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Article

The Born Rule Without a Measurement Postulate

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Abstract

The Born rule governs the probability of outcomes of measurements of quantum systems. Many attempts have been made to derive the Born rule from other postulates. We perform such an analysis in an Everett interpretation without any measurement postulates at all. We explain that probability is an ill defined concept and that an agent who nevertheless wishes to make approximate predictions will have no alternative measure to weigh the alternatives without subjecting herself to a dutch book. Our demonstration is complete for projective measurements of pure states in a finite dimensional Hilbert space and we discuss how it might be applied to generalized measurements. A similar demonstration is impossible for mixed states. Nevertheless, following the standard convention, probability for mixed states has the same validity and issues as it does in classical physics.

Keywords: quantum mechanics; quantum measurement; quantum foundations

1. Introduction

A naive application of the core postulates of quantum theory leads to counterintuitive results such as Schrödinger's cat states [1]. To avoid this, a seemingly arbitrary non-unitary measurement postulate is added, despite the fact that measurement devices are themselves physical objects subject to the laws of quantum mechanics. This postulate asserts that a measurement results in a single outcome even for a system in superposition, and that the various results occur with probabilities according to the Born rule. A complete quantum theory would explain measurement without resorting to either part of this postulate.

Our goal in this paper is to explain the second part (probabilities), but to do so we need to address the first part (collapse) in some fashion. We are not using a measurement postulate or proposing new physics, so we must accept that collapse is an illusion and the universe evolves unitarily. Then we would expect Schrödinger's cat states, as well as superpositions of Wigner's friend [2] to exist. This is known as the Everett, or many-worlds interpretation [3]. We might then ask why we do not experience such states. We will not resolve the measurement problem here. See [4] and [5] for different perspectives.

Fortunately we do not need to deal with macroscopic superpositions for our analysis, but their existence raises a challenging question. Specifically, why do observers believe in the Born probabilities when, in this picture at least, all measurement results actually occur. We will show that an observer who believes collapse is real and who wishes to make approximate predictions will have no alternative measure to weigh the alternatives without being subject to a dutch book, a set of bets on the result of an experiment that generate a loss of money regardless of the outcome. We emphasize that in this picture, the Born rule is not a behavior of nature, so our analysis is not a proof. It is an explanation of an illusion.

The paper is organized as follows: section 2 discusses concepts of classical probability including dutch books. Section 3 introduces Gleason's theorem and discusses what types of additional assumptions are needed for it to be relevant to quantum physics. Section 4 analyzes previous attempts. Section 5 describes our measurement formalism. Section 6 discusses contextuality. Section 7 contains the main analysis, showing that an agent making a projective measurement on a pure state must expect the

Born rule to avoid a dutch book. Section 8 considers generalized measurements. Section 9 analyzes what can be said about mixed states. Section 10 compares this approach to others, and includes a few philosophical comments. We conclude in section 11.

2. Classical Probability

Probability in classical physics is problematic because classical physics is deterministic. A statement about the final state of an experiment, such as “the coin will land on heads”, can be translated in principle to an equivalent statement about the initial conditions, such as “the coin was flipped in one of these ways...”. In practice, our physical knowledge of initial conditions is coarse grained so we can only specify a distribution of initial and thus final states. Furthermore, the final states can be highly sensitive to changes in initial conditions. Gambling houses design games using these two facts to prevent players from predicting outcomes. In this way, we can define concepts of chance, likelihood, etc. but these definitions are necessarily circular. Calculations always assume, at least implicitly, an *a priori* distribution.

It is useful to define conditions that ensure probability assignments are *coherent* or self-consistent. Specifically the measures of exclusive alternatives must be non-negative, they must add, and combined they must sum to 1. These conditions were first proposed by Ramsey [6]. He showed that any probability measure for a finite set of gambles that does not satisfy these conditions will be vulnerable to a *dutch book*, a set of wagers that will result in a profit for a bookmaker regardless of the actual events. These conditions were later independently found by Kolmogorov [7] for the infinite case, and they are known as the Kolmogorov axioms. Note that these axioms do not define what probability is.

Given that we lack a definition of probability, two main schools of thought on the interpretation of probability have arisen, which we will call the frequency interpretation and the Bayesian interpretation. The frequency interpretation is based on the assumption that the probabilities for an experiment can be explored by performing a large but finite number of trials, possibly implying the system has an inherent single-trial probability distribution or *propensity*. The Bayesian interpretation does not automatically assume inherent probabilities, but considers an agent’s statements of belief, *ie.* at what odds are they willing to gamble.

Using the frequency approach to define probabilities is inherently circular. For a large number of trials, the distribution of the mean result narrows, but it is still a distribution. Flipping a coin 100 times and getting 50 heads does not prove the coin is fair, or that it is probably fair, or even that it is close to fair. All it can prove is that its propensity is not 0 or 1. We can infer that the propensity is within a small range with some confidence, but that confidence is itself a probability.

The Bayesian approach sidesteps propensities altogether. Rather it is concerned with an agent’s expectations of probability and whether those expectations are coherent. There is nothing in this approach that guarantees an agent’s beliefs are correct. When put into practice, the Bayesian approach assumes a prior distribution, rendering this interpretation circular as well.

The concept of a dutch book can be understood as a game between an agent Alice and a bookmaker Bob. Bob is allowed to ask Alice what her expectations of probability are for any set of experiments. If he finds some discrepancy, he may propose a set of wagers. Alice is obliged to accept a wager if she believes she will at least break even on average. If Bob can gain money regardless of the outcome of the experiments, the set of wagers is called a dutch book. If no dutch book is possible, Alice’s probability beliefs are coherent. We will assume the wagers are implemented as contracts that pay off depending on the result of the experiment. Each contract has a price which Alice must pay to implement. If the price is negative Bob will pay Alice. Each contract has a vector of payouts, one for each possible result of the experiment. If the payout is negative, Alice pays Bob if that result happens. The expectation value of the payout of each contract, according to Alice, is greater than or equal to the price. Reference [8] has examples of dutch books for violations of each of the Kolmogorov axioms. Their examples are described in terms of “tickets”, which are easily converted to our contracts. For simplicity, if a dutch book is possible, our bookmaker will arrange that the sum of the payouts for all contracts is \$0

regardless of which experimental result occurs. He can do this because he has the freedom to adjust the cost of each contract and compensate by adjusting the payoff vector by the same amount. He performs the same adjustment on all contracts simultaneously to make the experimental result with maximum payout pay a total of \$0 across all contracts. He is still making money in all cases, so Alice must be paying a positive total amount to implement the contracts. He then offers a free contract that brings the total payout for each other result up to \$0.

It is important to understand that coherence does not imply correctness. Shown a fair coin, an agent may believe it will always come up heads. This belief is coherent and satisfies the Kolmogorov axioms. There is no dutch book since even a large number of coin flips might come up heads each time. Bob must gain money no matter the outcome of the experiments. The distinction between coherent and correct expectations will come up repeatedly here, and it's quantum analogy is critical as well.

3. Gleason's Theorem

Turning to the quantum case, Gleason's Theorem [9] states that any probability-like measure (PLM) of subspaces of a Hilbert space of dimension > 2 must be of the form

$$P(\Pi_i) = \text{tr}(\rho \Pi_i), \quad (1)$$

where Π_i is a projection operator, and ρ is a density operator. "probability-like" means the measures of exclusive alternatives (an orthogonal set of projectors) are non-negative, they add, and, if a complete set, they sum to 1. This set of assumptions coincides with the Kolmogorov axioms. Gleason did not require the measures to sum to 1, and did not require the density operator to have unit trace, but we will assume those normalizations.

We will rely on this theorem in only one subcase, but it is instructive to examine how a few additional assumptions could make contact with Physics and recover the Born rule from this purely mathematical statement. Our actual analysis will not make any of these assumptions *a priori*, but will touch upon each of them and we will refer to them by number. Any proof of the Born rule will have to address each of them in some fashion. The assumptions are:

1. The projection operators should be identified with testable propositions.
2. The density operator in (1) should be the one which represents the prepared state of the system.
3. The probabilities are non-contextual.
4. Experimental results are governed by probabilities, and thus a PLM.

For example, we might ask "what is the probability a system prepared in a pure state ϕ will be found upon measurement to be in a pure state ψ ?". Inserting the projection operator $|\psi\rangle\langle\psi|$ for the target state and the density operator $|\phi\rangle\langle\phi|$ for the prepared state gives

$$\begin{aligned} P(\psi|\phi) &= P_{\rho_\phi}(|\psi\rangle\langle\psi|) \\ &= \text{Tr}(|\phi\rangle\langle\phi||\psi\rangle\langle\psi|) \\ &= |\langle\psi|\phi\rangle|^2, \end{aligned} \quad (2)$$

which is the Born rule.

The identification in assumption 1 is simple for a projective, repeatable, or von Neumann measurement, as in the previous example. For such a measurement, the system is left in a state which is a projection of the original state, and a repeated measurement produces the same result. The phrase "the system is found to be in" should be familiar from elementary quantum mechanics textbooks. Gleason's theorem or a generalization can still be applied to generalized measurements [10,11], but more care is needed.

Assumption 2 is commonly missed. It is needed because Gleason's Theorem does not specify what density operator to use. Mathematically any density operator will do. For example, in a finite dimensional Hilbert space,

$$\rho = \hat{I} / \dim(\mathcal{H}) \quad (3)$$

says that a system is equally likely to be found in any state. Alternatively, in a field theory, the vacuum density operator could be assigned to all prepared states. Without making some contact with the prepared state, the complete Born rule is obviously unprovable.

For pure states, an example of a sufficient additional assumption is that a state is certain to be found in its own state if that is one of the choices, and zero probability to be found in an orthogonal state. This is sometimes postulated and we will refer to it as the exclusive measurement postulate or EMP. Working in a basis $\{|i\rangle\}$ where the prepared state is $\phi = |0\rangle$ and writing ρ for the unknown Gleason density operator associated with ϕ , we have

$$\begin{aligned} \delta_{i0} &= P(i|0) \\ &= \text{Tr}(\rho |i\rangle \langle i|) \\ &= \sum_{jkl} \langle j|k\rangle \rho_{kl} \langle l|i\rangle \langle i|j\rangle \\ &= \rho_{ii}. \end{aligned} \quad (4)$$

Then we have

$$\begin{aligned} \rho_{00} &= 1 \\ \rho_{ii} &= 0 \quad i \neq 0, \end{aligned} \quad (5)$$

and to ensure the operator is positive,

$$\rho_{ij} = 0 \quad i \neq j. \quad (6)$$

This is the usual density operator for the pure state ϕ .

Assumption 3 is related to assumption 1. Consider a 3 state system where we want to know if it is in the ground state using a projective measurement. We can perform a measurement with 3 possible results or we can perform a yes/no style measurement to see if it is the ground state vs. one of the other two. According to the Born rule, the probabilities are the same, but this cannot be assumed. *A priori*, the probability of the ground state could depend on the context of what other states are possible results of the measurement, *i.e.* the family of projectors corresponding to the experiment [12].

Even if, after the other assumptions, the PLM is unique, we still need assumption 4. Up to now, God seems to be playing dice [13] using the Born rule. At any time he may decide to abandon the dice and choose another way, such as selecting the most "likely" result each time. Furthermore, in a many-worlds multiverse, all results actually occur. Even if observers wrongly believe a collapse is occurring, it is not clear why they would believe one result occurs more often than another. They may instead view the unique Born values as a curious numerical artifact, similar to a conservation law.

The reader may consider the above to be a naive attempt at a proof of the Born rule. For our purposes these assumptions are too vague, and even unnecessary. But they are well worth studying to understand the core issues that need to be addressed.

4. Prior Efforts

It has been claimed [14] that the Born rule can be derived from Gleason's theorem together with assumption 3. Even if we accept assumptions 1 and 4 tacitly, assumption 2 must be addressed, and the way it is addressed depends on the interpretation. The claim can be proved if we also assume the other measurement postulates [15]. In fact many analyses based on Gleason's theorem exist [12,16–19]. All of them either acknowledge the gap of assumption 2 or ignore it. Our analysis fills the gap.

Deutsch suggested, using decision theory and symmetry arguments, that an agent's betting preferences, if rational, must follow the Born rule [20]. Unfortunately, the original argument relied on notational ambiguity at a critical point [14]. Subsequent analyses by Wallace and Greaves [21,22] assumed the Everett interpretation explicitly and attempted to make the argument more rigorous. Later Zurek presented a similar argument [23], but he deduced the necessary symmetries from the physics of entanglement, specifically enviance, rather than assumptions about agent reasoning. These treatments are complex. To make them accessible and demonstrate their weaknesses we will consider simple examples, starting with enviance.

Enviance is a pseudo-symmetry of entangled states, such as those that inevitably arise in a measurement process. Consider the Bell state $(|\uparrow_s\rangle |\uparrow_e\rangle - |\downarrow_s\rangle |\downarrow_e\rangle)/\sqrt{2}$, where the first ket in each pair represents a system and the second represents some state of the environment with which the system (and/or an apparatus) has interacted as part of a measurement process. A unitary operation (σ_x) which swaps $|\uparrow_s\rangle$ with $|\downarrow_s\rangle$ can be undone by a unitary operation on the environment so that the quantum state remains unchanged. Since the environment operation can have no effect on the future physical evolution of the system, the original swap cannot either. This demonstrates a symmetry for all physical consequences of the measurement. Zurek argues the probability the measurement will result in $\uparrow_s$ must then be $1/2$, and further analysis recovers the complete Born rule.

The problem with this argument is, the identification of probability as a physical consequence requires a natural process performing a selection. But Zurek is working explicitly in the Everett interpretation where there is no such physical process. If probability is not a physical consequence, then Zurek's conclusion is unwarranted. In fact the Born rule in the Everett interpretation can only be understood as an aspect of an agent's reasoning.

Wallace and Greaves take that approach from the start. Consider a concrete example from [22], modified for simplicity. A system qbit is prepared in a state $(|+\rangle + |-\rangle)/\sqrt{2}$ in some fixed basis. The qbit is to be measured in the same basis, and a reward paid to an agent (Alice) based on the result. Alice will receive \$1 if she predicts the result correctly, otherwise she gets nothing. Once she has made her prediction, the qbit is measured, and the state of a reward qbit is assigned secretly to indicate whether or not she gets the reward. If she predicts $|+\rangle$ the new state will be $(|+\rangle |1\rangle + |-\rangle |0\rangle)/\sqrt{2}$. Otherwise the new state will be $(|+\rangle |0\rangle + |-\rangle |1\rangle)/\sqrt{2}$. There is some peculiarity in this notation. After a measurement these should be described by density operators, but following Wallace, we write them as coherent superpositions. This does not affect the discussion significantly. Next an erasure operation is performed on the system qbit and, importantly, on Alice's memory of her prediction. This leaves the reward qbit in the state $(|0\rangle + |1\rangle)/\sqrt{2}$ regardless of her prediction.

Since we are trying to see if Alice is following the Born rule we can assume she is making her prediction based on a probability assignment for the post-measurement branches. If she believes the probability of measuring $|\pm\rangle$ is $P_{\pm}$ with $P_+ + P_- = 1$, then her expected rewards are $R_{\pm} = P_{\pm}$ depending on her prediction. But the final state is the same, so, according to Wallace, the expected rewards must be the same, so $P_+ = P_- = 1/2$. While this example is not rigorous enough for a proof, it is hard to take issue with it in this symmetric case. Indeed our argument in section 6 also depends on a symmetry.

In order to make this argument rigorous for all cases, symmetric and otherwise, Wallace proposes no less than 10 metaphysical "axioms". Wallace does not claim quantum mechanics should be supplemented with 10 new axioms. Instead, he seems to be implying they should be obvious. In fact, probabilities are not properties of Everettian branches, they are calculation tools of an agent's reasoning. The non-physical axioms are for the agent's thoughts. Wallace is effectively arguing that an agent would be irrational not to believe them. This seems dubious given that experts debate them at length [24–27]. The very existence of such a debate undermines conclusions about how agents must reason.

In contrast, the one assumption in our analysis is physical, not metaphysical, we will be explicit about exactly what our agents believe, and our gambler will lose money in all branches (but see section 6). A dutch book is the gold standard to demonstrate irrationality in an Everettian multiverse.

A partial dutch book argument has been found in the QBist interpretation. DeBroda *et al.* [8] have shown that a gambler (Alice) who wishes to avoid a dutch book must base her probabilities for quantum experiments on a set of probabilities associated with a density operator, echoing Gleason. Thier proof does not imply her probability assignments must be correct, but it does imply that they must be coherent, consistent with the QBist interpretation. For example, suppose she believes (coherently) the various results of any experiment are always equally likely. Then, using DeBroda's notation, $p(i) = 1/d^2$, $R(j|i) = 1/d$, and the "QBist version of the Born rule" (their equation 12) reads

$$\begin{aligned} q(j) &= \sum_{i=1}^{d^2} ((d+1)p(i) - \frac{1}{d})R(j|i) \\ &= \frac{1}{d}, \end{aligned} \quad (7)$$

and Alice feels vindicated.

It is worthwhile at this point to consider a concrete test case. Suppose we arrange a Stern Gerlach apparatus whose axis is tilted from the z-axis by $\theta = \arccos(1/3)$. We block off the lower path and send a spin through to prepare our state. Our detector will consist of another Stern Gerlach apparatus oriented along the z-axis. Particle detectors are used to determine which path the spin takes. Assuming 100% efficiency, what is the probability the upper detector will be the one to fire? Max Born would say 2/3. Alice believes the answer is 1/2. A QBist might or might not believe 2/3, but DeBroda's proof does not contradict Alice. We consider this version of the Born rule to be incomplete. Our analysis will show that Alice's expectations must be correct to avoid a dutch book, not just coherent.

There have been many other attempts to prove the Born rule. Reviewing all of them is beyond the scope of this article, but a summary from 2020 by Vaidman can be found in [28] where he states "I believe that in all frameworks there is no way to [derive] the Born rule from other axioms of standard quantum mechanics". We respectfully disagree.

5. Measurement Formalism

We model measurements as follows. An apparatus implements a unitary evolution that entangles a system and a result register. Following [10], the entangled resulting state is

$$\rho_{sr} = \sum_{aa'} A_a \rho_s A_a^\dagger \otimes |a\rangle_r \langle a'|_r, \quad (8)$$

where ρ_s is the prepared state of the system, A_a is a measurement operator, and $|a\rangle_r$ is a register state. In the case of a projective measurement, $A_a = \Pi_a$, a hermitian projection operator.

If the starting state is not pure, the interpretation of the density operator depends on how it is prepared. It may represent an ensemble from a prior distribution, as if we flipped a classical coin when we prepared it (see section 2), or it may result from a previous measurement. For now our only requirement is that we do not interpret it via probabilities.

At this point the description of a projective measurement diverges. In the Everett interpretation, interactions with the environment result in decoherence. The details are controversial, and we do not address them here. For simplicity, we will explicitly release the system and apparatus to the environment. Imagine we melt them down. The result is a register in the effective state

$$\begin{aligned} \rho_r &= \sum_a B_a |a\rangle_r \langle a|_r \\ B_a &= \text{tr}(\rho_s \Pi_a), \end{aligned} \quad (9)$$

where the Born value B_a is just a numerical coefficient, not interpreted as a probability. Note that we have made use of trace reduction of the system Hilbert space. This is normally derived using the Born rule implicitly, but it can also be justified by asserting the proper unitary evolution of the decoupled Hilbert spaces [29].

In the Copenhagen interpretation [30] a non-unitary transition occurs and the register state becomes

$$\begin{aligned}\rho_r &= |a\rangle_r \langle a|_r \\ P_a &= \text{tr}(\rho_s \Pi_a),\end{aligned}\tag{10}$$

where P_a is the probability of that result.

For a generalized measurement, we add a probe layer between the system and the register. The interaction between the system and probe is fully unitary and the measurement operators are not projectors. We then perform a projective measurement of the probe. The result in the Everett interpretation is

$$\begin{aligned}\rho_r &= \sum_a B_a |a\rangle_r \langle a|_r \\ B_a &= \text{tr}(A_a \rho_s A_a^\dagger),\end{aligned}\tag{11}$$

with an analogous result in Copenhagen.

6. Contextuality

We turn now to the issue of contextuality. Contextuality here is slightly different from the more familiar version used to restrict hidden variable theories. Here we are comparing probabilities for one common projector using measurements that differ only in the set of other projectors measured by the apparatus. In the familiar version, multiple projectors are measuring a common observable, but differ with respect to other observables. In the latter case quantum mechanics is contextual, so a hidden variable theory is severely restricted if it wants to match predictions. In our case (a single projector) the Born rule is non-contextual, but we can't assume that.

Our analyses will be presented as discussions or games between Bob and Alice. We (and Bob and Alice) assume a Hilbert space, a state represented by a vector or density operator, and a unitary time evolution via a Hamiltonian. We (and Bob) assume measurements are unitary processes resulting in macroscopic incoherent superpositions, but we remain agnostic on the exact mechanisms of decoherence. Alice believes measurement includes a non-unitary transition, as in section 5, that produces one result by chance via some probabilities, but with the Born rule for probabilities distorted in some way.

Bob's strategy in section 7 will make use of Gleason's theorem, so Bob needs to make sure Alice believes her probability assignments should be non-contextual. To convince her, he takes an empirical approach and, for our benefit, does not assume the Born rule. He is allowed to question her about any experiment, so he finds one where her probabilities for one result of a projective measurement are contextual. For example, she may believe a measurement with n possible results always has probability $1/n$ for each result. He sets up two versions of the experiment and repeats each of them N times. For example, one apparatus determines if an atom, prepared in a specific way, is in the ground state vs. an excited state, and one determines the ground state vs. the first excited state vs. higher states. In the example, Alice believes the probabilities for the ground state are $1/2$ and $1/3$ respectively. Each run prints out a slip of paper with "yes" if the system is found in the state corresponding to the common projector and "no" otherwise. The apparatus is then released to the environment.

Labeling the common projector with index 0, the state of each result is

$$\rho_{yn} = B_0 |yes\rangle \langle yes| + (1 - B_0) |no\rangle \langle no|,\tag{12}$$

regardless of which apparatus was used. In this sense, Everettian branches (the common branch in particular) are non-contextual.

Bob then sorts the slips into two groups based on which version of the experiment was run. He then asks Alice which group corresponds to which experiment. Alice has no basis to decide and she cannot justify contextuality.

In the unhappy event that the first group comes out with $\sim N/2$ “yes” results and the second with $\sim N/3$ (or vice versa!) Alice will be undeterred. Quantum mechanics predicts that such things do happen, and we are surprised when they do, even if Alice is not.

7. Projective Measurements of Pure States

The above empirical approach does not work to derive the quantitative Born rule (at least not without many additional axioms, *a la* Wallace). Instead we make use of Gleason’s theorem and address the remaining assumptions 1 and 2.

Bob prepares a pure state. He explains to Alice how he prepared it and outlines the projective measurement setup. He asks Alice questions about her expectations for probabilities. We divide the analysis into cases.

Case 1: Her probabilities can be expressed as a density operator using (1) where the operator accurately describes the correct pure state (let us say Bob uses standard conventions for density matrices for his own analyses). In this case the game is over. Alice is following the Born rule.

Case 2: Her probabilities can be described by a density operator, but it is not the prepared state. Let $p > 0$ be Alice’s expected probability that the system will *not* be found in the starting state. Bob offers Alice a contract for \$1 that pays at least $\$1/p$ if the state is not found in the starting state. Alice is obligated to purchase this worthless contract. Then Bob constructs an experiment to perform the test. “The system is found in...” is loose language and we will define the specific measurement operators below.

At this point, we need to fulfill assumption 1. We are considering only projective measurements of a system with a finite dimensional Hilbert space, so we will make assumption A: *for any complete set of orthogonal projectors for a finite dimensional Hilbert space, there exists an apparatus that implements a projective measurement, up to but not including the collapse projection*. Such observations are possible in principle, and in an age where, for example, arbitrary unitary operations can be performed on qbits, this assumption is not unreasonable when projective measurements are possible.

In the case at hand (Alice’s expectations are based on the wrong density operator), Bob will employ a set of two Measurement operators. One coincides with the prepared density operator ρ_s . The other is its orthogonal complement. The Born value for the first is 1 so there is only one term in the register state, even though we have not applied a collapse projection. We can then print out the single result on a slip of paper.

At this point we consider assumption 2. But instead of an EMP, we simply assert that an Everettian state with a single macroscopic classical branch is *defined* to be found in that classical state. For all intents and purposes it acts like that state. For example, Alice knows her contract is worthless and she has lost \$1. This is how we avoid a measurement postulate. Bob creates a situation where a macroscopic observer sees only one result with purely unitary evolution.

Case 3: Her probabilities cannot be derived from a density operator. Then by Gleason’s theorem there exists a set of projectors whose probabilities violate the Komolgorov axioms. Bob proposes the bets, and if Alice does not capitulate, he constructs an apparatus to produce a result slip from these projectors. In this case there is an additional step. After the system interacts with the register, unitary operations should be performed to compute the final sum of the value of the contracts. Since Bob arranged for the value of the contracts to sum to zero in all cases, Alice will know the set of contracts is worthless and she lost the net amount she paid for them. For pure states at least, if Alice does not use the correct density operator, she is vulnerable to a dutch book.

Note, if the system is a qbit, we can add an ancillary qbit and consider projectors in the resulting 4 dimensional Hilbert space so that Gleason's theorem applies.

This completes the analysis of projective measurements of pure states. Note that we have not made assumption 4. In fact without a measurement postulate there is no such thing as probability. All possible results actually occur in the final state.

8. Generalized Measurements

In practice most measurements are not projective. The system may change in a way which is non-recoverable. The propositions of assumption 1 are that the probe, rather than the system, will be left in one of its distinguishable states.

Attempting the above analysis is problematic here. If the prepared state is pure, the entangled state of system and probe will be pure as well, but we are performing a generalized measurement because a projective measurement of the system is not practical. A measurement operator acting on the probe is

$$\bar{\Pi}_a = \mathbb{I}_s \otimes \Pi_a , \quad (13)$$

where $\mathbb{I}_s$ is the identity in the system Hilbert space and Π_a is the projector in the probe Hilbert space. The projectors are only a finite subset of the full space of projectors on the tensor product space. Furthermore, the "a" states are specifically designed to represent the desired measurement results, and not the arbitrary states that the analysis requires.

Suppose all "a" states are actually physically possible (they have non-zero Born values) and Bob asks Alice what are her expected probabilities. If her answers sum to 1, he knows there is in general no experiment he can actually perform to implement a dutch book. Instead, he could ask Alice to give her probabilities for experiments that can be performed *in principle*. Specifically, projective measurements are always possible in principle for any set of projectors, even on the tensor product Hilbert space. Bob may inform Alice that he will delay the experiment until new technology is available. Alice, having been stung by the first set of experiments, might be convinced to use the correct density operator (8) to compute her probabilities, and thus answer with the Born result

$$P_a = \text{Tr}(A_a^\dagger \rho_s A_a) . \quad (14)$$

9. Mixed States

Our analysis does not work for mixed states. Consider a system represented by a diagonal density matrix whose first two Born values (diagonal elements) are p and $1 - p$. By now, Alice has learned that her expectations must derive from a density operator. If her density matrix has any components outside the 2×2 upper left submatrix in the same basis, she is vulnerable to a dutch book. Just as in the pure case, she expects a non-zero probability for a result that will never happen.

But if the support of her density operator is the same subspace (or less) as the support of the system state density operator, then no dutch book is possible.

Proof: For Alice to accept contract i , it must satisfy

$$C_c \leq \sum_{i=1}^N p_i P_{ic} , \quad (15)$$

where C_c is the cost of contract c , N is the number of possible experimental results, p_i is her expected probability for experimental result i , and P_{ic} is the payout for contract c if result i occurs. Summing over all contracts we have

$$\begin{aligned}\bar{C} &\leq \sum_{i=1}^N p_i \bar{P}_i \\ &\leq \sum_{i=1}^N p_i \bar{P}_{\max},\end{aligned}\tag{16}$$

where $\bar{P}_{\max}$ is the best payoff for Alice among the various experimental results. If Alice's expectations are coherent, $\sum p_i = 1$ and $\bar{P}_{\max}$ is at least as much as the cost.

There remains a loophole that we have exploited previously. Bob may assign a large payout for a result that cannot physically occur. Thus it remains to prove that Alice will not assign a positive probability to such a result. Results associated with projectors with support outside our 2D subspace would be rejected by Alice. Furthermore, any projector in the 2D subspace will be associated with a non-zero Born value in (1) for one or both components of ρ so that result can physically occur. This completes the proof. More generally Alice can evade a dutch book if the system density operator has rank $R > 1$, ie. a mixed state.

This is all a little abstract, but it may be understood in terms of a coin flip. Alice assigns a probability p to heads and a probability $1 - p$ to tails. No dutch book is possible. If the "true" probability of heads is less than p , Bob can apply a high payout to this result and make money on average, but cannot guarantee making money because heads might occur many times despite its low probability. Furthermore, in the dutch book game, Alice could be allowed to choose her result (from physically possible ones), so she is free to assign probability 1 to one result and reject any wager that loses money in that one case.

Bob only cares that he makes money in all cases. He may consider the p parameter of the quantum state to be meaningless. Instead the only meaningful part of the state to him is the subspace in which the density operator has support. It would be interesting to formulate quantum mechanics (without collapse or probability) using this state representation.

Alice disagrees. Probability is meaningful to her, and it is convenient to identify the p parameter as an *a priori* probability. With this convention she can use (1) or (14) to calculate final probabilities. This is a perfect analogy with the classical circular definitions, and indeed the density operator can be used to model superpositions, mixtures, distributions, and ensembles of classical and quantum states.

10. Discussion

The Everett interpretation has been important for this analysis. Compared to other interpretations it may have the disadvantage that probability cannot exist as a physical phenomenon, but this is solved by considering agents' reasoning. A harder problem in a non-Everett approach is dealing with the arbitrary non-unitary transition required by the collapse effect. *A priori* this transition has no restrictions at all so additional postulates are needed such as the Lüders rule [31] and the EMP. Even the requirement that probabilities converge must be postulated. If god is playing dice, he can arrange the selections to prevent convergence.

Among approaches within the Everett interpretation, our analysis has the advantage that it can make use of Gleason's powerful theorem. Furthermore, recognizing that probability is an illusion led us to consider agents with different beliefs. This avoids the problem where Everettian agents don't believe in probability and simply cannot reproduce the illusion. Finally, we were able to construct a dutch book, which is required to confidently demonstrate irrationality in an Everettian multiverse.

On the other hand, this Everettian picture is very counterintuitive. Even a classical experiment such as flipping a coin 25 times is subject to quantum fluctuations, and there will be a "branch" where the coin land heads every time. In that branch, the agent, with her expectations of probability, will be very surprised. Nature in our picture assigns a tiny Born value to that branch of the superposition,

but makes no judgement about “likelihood” or “frequency”. But we can say this: extremely unlikely events do occur, and when they do we are surprised. In this sense the predictions of quantum theory are accurate.

Furthermore, there are branches where our non-contextuality demonstration fails, or worse, every event in history had a chaotic result. It would be impossible to learn Physics in such a branch, and the inhabitants would not be able to have this discussion. Anthropic reasoning may help exclude that case.

11. Conclusions

Despite considerable effort, there has not been a universally accepted proof of the complete Born rule from the other postulates of quantum mechanics. The complete Born rule requires probabilities to be correct and not just coherent.

Probability is an ill defined concept in a deterministic universe. The classical universe is deterministic, and without a measurement postulate or new physics, a quantum universe is as well. It follows that macroscopic superpositions must exist, and given that all possible experimental results actually occur, there is no intrinsic concept of probability. But an agent who believes collapse occurs and wishes to make approximate predictions will have no alternative measure to weigh the alternatives without subjecting herself to a dutch book. Agents may deviate from the Born rule for generalized measurements at their own risk.

Our demonstration is complete for projective measurements of pure states in a finite dimensional Hilbert space without resorting to any measurement postulate. A similar demonstration is impossible in the general case of mixed states. Nevertheless, with the usual convention for diagonal density matrix elements, probabilities for mixed states have the same validity and issues as they do in classical physics. With such a convention, (1) applies to all experiments, both quantum and classical.

In this picture, the Born rule is not a behavior of the universe. It is a rule of thumb used by observers who wish to make sense of the apparent randomness that arises from the collapse effect.

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