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Article

A Note on Generalized Parabolic Marcinkiewicz Integrals with Grafakos-Stefanov Kernels

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Abstract: This paper focuses on studying the generalized Marcinkiewicz integral operators with mixed homogeneity. Under the Grafakos-Stefanov condition on their kernel functions, the L^p boundedness of these operators over symmetric spaces is established. The main results generalize and improve some previously known results.

Keywords: Parabolic Marcinkiewicz integral; mixed homogeneity space; Grafakos-Stefanov class; Triebel–Lizorkin space

MSC: 42B20; 42B25; 42B35

1. Introduction

Let $d \geq 2$ and $\mathbb{S}^{d-1}$ be the unit sphere in the Euclidean space $\mathbb{R}^d$ equipped with the induced Lebesgue surface measure $d\sigma_d(w')$. For $j \in \{1, 2, \dots, d\}$, let $\kappa_j \geq 1$ be fixed numbers. Define the real mapping Γ on $\mathbb{R}_+ \times \mathbb{R}^d$ by $\Gamma(\gamma, w) = \sum_{j=1}^d w_j^2 \gamma^{-2\kappa_j}$ with $w = (w_1, w_2, \dots, w_d) \in \mathbb{R}^d$. For each fixed $w \in \mathbb{R}^d$, we denote the unique solution to the equation $\Gamma(\gamma, w) = 1$ by $\gamma \equiv \gamma_d(w)$. The metric space $(\mathbb{R}^d, \gamma)$ is known by the mixed homogeneity space associated to $\{\kappa_j\}_{j=1}^d$.

The change of variables concerning the space $(\mathbb{R}^d, \gamma)$ is presented in the following transformation:

$$\begin{aligned} w_1 &= \gamma^{\kappa_1} \cos v_1 \cdots \cos v_{d-2} \cos v_{d-1}, \\ w_2 &= \gamma^{\kappa_2} \cos v_1 \cdots \cos v_{d-2} \sin v_{d-1}, \\ &\vdots \\ w_{d-1} &= \gamma^{\kappa_{d-1}} \cos v_1 \sin v_2, \\ w_d &= \gamma^{\kappa_d} \sin v_1. \end{aligned}$$

So, $dw = \gamma^{\kappa-1} J(w') d\gamma d\sigma_d(w')$, where $\kappa = \sum_{j=1}^d \kappa_j$, $J(w') = \sum_{j=1}^d \kappa_j (w'_j)^2$, $w' = M_{\gamma^{-1}} w \in \mathbb{S}^{d-1}$, $\gamma^{\kappa-1} J(w')$ is the Jacobian of the transformation, and M_γ is the $d \times d$ diagonal matrix given by

$$M_\gamma = \begin{bmatrix} \gamma^{\kappa_1} & & 0 \\ & \ddots & \\ 0 & & \gamma^{\kappa_d} \end{bmatrix}.$$

In [1], it was proved that $J(w') \in \mathcal{C}^\infty(\mathbb{S}^{d-1})$ and $J(w') \in [1, C]$ for some real constant $C \geq 1$. Let $\mathcal{U} \in L^1(\mathbb{S}^{d-1})$ be a measurable mapping satisfying:

$$\mathcal{U}(M_\gamma w) = \mathcal{U}(w), \quad \forall \gamma > 0 \quad (1)$$

and

$$\int_{\mathbb{S}^{d-1}} \mathcal{U}(w') J(w') d\sigma_d(w') = 0. \quad (2)$$

For $f \in \mathcal{S}(\mathbb{R}^d)$ and $\tau \in (1, \infty)$, we define the generalized parabolic Marcinkiewicz integral $\mathcal{G}_{\mathcal{U}}^{(\tau)}$ by

$$\mathcal{G}_{\mathcal{U}}^{(\tau)}(f)(x) = \left(\int_{\mathbb{R}_+} \left| \frac{1}{r} \int_{\gamma(v) \leq r} f(x-v) \frac{\mathcal{U}(v)}{(\gamma(v))^{\kappa-1}} dv \right|^\tau \frac{dr}{r} \right)^{1/\tau}.$$

We point out when $\kappa_1 = \kappa_2 = \dots = \kappa_d = 1$, we have $\kappa = d$, $\gamma(v) = |v|$ and $(\mathbb{R}^d, \gamma) = (\mathbb{R}^d, |\cdot|)$. In this case, we denote $\mathcal{G}_{\mathcal{U}}^{(\tau)}$ by $\mathcal{G}_{\mathcal{U}}^{(\tau),c}$. Moreover, when $\tau = 2$, the operator $\mathcal{G}_{\mathcal{U}}^{(\tau),c}$ is reduced to the classical Marcinkiewicz integral operator, which is denoted by $\mathcal{M}_{\mathcal{U}}$. We remark that the operator $\mathcal{M}_{\mathcal{U}}$ was introduced by E. Stein in [2] in which he proved that the L^p boundedness of $\mathcal{M}_{\mathcal{U}}$ for $p \in (1, 2)$ holds under the condition $\mathcal{U} \in Lip_\nu(\mathbb{S}^{d-1})$ with $0 < \nu \leq 1$. Subsequently, the operator $\mathcal{M}_{\mathcal{U}}$ has been investigated by many authors. For example, Walsh [3] proved the L^2 boundedness of $\mathcal{M}_{\mathcal{U}}$ provided that $\mathcal{U} \in L(\log L)^{1/2}(\mathbb{S}^{d-1})$, and proved that the condition $\mathcal{U} \in L(\log L)^{1/2}(\mathbb{S}^{d-1})$ is nearly optimal. Later, this result was extended and improved in [4] where the authors obtained that $\mathcal{M}_{\mathcal{U}}$ is of type (p, p) for all $p \in (1, \infty)$ if $\mathcal{U} \in L(\log L)^{1/2}(\mathbb{S}^{d-1})$. If $\mathcal{U}$ belongs to a certain class of block spaces, the authors in [5] proved that if $\mathcal{U}$ belongs to the block space $B_q^{(0,-1/2)}(\mathbb{S}^{d-1})$, then $\mathcal{M}_{\mathcal{U}}$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$ and this condition on $\mathcal{U}$ is nearly optimal.

On the other hand, the authors of [7] proved that whenever $\mathcal{U}$ belongs to the Grafakos-Stefanov class $\mathcal{F}_\alpha(\mathbb{S}^{d-1})$ with $\alpha > 0$, then the operator $\mathcal{M}_{\mathcal{U}}$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$. Readers may consult: For a more background information on $\mathcal{M}_{\mathcal{U}}$ [8–11], for its extensions and developments [12–17] and for recent advances [18–20].

The generalized Marcinkiewicz operator $\mathcal{G}_{\mathcal{U}}^{(\tau),c}$ was introduced by the authors of [21]. They proved that if $\mathcal{U} \in L^q(\mathbb{S}^{d-1})$ with $q > 1$, then the inequality

$$\left\| \mathcal{G}_{\mathcal{U}}^{(\tau),c}(f) \right\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{F_p^{0,\tau}(\mathbb{R}^d)} \quad (3)$$

holds for all $p, \tau \in (1, \infty)$. Later, Le improved this result in [22] under the weaker condition $\mathcal{U} \in L(\log L)(\mathbb{S}^{d-1})$. These results were extended and improved by the authors of [23] who showed that if $\mathcal{U}$ is in either $L(\log L)^{1/\tau}(\mathbb{S}^{d-1})$ or in $B_q^{(0, \frac{1}{\tau})}(\mathbb{S}^{d-1})$, then $\mathcal{G}_{\mathcal{U}}^{(\tau),c}$ is bounded on $L^p(\mathbb{R}^d)$ for $p, \tau \in (1, \infty)$. The authors of [24] proved the inequality (3) for all $p, \tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$ provided that $\mathcal{U} \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$ for some $\alpha > 0$. For relevant results, we refer the reader to [25–27].

We point out that the Grafakos-Stefanov class $\mathcal{F}_\alpha(\mathbb{S}^{d-1})$ (for $\alpha > 0$), which is the set of all integrable functions $\mathcal{U}$ over $\mathbb{S}^{d-1}$ such that

$$\sup_{\zeta \in \mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \log^{\alpha+1}(|\zeta \cdot u|^{-1}) |\mathcal{U}(u)| d\sigma_d(u) < \infty, \quad (4)$$

was introduced in [3] and then developed in [28].

Now let us recall the definition of homogeneous Triebel-Lizorkin space $\dot{F}_p^{\epsilon, \tau}(\mathbb{R}^d)$. For $\epsilon \in \mathbb{R}$ and $\tau, p \in (1, \infty)$, the space $\dot{F}_p^{\epsilon, \tau}(\mathbb{R}^d)$ is the collection of all tempered distribution functions f defined on $\mathbb{R}^d$ such that

$$\|f\|_{\dot{F}_p^{\epsilon, \tau}(\mathbb{R}^d)} = \left\| \left(\sum_{j \in \mathbb{Z}} 2^{j\epsilon\tau} |\Psi_j * f|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)} < \infty,$$

where $\widehat{\Psi_j}(\zeta) = \Gamma(2^{-j}\zeta)$ and $\Gamma \in C_0^\infty(\mathbb{R}^\eta)$ is a radial mapping satisfying the following:

$$(1) \ 0 \leq \Gamma(\zeta) \leq 1,$$

- (2) $\text{supp}(\Gamma) \subset \left\{ \zeta : |\zeta| \in [\frac{1}{2}, 2] \right\}$,
 (3) For $|\zeta| \in [\frac{3}{5}, \frac{5}{3}]$, there is a positive constant C such that $\Gamma(\zeta) \geq C$,
 (4) For $\zeta \neq 0$, we have $\sum_{j \in \mathbb{Z}} \Gamma(2^{-j}\zeta) = 1$.

The authors of [25] proved the following:

- (a) The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is dense in $\dot{F}_p^{\epsilon, \tau}(\mathbb{R}^d)$,
 (b) For $p \in (1, \infty)$, $\dot{F}_p^{0,2}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$,
 (c) $\dot{F}_p^{\epsilon, \tau_1}(\mathbb{R}^d) \subseteq \dot{F}_p^{\epsilon, \tau_2}(\mathbb{R}^d)$ whenever $\tau_1 \leq \tau_2$.

Motivated by the work done in [24] regarding the L^p boundedness of the generalized Marcinkiewicz integral $\mathcal{G}_U^{(\tau), c}$ in the classical form whenever $U \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$, and by the work done in [16] regarding the L^p boundedness of the parabolic Marcinkiewicz integral $\mathcal{G}_U^{(2)}$ whenever $U \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$, we shall study the L^p boundedness of the generalized parabolic Marcinkiewicz integral $\mathcal{G}_U^{(\tau)}$ provided that $U \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$. In fact, we shall prove the following:

Theorem 1. Let $U \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$ for some $\alpha > 0$ and satisfy (1)–(2). Then there exists a positive constant $C_p > 0$ such that

$$\left\| \mathcal{G}_U^{(\tau)}(f) \right\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{\dot{F}_p^{\vec{0}, \tau}(\mathbb{R}^d)}$$

for all $p \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$ and $\tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$.

Remarks

- (1) For the cases $\kappa_j = 1$ ($1 \leq j \leq d$) and $\tau = 2$, the L^p boundedness of $\mathcal{G}_U^{(\tau)}$ was proved in [2] for $p \in (1, 2)$ provided that $U \in \text{Lip}_\nu(\mathbb{S}^{d-1}) \subsetneq \mathcal{F}_\alpha(\mathbb{S}^{d-1})$. Hence, our result generalizes and improves the result in [2].
- (2) Since $\bigcup_{q>1} L^q(\mathbb{S}^{d-1}) \subsetneq \mathcal{F}_\alpha(\mathbb{S}^{d-1})$, Theorem 1 generalizes and improves the result in [14] for the case $\tau = 2$.
- (3) If we take $\tau = 2$ in Theorem 1, the main result in [16] is obtained.
- (4) The authors of [17,19] proved the boundedness of $\mathcal{G}_U^{(\tau)}$ whenever U belongs to the space $L(\log L)^{1/2}(\mathbb{S}^{d-1})$ and the space $B_q^{(0, -\frac{1}{2})}(\mathbb{S}^{d-1})$, respectively, which are totally different from the space $\mathcal{F}_\alpha(\mathbb{S}^{d-1})$.
- (5) For the special case $\kappa_j = 1$ ($1 \leq j \leq d$), we obtain the main result in [24].

Henceforth, the letter C will denote a positive constant, the value of which may vary at each occurrence but remains independent of the essential variables.

2. Auxiliary Lemmas

We begin this section by presenting several definitions and lemmas. Define the family of measures $\{\lambda_{U,r,j} := \lambda_{r,j} : r \in \mathbb{R}_+, j \in \mathbb{Z}\}$ and its corresponding maximal operator λ^* on $\mathbb{R}^d$ by

$$\int_{\mathbb{R}^d} f d\lambda_{r,j} = \frac{1}{2^j r} \int_{2^{j-1}r \leq \gamma(u) \leq 2^j r} U(u) f(u) du,$$

and

$$\lambda^*(f) = \sup_{j \in \mathbb{Z}} \sup_{r \in \mathbb{R}_+} |\lambda_{r,j}| * f|,$$

where $|\lambda_{r,j}|$ is defined in the same way as $\lambda_{r,j}$ but U is replaced by $|U|$.

We shall need the following lemma from [29].

Lemma 1. Let $P(\gamma) = (v_1\gamma^{\kappa_1}, v_2\gamma^{\kappa_2}, \dots, v_d\gamma^{\kappa_d})$, where $v_j, \kappa_j \in \mathbb{R}$ and $j \in \{1, 2, \dots, d\}$. Suppose that $\mathcal{M}_P$ is the maximal function defined on $\mathbb{R}^d$ by

$$\mathcal{M}_P f(w) = \sup_{\rho > 0} \frac{1}{\rho} \int_0^\rho |f(w - P(\gamma))| d\gamma.$$

Then, there a positive C_p (independent of v_j 's) exists such that the estimate

$$\|\mathcal{M}_P(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$

holds for all $1 < p \leq \infty$.

We shall need the following lemma which is due to Chen and Ding [15].

Lemma 2. Assume that v indicates to the distinct numbers of $\{\kappa_j\}_{j=1}^d$ and that $\delta \in [0, 1]$. Then there exists $C > 0$ such that for $u \in \mathbb{S}^{d-1}$ and $\zeta \in \mathbb{R}^d$,

$$\left| \int_{1/2}^1 e^{-iM_\gamma u \cdot \zeta} \frac{d\gamma}{\gamma} \right| \leq C |u \cdot \zeta|^{-\frac{\delta}{v}}.$$

By using Lemma 1, we directly deduce the following result.

Lemma 3. Let $\mathcal{U} \in L^1(\mathbb{S}^{d-1})$ and $f \in L^p(\mathbb{R}^d)$. Then for each $p \in (1, \infty)$, there a positive C_p such that

$$\|\lambda^*(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \|f\|_{L^p(\mathbb{R}^d)}. \quad (5)$$

Lemma 4. Let $\mathcal{U} \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$ for some $\alpha > 0$ and satisfy (1)–(2). Then there is a positive constant C such that

$$\|\lambda_{r,j}\| \leq C, \quad (6)$$

$$|\hat{\lambda}_{r,j}(\zeta)| \leq C \min \left\{ |M_{2^j r} \zeta|, (\log |M_{2^j r} \zeta|)^{-(\alpha+1)} \right\}, \quad (7)$$

where $\|\lambda_r\|$ is the total variation of λ_r .

Proof. By the definition of $\lambda_{r,j}$, it is easy to verify that (6) holds. By Hölder's inequality, we get

$$\begin{aligned} |\hat{\lambda}_{r,j}(\zeta)| &\leq C \int_{1/2}^1 \left| \int_{\mathbb{S}^{d-1}} \mathcal{U}(u) J(u) e^{-iM_{2^j r \gamma} u \cdot \zeta} d\sigma_d(u) \right| \frac{d\gamma}{\gamma} \\ &\leq C \int_{\mathbb{S}^{d-1}} |\mathcal{U}(u)| \int_{1/2}^1 I(\zeta, u, r) \frac{d\gamma}{\gamma} d\sigma_d(u), \end{aligned} \quad (8)$$

where

$$I(\zeta, u, r) = \int_{1/2}^1 e^{-iM_{2^j r \gamma} u \cdot \zeta} \frac{d\gamma}{\gamma}.$$

Thanks to Lemma 2, we deduce that

$$I(\zeta, u, r) \leq |M_{2^j r} u \cdot \zeta|^{-\frac{\delta}{v}}$$

which when combined with the trivial estimate $|I(\zeta, u, r)| \leq 1$ and using the fact $(t / \log^\alpha t)$ is increasing over the interval $(2^\alpha, \infty)$, we have that

$$|I(\zeta, u, r)| \leq C \frac{\log^{\alpha+1}(|\eta \cdot u|^{-1})}{\log^{\alpha+1}(|M_{2^j r} \zeta|)} \quad \text{if } |M_{2^j r} \zeta| > 1, \quad (9)$$

where $\eta = \frac{M_{2^j r} \zeta}{|M_{2^j r} \zeta|}$. Thus, by (8)-(9) we get

$$\begin{aligned} |\hat{\lambda}_{r,j}(\zeta)| &\leq C (\log |M_{2^j r} \zeta|)^{-\alpha-1} \int_{\mathbb{S}^{d-1}} |\mathcal{U}(u)| \log^\alpha(|\eta \cdot u|^{-1}) d\sigma_d(u) \\ &\leq C (\log |M_{2^j r} \zeta|)^{-\alpha-1} \quad \text{if } |M_{2^j r} \zeta| > 1. \end{aligned} \quad (10)$$

Now by the condition (2), we obtain

$$\begin{aligned} |\hat{\lambda}_{r,j}(\zeta)| &\leq C \int_{\mathbb{S}^{d-1}} |\mathcal{U}(u)| \int_{1/2}^1 |J(\zeta, u, r) - 1| \frac{d\gamma}{\gamma} d\sigma_d(u) \\ &\leq C |M_{2^j r} \zeta|. \end{aligned} \quad (11)$$

Consequently, by (10)–(11), we get (7) which ends the proof the lemma. $\square$

Now we need to prove the following result:

Lemma 5. Let $\mathcal{U} \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$ for some $\alpha > 0$ satisfy (1)–(2). Then for any class of functions $\{\mathcal{H}_j(\cdot), j \in \mathbb{Z}\}$ defined on $\mathbb{R}^d$, we have

$$\left\| \left(\sum_{j \in \mathbb{Z}} \int_1^2 |\lambda_{r,j} * \mathcal{H}_j|^\tau \frac{dr}{r} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)} \leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \left\| \left(\sum_{j \in \mathbb{Z}} |\mathcal{H}_j|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)} \quad (12)$$

for all $1 < p, \tau < \infty$

Proof. As $p > 1$, by duality, a non-negative mapping $\varphi \in L^{p'}(\mathbb{R}^d)$ exists such that $\|\varphi\|_{L^{p'}(\mathbb{R}^d)} \leq 1$ and

$$\begin{aligned} \left\| \left(\sum_{j \in \mathbb{Z}} \int_1^2 |\lambda_{r,j} * \mathcal{H}_j| \frac{dr}{r} \right) \right\|_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \sum_{j \in \mathbb{Z}} \int_1^2 |\lambda_{r,j} * \mathcal{H}_j(w)| \frac{dr}{r} \varphi(w) dw \\ &\leq \int_{\mathbb{R}^d} \sum_{j \in \mathbb{Z}} |\mathcal{H}_j(w)| \lambda^*(\tilde{\varphi})(-w) dw, \end{aligned} \quad (13)$$

where $\tilde{\varphi}(w) = \varphi(-w)$. Thus, by Lemma 3, Hölder's inequality and (13), we obtain that

$$\left\| \left(\sum_{j \in \mathbb{Z}} \int_1^2 |\lambda_{r,j} * \mathcal{H}_j| \frac{dr}{r} \right) \right\|_{L^p(\mathbb{R}^d)} \leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \left\| \sum_{j \in \mathbb{Z}} |\mathcal{H}_j| \right\|_{L^p(\mathbb{R}^d)} \quad (14)$$

for $1 < p < \infty$. On the other hand, by the inequality (5), we get

$$\begin{aligned} \left\| \sup_{j \in \mathbb{Z}} \sup_{r \in [1,2]} |\lambda_{r,j} * \mathcal{H}_j| \right\|_{L^p(\mathbb{R}^d)} &\leq \left\| \lambda^* \left(\sup_{j \in \mathbb{Z}} |\mathcal{H}_j| \right) \right\|_{L^p(\mathbb{R}^d)} \\ &\leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \left\| \sup_{j \in \mathbb{Z}} |\mathcal{H}_j| \right\|_{L^p(\mathbb{R}^d)} \end{aligned} \quad (15)$$

for all $p \in (1, \infty)$. Therefore, by interpolating between (13) and (15) we have

$$\left\| \left(\sum_{j \in \mathbb{Z}} \int_1^2 |\lambda_{r,j} * \mathcal{H}_j|^\tau \frac{dr}{r} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)} \leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \left\| \left(\sum_{j \in \mathbb{Z}} |\mathcal{H}_j|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)}$$

for all $p, \tau \in (1, \infty)$. $\square$

3. Proof of Theorem 1

Assume that $\mathcal{U} \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$ for some $\alpha > 0$. By Minkowski's inequality, we obtain

$$\begin{aligned} \mathcal{G}_{\mathcal{U}}^{(\tau)}(f)(x) &= \left(\int_{\mathbb{R}_+} \left| \sum_{j=0}^{\infty} \frac{1}{r} \int_{2^{-j-1}r < \gamma(v) \leq 2^{-j}r} f(x-v) \frac{\mathcal{U}(v)}{(\gamma(v))^{\kappa-1}} dv \right|^\tau \frac{d\gamma}{\gamma} \right)^{1/\tau} \\ &\leq \sum_{j=0}^{\infty} \left(\int_{\mathbb{R}_+} \left| \frac{1}{r} \int_{2^{-j-1}r < \gamma(v) \leq 2^{-j}r} f(x-v) \frac{\mathcal{U}(v)}{(\gamma(v))^{\kappa-1}} dv \right|^\tau \frac{d\gamma}{\gamma} \right)^{1/\tau} \\ &\leq C \left(\sum_{j=0}^{\infty} \int_1^2 |\lambda_{r,j} * f(x)|^\tau \frac{d\gamma}{\gamma} \right)^{1/\tau}. \end{aligned} \quad (16)$$

Choose a set of smooth mappings $\{\Lambda_j\}_{j \in \mathbb{Z}}$ on $\mathbb{R}_+$ satisfying the following:

$$\begin{aligned} \Lambda_j(w) &= \Lambda_j(\gamma(w)), \quad \Lambda_j \subset [0, 1], \quad \sum_{j \in \mathbb{Z}} \Lambda_j(v)^2 = 1, \\ \text{supp } (\Lambda_j) &\subset [2^{-1-j}, 2^{1-j}], \quad \text{and} \quad \left| \frac{d^m \Lambda_j(v)}{dv^m} \right| \leq \frac{C_m}{v^m}. \end{aligned}$$

For $\zeta \in \mathbb{R}^d$, define the operators $(\widehat{\Psi}_j(\zeta)) = \Lambda_j(\gamma(\zeta))$ and $S_j(\zeta) = (\Psi_j * f)(\zeta)$. Hence, for any $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\left(\sum_{j=0}^{\infty} \int_1^2 |\lambda_{r,j} * f(x)|^\tau \frac{d\gamma}{\gamma} \right)^{1/\tau} \leq \left(\sum_{j \in \mathbb{Z}} \int_1^2 \left| \sum_{n \in \mathbb{Z}} S_{j+n}(\lambda_{r,j} * (S_{j+n}f))(x) \right|^\tau \frac{d\gamma}{\gamma} \right)^{1/\tau} := D(f)(x).$$

Thus, to prove our main result, it suffices to prove that

$$\|D(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{\dot{F}_p^{\vec{0}, \tau}(\mathbb{R}^d)} \quad (17)$$

for all $p \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$ and $\tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$. Define the mapping T by

$$T : \{\mathcal{H}_{j,n}(x)\}_{n,j \in \mathbb{Z}} \longrightarrow \left\{ \sum_{n=1}^{\infty} S_{j+n}(\mathcal{H}_{j,n})(x) \right\}_{j \in \mathbb{Z}}.$$

Then, it is easy to show:

(i) For $1 < p < \tau$ and $1 < s < p$,

$$\left\| \left(\sum_{j \in \mathbb{Z}} \int_1^2 \left| \sum_{n \in \mathbb{Z}} S_{j+n}(\mathcal{H}_{j,n}) \right|^\tau \frac{d\gamma}{\gamma} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)}$$

$$\leq C \left(\sum_{n \in \mathbb{Z}} \left\| \left(\sum_{j \in \mathbb{Z}} \int_1^2 |\mathcal{H}_{j,n}|^\tau \frac{d\gamma}{\gamma} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)}^s \right)^{1/s}. \quad (18)$$

(ii) For $\tau < p < \infty$ and $1 < s < p'$,

$$\begin{aligned} & \left\| \left(\sum_{j \in \mathbb{Z}} \int_1^2 \left| \sum_{n \in \mathbb{Z}} S_{j+n}(\mathcal{H}_{j,n}) \right|^\tau \frac{d\gamma}{\gamma} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)} \\ & \leq C \left(\sum_{n \in \mathbb{Z}} \left(\int_1^2 \left\| \left(\sum_{j \in \mathbb{Z}} |\mathcal{H}_{j,n}|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)}^\tau \frac{d\gamma}{\gamma} \right)^{s/\tau} \right)^{1/s}. \end{aligned} \quad (19)$$

To prove (17), we need to consider three cases:

Case 1. $p \in (\frac{2+2\alpha}{1+2\alpha}, \tau)$ and $\tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$. By (18), we deduce that

$$\begin{aligned} \|D(f)\|_{L^p(\mathbb{R}^d)} & \leq C \left(\sum_{n \in \mathbb{Z}} \left\| \left(\sum_{j \in \mathbb{Z}} \int_1^2 |\lambda_{r,j} * (S_{j+n}f)|^\tau \frac{d\gamma}{\gamma} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)}^s \right)^{1/s} \\ & \leq C \left(\sum_{n \in \mathbb{Z}} \|A_{n,\tau}f\|_{L^p(\mathbb{R}^d)}^s \right)^{1/s} \end{aligned} \quad (20)$$

for all $1 < s < p$, where

$$A_{n,\tau}f = \left(\sum_{j \in \mathbb{Z}} \int_1^2 |\lambda_{r,j} * (S_{j+n}f)|^\tau \frac{d\gamma}{\gamma} \right)^{1/\tau}. \quad (21)$$

Let us estimate the L^p norm of $A_{n,\tau}f$ whenever $p = \tau = 2$. For this case, we have that $\dot{F}_2^{0,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Thus, by Plancherel's theorem along with Fubini's theorem and (7), we get

$$\begin{aligned} \|A_{n,\tau}f\|_{L^2(\mathbb{R}^d)}^2 & \leq \sum_{j \in \mathbb{Z}} \int_{\mathcal{O}_{n+j}} \left(\int_1^2 |\widehat{\lambda_{r,j}}(\zeta)|^2 \frac{d\gamma}{\gamma} \right) |\hat{f}(\zeta)|^2 d\zeta \\ & \leq C \sum_{j \in \mathbb{Z}} \int_{\mathcal{O}_{n+j}} |\hat{f}(\zeta)|^2 \min\{|M_{2^j r} \zeta|^2, (\log |M_{2^j r} \zeta|)^{-2(\alpha+1)}\} d\zeta \\ & \leq C(1+|n|)^{-2(1+\alpha)} \|f\|_{L^2(\mathbb{R}^d)}^2 \\ & = C(1+|n|)^{-2(1+\alpha)} \|f\|_{\dot{F}_2^{0,2}(\mathbb{R}^d)}^2, \end{aligned} \quad (22)$$

where $\mathcal{O}_n = \{\zeta \in \mathbb{R}^d : |\zeta| \in [2^{-1-n}, 2^{1-n}]\}$.

It is clear that by Lemma 5, we have

$$\|A_{n,\tau}f\|_{L^p(\mathbb{R}^d)} \leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \left\| \left(\sum_{j \in \mathbb{Z}} |\Psi_j * f|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)} \sim C \|f\|_{\dot{F}_p^{0,\tau}(\mathbb{R}^d)} \quad (23)$$

for all $1 < p, \tau < \infty$. Therefore, by interpolation (22) with (23), there exists a constant $\theta \in (1/(1+\alpha), 1)$ such that for all $p \in (\frac{2+2\alpha}{1+2\alpha}, \tau)$,

$$\|A_{n,\tau}f\|_{L^p(\mathbb{R}^d)} \leq C(1+|n|)^\theta \|f\|_{\dot{F}_p^{\vec{0},\tau}(\mathbb{R}^d)}. \quad (24)$$

For fix $p \in (\frac{2+2\alpha}{1+2\alpha}, \tau)$, choose $1 < s < p$ so that $s\theta > 1/(\alpha+1)$ and employing (20) along with (24), we confirm (17) which by (16) proves our main result for the case $p \in (\frac{2+2\alpha}{1+2\alpha}, \tau)$ and $\tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$.

Case 2. $p \in (\tau, 2\alpha+2)$ and $\tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$. We construct the proof of Theorem 1 by following a similar argument as above except that we need to invoke (19) instead of (18). The details are omitted.

Case 3. $p = \tau$ and $\tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$. By the definition of $D(f)$ and Lemma 1, we have

$$\begin{aligned} \|Df\|_{L^p(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} \sum_{j \in \mathbb{Z}} \int_1^2 \left| \sum_{n \in \mathbb{Z}} S_{j+n}(\lambda_{r,j} * (S_{j+n}f))(x) \right|^\tau \frac{dr}{r} dx \right)^{1/\tau} \\ &\leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \left(\sum_{j \in \mathbb{Z}} \int_1^2 \|\mathcal{M}_P(|S_{j+n}f|)\|_{L^p(\mathbb{R}^d)}^\tau \frac{dr}{r} \right)^{1/\tau} \\ &\leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \left\| \left(\sum_{j \in \mathbb{Z}} |\Psi_j * f|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)} \\ &\leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \|f\|_{\dot{F}_p^{\vec{0},\tau}(\mathbb{R}^d)}. \end{aligned}$$

Consequently, the proof of Theorem 1 is complete.

4. Conclusions

In this work, we introduced the generalized parabolic Marcinkiewicz operator $\mathcal{G}_U^{(\tau)}$ under a very weak condition on the rough kernel $\mathcal{U}$. Whenever $\mathcal{U}$ belongs to the Grafakos-Stefanov class $\mathcal{F}_\alpha(\mathbb{S}^{d-1})$ for some $\alpha > 0$, we proved that the operator $\mathcal{G}_U^{(\tau)}$ is bounded from homogeneous Triebel-Lizorkin space $\dot{F}_p^{0,\tau}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ space for all $p, \tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$. The results obtained in this paper improve and generalize a number of previously known results, see [2,7,14,16,21,24].

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