207})$$

We solve the maximization problem as follows:

$$0 = \frac{\partial \mathcal{L}(\rho_1, \dots, \rho_i, \dots, \rho_n)}{\partial \rho_i} \quad (\text{A208})$$

$$= -\ln \rho_i - 1 - \lambda - \beta E_i \quad (\text{A209})$$

$$= \ln \rho_i + 1 + \lambda + \beta E_i \quad (\text{A210})$$

$$\implies \ln \rho_i = -1 - \lambda - \beta E_i \quad (\text{A211})$$

$$\implies \rho_i = \exp(-1 - \lambda) \exp(-\beta E_i) \quad (\text{A212})$$

$$= \frac{1}{Z(\tau)} \exp(-\beta E_i) \quad (\text{A213})$$

The partition function, is obtained as follows:

$$1 = \sum_j \exp(-1 - \lambda) \exp(-\beta E_j) \quad (\text{A214})$$

$$\implies (\exp(-1 - \lambda))^{-1} = \sum_j \exp(-\beta E_j) \quad (\text{A215})$$

$$Z(\tau) = \sum_j \exp(-\beta E_j) \quad (\text{A216})$$

Finally, the probability measure is:

$$\rho_i = \frac{1}{\sum_j \exp(-\beta E_j)} \exp(-\beta E_i) \quad (\text{A217})$$

Appendix F SageMath program showing $[\mathbf{u}^\dagger \mathbf{u}]_{3,4} \mathbf{u}^\dagger \mathbf{u} = \det \mathbf{M}_{\mathbf{u}}$

```
from sage.algebras.clifford_algebra import CliffordAlgebra
from sage.quadratic_forms.quadratic_form import QuadraticForm
from sage.symbolic.ring import SR
from sage.matrix.constructor import Matrix
```

```
# Define the quadratic form for GA(3,1) over the Symbolic Ring
Q = QuadraticForm(SR, 4, [-1, 0, 0, 0, 1, 0, 0, 1, 0, 1])
```

```
# Initialize the GA(3,1) algebra over the Symbolic Ring
algebra = CliffordAlgebra(Q)
```

```

# Define the basis vectors
e0, e1, e2, e3 = algebra.gens()

# Define the scalar variables for each basis element
a = var('a')
t, x, y, z = var('t x y z')
f01, f02, f03, f12, f23, f13 = var('f01 f02 f03 f12 f23 f13')
v, w, q, p = var('v w q p')
b = var('b')

# Create a general multivector
udegree0=a
udegree1=t*e0+x*e1+y*e2+z*e3
udegree2=f01*e0*e1+f02*e0*e2+f03*e0*e3+f12*e1*e2+f13*e1*e3+f23*e2*e3
udegree3=v*e0*e1*e2+w*e0*e1*e3+q*e0*e2*e3+p*e1*e2*e3
udegree4=b*e0*e1*e2*e3
u=udegree0+udegree1+udegree2+udegree3+udegree4

u2 = u.clifford_conjugate()*u

u2degree0 = sum(x for x in u2.terms() if x.degree() == 0)
u2degree1 = sum(x for x in u2.terms() if x.degree() == 1)
u2degree2 = sum(x for x in u2.terms() if x.degree() == 2)
u2degree3 = sum(x for x in u2.terms() if x.degree() == 3)
u2degree4 = sum(x for x in u2.terms() if x.degree() == 4)
u2conj34 = u2degree0+u2degree1+u2degree2-u2degree3-u2degree4

I = Matrix(SR, [[1, 0, 0, 0],
                [0, 1, 0, 0],
                [0, 0, 1, 0],
                [0, 0, 0, 1]])

#MAJORANA MATRICES
y0 = Matrix(SR, [[0, 0, 0, 1],
                [0, 0, -1, 0],
                [0, 1, 0, 0],
                [-1, 0, 0, 0]])

y1 = Matrix(SR, [[0, -1, 0, 0],
                [-1, 0, 0, 0],
                [0, 0, 0, -1],
                [0, 0, -1, 0]])

y2 = Matrix(SR, [[0, 0, 0, 1],
                [0, 0, -1, 0],
                [0, -1, 0, 0],
                [1, 0, 0, 0]])

```

```

y3 = Matrix(SR, [[-1, 0, 0, 0],
                 [0, 1, 0, 0],
                 [0, 0, -1, 0],
                 [0, 0, 0, 1]])

mdegree0 = a
mdegree1 = t*y0+x*y1+y*y2+z*y3
mdegree2 = f01*y0*y1+f02*y0*y2+f03*y0*y3+f12*y1*y2+f13*y1*y3+f23*y2*y3
mdegree3 = v*y0*y1*y2+w*y0*y1*y3+q*y0*y2*y3+p*y1*y2*y3
mdegree4 = b*y0*y1*y2*y3
m=mdegree0+mdegree1+mdegree2+mdegree3+mdegree4

print(u2conj34*u2 == m.det())

```

The program outputs

True

showing, by computer assisted symbolic manipulations, that the determinant of the real Majorana representation of a multivector u is equal to the double-product: $\det \mathbf{M}_u = [\mathbf{u}^\dagger \mathbf{u}]_{3,4} \mathbf{u}^\dagger \mathbf{u}$.

References

1. Jaynes, E.T. Information theory and statistical mechanics. *Physical review* **1957**, *106*, 620.
2. Jaynes, E.T. Information theory and statistical mechanics. II. *Physical review* **1957**, *108*, 171.
3. Dirac, P.A.M. *The principles of quantum mechanics*; Number 27, Oxford university press, 1981.
4. Von Neumann, J. *Mathematical foundations of quantum mechanics: New edition*; Vol. 53, Princeton university press, 2018.
5. Hestenes, D. Spacetime physics with geometric algebra. *American Journal of Physics* **2003**, *71*, 691–714.
6. Lundholm, D. Geometric (Clifford) algebra and its applications. *arXiv preprint math/0605280* **2006**.
7. Acus, A.; Dargys, A. Inverse of multivector: Beyond $p+q=5$ threshold. *arXiv preprint arXiv:1712.05204* **2017**.

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