Article

Extended Field Equations for Conformally Curved Spacetime

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Abstract: The recent Planck Legacy release has confirmed the presence of an enhanced lensing amplitude in the cosmic microwave background power spectra, which prefers a positively curved early Universe with a confidence level greater than 99%. In addition, the spacetime curvature of the entire galaxy differs from one galaxy to another due to their diverse energy densities. This study considers both the implied positive curvature of the early Universe and the curvature across the entire galaxy as the curvature of 'the background or the 4D bulk' and distinguishes it from the localized curvature that is induced in the bulk by the presence of comparably smaller celestial objects that are regarded as 'relativistic 4D branes'. Branes in different galaxies experience different bulk curvatures, thus their background or bulk curvature should be taken into consideration along with their energy densities when finding their induced curvatures. To account for the interaction between the bulk and branes, this paper presents extended field equations in terms of brane-world modified gravity consisting of conformal Einstein field equations with a boundary term, which could remove the singularities and satisfy a conformal invariance theory. A visualization of the evolution of the 4D relativistic branes over the conformal space-time of the 4D bulk is presented.

Keywords: Conformal Einstein Field Equations; Boundary Terms; Brane-World.

1. Introduction

The recent Planck Legacy (PL18) release has confirmed the presence of an enhanced lensing amplitude in the cosmic microwave background (CMB) power spectra, which is higher than that estimated by the lambda cold dark matter model (ΛCDM). This prefers the positive curvature of the early Universe with a confidence level greater than 99% [1]. In addition, the PL18 release has a moderate preference for a closed Universe by not including the CMB lensing or the baryon acoustic oscillation [2].

Considerable efforts have been devoted to modifying gravity, which aim to elucidate possible existence or the nature of dark matter and dark energy, achieve a better description of observation data, verify theoretical restrictions in the strong curvature regime such as within back holes and to formulate quantum gravity [3–12]. To achieve an efficient action for quantum corrections, several theories have been formulated on the modification of Lagrangian gravitational fields and curvature terms. Such modifications appear to be inevitable, which have included higher-order curvature terms as well as non-minimally coupled scalar fields [13–15].

Considering said motivations, this study presents extended field equations in terms of brane-world modified gravity. It considers both the implied positive curvature of the early Universe and the curvature across the entire galaxy as the curvature of the background or the 4D bulk and distinguishes the bulk curvature from the localized curvature that is induced into the bulk by the presence of comparably smaller celestial objects that are regarded as 'relativistic 4D branes'. The paper is organised as follows. Section 2 discusses the mathematical derivations of the extended field equations. Section 3 visualizes the field equations while Section 4 summarises the conclusions and suggests the future development of this work.

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2. Extended Field Equations

The spacetime curvature of the entire galaxy differs from one galaxy to another due to their diverse energy densities, where celestial objects experience different background or bulk curvatures. Additionally, the PL18 release has preferred a positively curved early Universe, that is, is a sign of a background curvature or a curved bulk. To incorporate the bulk curvature and its evolution over cosmic time, a modulus of spacetime deformation, E_D, is introduced based on the theory of elasticity [16]. By using Einstein field equations, $E_D =$ (stress/strain) in terms of energy density is

$$E_D = \frac{T_{\mu\nu} - T g_{\mu\nu}/2}{R_{\mu\nu}/\mathcal{R}} \tag{1}$$

where the stress is signified by the stress-energy tensor $T_{\mu\nu}$ of trace T while the strain is signified by the Ricci curvature tensor $R_{\mu\nu}$ as the change in the curvature divided by the scalar of the bulk curvature $\mathcal{R}$. The Einstein–Hilbert action can be extended to

$$S = \int E_D \left[\frac{R}{\mathcal{R}} + \frac{L}{\mathcal{L}}\right] \sqrt{-g}\, d^4\sigma \tag{2}$$

This action is visualized in Section 3, where R represents the Ricci scalar curvature as a localized curvature induced by a celestial object that is regarded as a 4D relativistic brane of metric g_{uv} and Lagrangian density L. $\mathcal{R}$ represents the scalar curvature of the 4D bulk of metric $\tilde{g}_{\mu\nu}$ while $\mathcal{L}$ is the bulk's Lagrangian density as a manifestation of its internal stresses reflecting its curvature. The evolution of the bulk curvature over cosmic time from a preferred positive curvature at the early Universe of metric $g_{\mu\nu}$ to current spacetime spatial flatness can be characterized by Weyl's conformal transformation as $\tilde{q}_{\mu\nu} = q_{\mu\nu}\Omega^2$ where Ω^2 is a conformal function [17,18]. The variation in the action is

$$\delta S = \int E_D \left[\left(\frac{\delta R\sqrt{-g}}{\mathcal{R}} - \frac{\delta\mathcal{R}\sqrt{-g}}{\mathcal{R}^2}R + \frac{\delta\sqrt{-g}}{\mathcal{R}}R\right) + \left(\frac{\delta L\sqrt{-g}}{\mathcal{L}} - \frac{\delta\mathcal{L}\sqrt{-g}}{\mathcal{L}^2}L + \frac{\delta\sqrt{-g}}{\mathcal{L}}L\right)\right] d^4\sigma \tag{3}$$

By utilizing Jacobi's formula: $\delta\sqrt{-g} = -\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}/2$ [19]. Hence, the variation is

$$\delta S = \int E_D \left[\begin{array}{l}\left(\dfrac{R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}}{\mathcal{R}} - \dfrac{\mathcal{R}_{\mu\nu}\delta\tilde{g}^{\mu\nu} + \tilde{g}^{\mu\nu}\delta\mathcal{R}_{\mu\nu}}{\mathcal{R}^2}R - \dfrac{g_{\mu\nu}\delta g^{\mu\nu}}{2\mathcal{R}}R\right) + \\ \left(\dfrac{L_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta L_{\mu\nu}}{\mathcal{L}} - \dfrac{\mathcal{L}_{\mu\nu}\delta\tilde{g}^{\mu\nu} + \tilde{g}^{\mu\nu}\delta\mathcal{L}_{\mu\nu}}{\mathcal{L}^2}L - \dfrac{g_{\mu\nu}\delta g^{\mu\nu}}{2\mathcal{L}}L\right)\end{array}\right]\sqrt{-g}\, d^4\sigma \tag{4}$$

where the Lagrangian density is handled as a tensor, thus, $\delta\mathcal{L} = \mathcal{L}_{\mu\nu}\delta\tilde{g}^{\mu\nu} + \tilde{g}^{\mu\nu}\delta\mathcal{L}_{\mu\nu}$.

By considering the first boundary term, $\int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}\, d^4\sigma$, the variation in the Ricci curvature tensor δR_{uv} can be expressed in terms of the covariant derivative of the difference between two Levi-Civita connections, the Palatini identity: $\delta R_{\mu\nu} = \nabla_\rho\left(\delta\Gamma^\rho{}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^\rho{}_{\rho\mu}\right)$, where this variation with respect to the inverse metric $g^{\mu\nu}$ can be obtained by using the metric compatibility of the covariant derivative, $\nabla_\rho g^{\mu\nu} = 0$ [19], as $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\rho\left(g^{\mu\nu}\delta\Gamma^\rho_{\nu\mu} - g^{\mu\rho}\delta\Gamma^\sigma_{\sigma\mu}\right)$. Thus, the first boundary term as a total derivative for any tensor density is transformed based on Stokes' theorem with renaming the dummy indices as

$$\begin{aligned}\frac{E_D}{\mathcal{R}}\int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}\, d^4\sigma &= \frac{E_D}{\mathcal{R}}\int \nabla_\rho\left(g^{\mu\nu}\delta\Gamma^\rho_{\nu\mu} - g^{\mu\rho}\delta\Gamma^\sigma_{\sigma\mu}\right)\sqrt{-g}\, d^4\sigma \\ &\equiv \frac{E_D}{\mathcal{R}}\int_{\mathcal{M}} \nabla_\mu A^\mu\sqrt{-g}\, d^4\sigma = \frac{E_D}{\mathcal{R}}\int_{\partial\mathcal{M}} K\epsilon\sqrt{|q|}\, d^3\rho\end{aligned} \tag{5}$$

where the non-boundary term $\mathrm{E_D}/\mathcal{R}$ is left outside the integral transformation as it only

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acts as a scalar to the integral is called S_{GHY} [20,21]. The same is applied to the bulk and the Lagrangian boundary terms. Accordingly, the variation in the action is expressed as

$$\delta S=\int E_D\begin{bmatrix}\left(\frac{R_{\mu\nu}\delta g^{\mu\nu}}{\mathcal{R}}-\frac{\mathcal{R}_{\mu\nu}\delta\tilde{g}^{\mu\nu}}{\mathcal{R}^2}R-\frac{g_{\mu\nu}\delta g^{\mu\nu}}{2\mathcal{R}}R\right)+\\ \left(\frac{L_{\mu\nu}\delta g^{\mu\nu}}{\mathcal{L}}-\frac{\mathcal{L}_{\mu\nu}\delta\tilde{g}^{\mu\nu}}{\mathcal{L}^2}L-\frac{g_{\mu\nu}\delta g^{\mu\nu}}{2\mathcal{L}}L\right)\end{bmatrix}\sqrt{-g}\,d^4\sigma+\int E_D\begin{bmatrix}\epsilon\left(\frac{K\sqrt{|q|}}{\mathcal{R}}-\frac{\mathcal{K}\sqrt{|\mathcal{q}|}}{\mathcal{R}^2}R\right)+\\ \epsilon\left(\frac{T\sqrt{|q|}}{\mathcal{L}}-\frac{\mathcal{T}\sqrt{|\mathcal{q}|}}{\mathcal{L}^2}L\right)\end{bmatrix}d^3\rho \quad (6)$$

where K and $\mathcal{K}$ are the traces of the brane and the bulk extrinsic curvatures, T and $\mathcal{T}$ are the extrinsic traces of the Lagrangian density on the brane and the bulk boundaries, q and $\mathcal{q}$ are the determinants of their induced metrics respectively, and ϵ equals 1 when the normal $\hat{n}_u$ is a spacelike entity and equals -1 when it is a timelike entity. It is worth noting that all the terms are satisfying the criteria that the variation in the action δS is with respect to the variation in the inverse metric $\delta g^{\mu\nu}$ excluding the boundary terms that still lack this feature. Thus, to achieve the consistency of the action, the variation in the boundary action has to be determined. The variation in the brane boundary term is

$$\frac{E_D\epsilon}{\mathcal{R}}\int\left(K_{\mu\nu}\delta q^{\mu\nu}+q^{\mu\nu}\delta K_{\mu\nu}+K\frac{\delta\sqrt{|q|}}{\sqrt{|q|}}\right)\sqrt{|q|}\,d^3\rho \quad (7)$$

where $K=K_{\mu\nu}q^{\mu\nu}$. The non-boundary term $E_D/\mathcal{R}$ is left outside, where it can be considered as a scalar. Otherwise, its high-order variational terms can be incorporated into the conformal transformation function Ω^2 as follows. By utilising Jacobi's formula for the determinant differentiation; thus, $\delta\sqrt{|q|}=-\sqrt{|q|}\,q_{\mu\nu}\delta q^{\mu\nu}/2$ and by utilising the variation in the $q^{\mu\nu}q_{\mu\nu}=\delta^{\mu}_{\nu}$ as $q^{\mu\nu}=-q_{\mu\nu}\delta q^{\mu\nu}/\delta q_{\mu\nu}$; thus, the brane boundary term is

$$\frac{E_D\epsilon}{\mathcal{R}}\int\left(K_{\mu\nu}\delta q^{\mu\nu}-\frac{1}{2}K\left(q_{\mu\nu}\delta q^{\mu\nu}+2q_{\mu\nu}\frac{\delta K_{\mu\nu}}{\delta q_{\mu\nu}K}\delta q^{\mu\nu}\right)\right)\sqrt{|q|}\,d^3\rho \quad (8)$$

here $\delta K_{\mu\nu}/\delta q_{\mu\nu}K=(\delta K_{\mu\nu}/K_{\mu\nu})(q_{\mu\nu}/\delta q_{\mu\nu})=\delta\ln K_{\mu\nu}/\delta\ln q_{\mu\nu}$ resembles the Ricci flow in a normalised form reflecting the conformal distortion in the boundary, which can be expressed as a positive function Ω^2 based on Weyl's conformal transformation [18] as $\tilde{q}_{\mu\nu}=q_{\mu\nu}\Omega^2$. Thus, Equation (8) is expressed as $E_D\epsilon/\mathcal{R}\int(K_{\mu\nu}-K\hat{q}_{\mu\nu}/2)\delta q^{\mu\nu}\sqrt{|q|}\,d^3\rho$ where $\hat{q}_{\mu\nu}=q_{\mu\nu}+2\tilde{q}_{\mu\nu}$ denoting the conformally transformed induced metric tensor on the boundary whereas Einstein spaces are a subclass of the conformal space [17]. The same is applied for the bulk and Lagrangian boundary terms, thus, the variation in the action is

$$\delta S=\int\begin{matrix}\left[E_D\left(\frac{R_{\mu\nu}\delta g^{\mu\nu}}{\mathcal{R}}-\frac{\tilde{g}_{\mu\nu}\delta\tilde{g}^{\mu\nu}}{\mathcal{R}}R-\frac{g_{\mu\nu}\delta g^{\mu\nu}}{2\mathcal{R}}R\right)+\frac{E_D}{\mathcal{L}}\left(L_{\mu\nu}\delta g^{\mu\nu}-L\left(\tilde{g}_{\mu\nu}\delta\tilde{g}^{\mu\nu}-\frac{g_{\mu\nu}\delta g^{\mu\nu}}{2}\right)\right)\right]\sqrt{-g}\,d^4\sigma+\\ \left[E_D\epsilon\frac{K_{\mu\nu}-\frac{K}{2}\hat{q}_{\mu\nu}}{\mathcal{R}}\right]\delta q^{\mu\nu}\sqrt{|q|}d^3\rho-\left[E_D\epsilon\frac{\mathcal{K}_{\mu\nu}-\frac{\mathcal{K}}{2}\hat{\mathcal{q}}_{\mu\nu}}{\mathcal{R}^2}R\right]\delta\mathcal{q}^{\mu\nu}\sqrt{|\mathcal{q}|}d^3\rho+\frac{E_D\epsilon}{\mathcal{L}}\left[\frac{\delta L_b}{\delta q^{\mu\nu}}\right]\delta q^{\mu\nu}\sqrt{|q|}d^3\rho\end{matrix} \quad (9)$$

where $\tilde{g}_{\mu\nu}=\mathcal{R}_{\mu\nu}/\mathcal{R}=\mathcal{R}_{\mu\nu}/\mathcal{R}_{\mu\nu}\tilde{g}^{\mu\nu}$ is the bulk metric and $\delta L_b/\delta q^{\mu\nu}$ denotes Lagrangian density variation on the bulk and brane boundaries. According to the elasticity theory, E_D is in terms of energy density and represents the resistance of the continuum (spacetime continuum) to deformation [16]. Equation (1) states the modulus $E_D=\mathcal{R}c^4/8\pi G_t$ that is proportional to the fourth-power of the speed of light that in turn is directly proportional to the frequency; which can be in accordance with the frequency cut-off predictions of the vacuum energy density in the quantum field theory [22,23]; therefore, E_D characterizes spacetime (bulk) resistance to curvature and can represent vacuum energy density.

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An extended stress-energy tensor, $\bar{T}_{\mu\nu}$, can be defined as proportional to the Lagrangian term as $\bar{T}_{\mu\nu} := \hat{g}_{\mu\nu}L/2 - L_{\mu\nu} - \delta L_b/\delta q^{\mu\nu}$ by considering the new Lagrangian density variation on the boundaries, $\delta L_b/\delta q^{\mu\nu}$, and the conformally transformed metric, $\hat{g}_{\mu\nu} = g_{\mu\nu} + 2\tilde{g}_{\mu\nu}$, while regarding $E_D/\mathcal{L} \equiv 1$ as the variation in the bulk Lagrangian density $\mathcal{L}$ from its constant energy density E_D is insignificant due to its globality. By choosing ϵ as a timelike entity and applying the principle of stationary action yields

$$\frac{R_{\mu\nu}}{\mathcal{R}} - \frac{1}{2}\frac{R}{\mathcal{R}}g_{\mu\nu} - \frac{\mathcal{R}_{\mu\nu}}{\mathcal{R}^2}R + \frac{R\left(\mathcal{K}_{\mu\nu} - \frac{1}{2}\mathcal{K}\hat{q}_{\mu\nu}\right) - \mathcal{R}\left(K_{\mu\nu} - \frac{1}{2}K\hat{q}_{\mu\nu}\right)}{\mathcal{R}^2} = \frac{\bar{T}_{\mu\nu}}{E_D} \tag{10}$$

The extended field equations can be interpreted as indicating that the induced curvature over the bulk (background or pre-existing) curvature equals the ratio of the imposed energy density and its flux to the vacuum energy density and its flux throughout the expanding/contracting Universe. Since Equation (1) states $E_D = \mathcal{R}c^4/8\pi G_t$ which shows an inverse proportionality between the gravitational 'constant' G_t and r_t as the global Universe radius of curvature where $\mathcal{R} = 1/r_r^2$. Thus, G_t follows the inverse square law with respect to the Universe radius of curvature. The evolution in G_t is preferred to reduce the conflict of matter power spectrum amplitude with Planck datasets [24–26] while the decrease in star formation rate over the Universe age [27] can be due to the decrease in G_t over cosmic time. Experimental measurements of G_t suggested its change over time [28] while the gradual evolution in the fine structure 'constant' [29–31] could reveal that the presumed fundamental constants could rely on other Universe properties. By substituting Equation (1) to Equation (10), the field equations are simplified to

$$R_{\mu\nu} - \frac{1}{2}R\hat{g}_{\mu\nu} + \frac{R\left(\mathcal{K}_{\mu\nu} - \frac{1}{2}\mathcal{K}\hat{q}_{\mu\nu}\right) - \mathcal{R}\left(K_{\mu\nu} - \frac{1}{2}K\hat{q}_{\mu\nu}\right)}{\mathcal{R}} = \frac{8\pi G_t}{c^4}\bar{T}_{\mu\nu} \tag{11}$$

The conformally transformed metric accounts for the bulk curvature evolution over cosmic time where the conformal transformation could describe the tidal distortion and gravitational waves in the absence of matter [32]. The new boundary term is only significant at high-energy limits such as within black holes [20]. The evolution in G_t can accommodate the bulk or background curvature evolution over cosmic time against constant G for a special flat spacetime case. The boundary term comprising $\mathcal{K}_{\mu\nu}$ accounts for the extrinsic curvature of the bulk while the term comprising $K_{\mu\nu}$ accounts for the extrinsic curvature induced by local relativistic celestial objects. The extended field equations could remove singularities and satisfy a conformal invariance theory. This is because while the preferred length scale of any given mass changes proportionally to the change in the conformal bulk metric, the corresponding change in G_t accommodates the effective gravitational forces of the given mass.

3. Evolution of 4D Relativistic Branes travelling in the 4D Conformal Bulk

This section aims to visualize the evolution of the 4D relativistic branes over the conformal space-time of the 4D bulk. Galaxy formation and evolution as a 4D relativistic brane travelling throughout a 4D conformal bulk is considered. This scenario reveals the galaxy formation as a forced vortex due to the curved bulk background, which could resemble galaxy rotational curves. The mathematical derivations are presented in [33].

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Figure 1 shows the evolution of the metric tensor of the 4D brane through its travel in the conformal space-time of the 4D bulk.

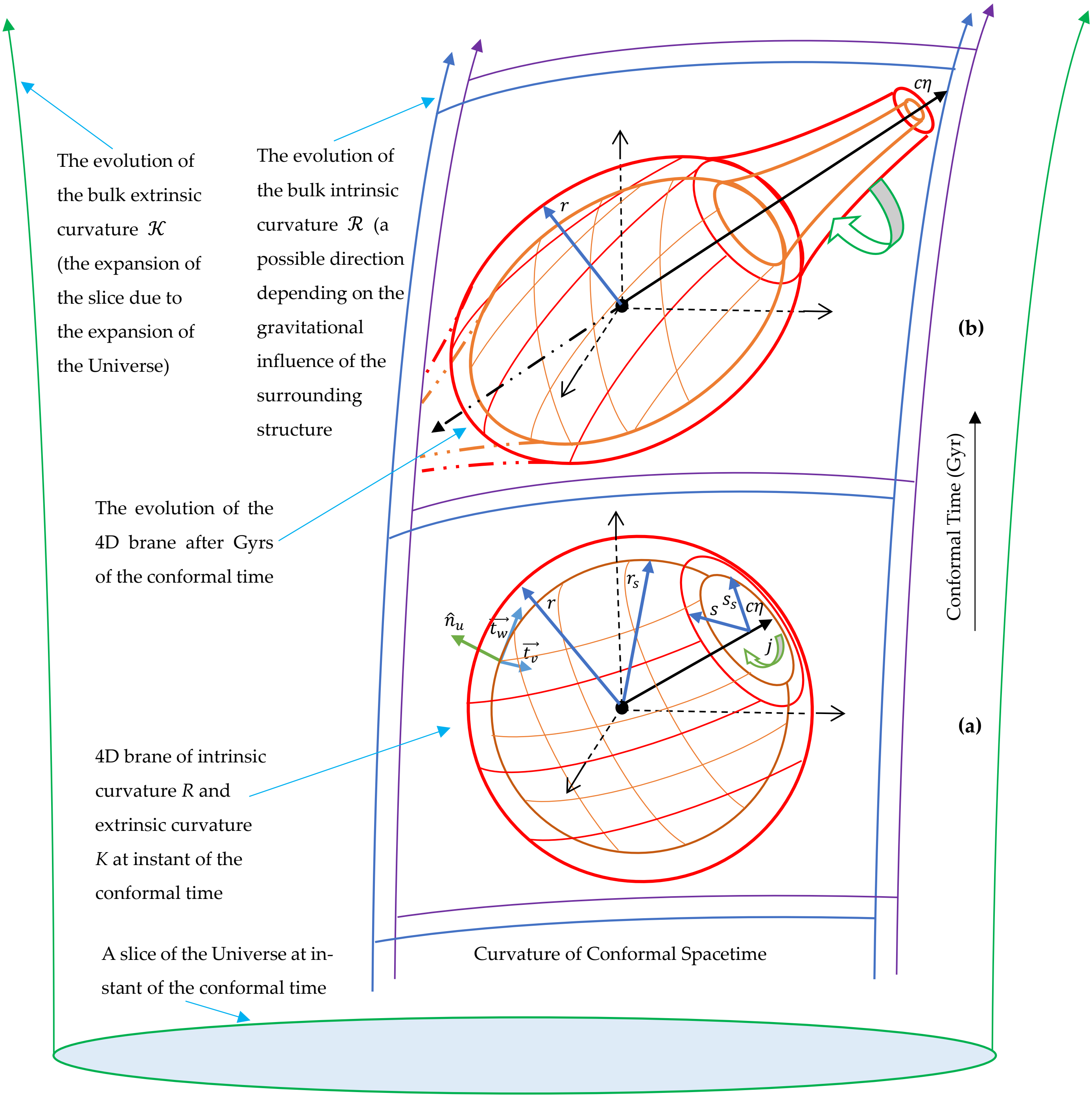

Figure 1. The hypersphere of a compact core of a galaxy (the red-orange 4D hypersphere/brane) along with its travel and spin through the conformal space-time (the blue-purple 4D bulk representing the independent background of distinctive curvature evolving over cosmic time.

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4. Conclusions and Future Works

This study has presented extended field equations in terms of brane-world modified gravity consisting of conformal Einstein field equations with a boundary term. The study has considered both the implied positive curvature of the early Universe and the curvature across the entire galaxy as the curvature of the background or the 4D bulk and distinguished it from the localized curvature that is induced in the bulk by the presence of comparably smaller celestial objects that are regarded as 'relativistic 4D branes', where branes in different galaxies experience different bulk curvatures; similarly, galaxies at different conformal time interval experience different bulk curvatures. Thus, the background or bulk curvature should be taken into consideration along with the energy densities of the branes when finding their induced curvatures.

The new boundary term is only significant at high-energy limits such as within black holes and it can remove the singularities from the theory. Additionally, since the preferred length scale of any given mass changes proportionally to the change in the conformal bulk metric, the corresponding change in G_t accommodates the effective gravitational forces of the given mass, which could satisfy a conformal invariance theory.

This work will be utilized to study the evolution of the Universe and the formation of galaxies and their rotational curves. Finally, this theoretical work will be tested using observational data in future works.

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