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Article

Entropy and Variational Formulation of Relativistic Fluid Dynamics

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Abstract

Simplifying the representation of a dynamical system allows us to identify its genuine degrees of freedom and, consequently, the types of instabilities it can exhibit. In this work, we extend earlier variational analysis of classical non-barotropic flows to the more general relativistic non-barotropic case. Specifically, we present a new Eulerian variational formulation for relativistic non-barotropic flows, based on six functions.

Keywords: non-barotropic flows; variational analysis; relativistic flows

1. Introduction

A detailed overview of the variational formulation of non-relativistic fluid dynamics and quantum mechanics, along with their profound interrelations, can be found in [1,2]; therefore, it will not be repeated here.

The pioneering studies by Clebsch [3,4] and the subsequent related works all consider non-relativistic fluids, where the flow velocity is much smaller than the speed of light in vacuum, c . This is naturally understandable, since Clebsch's research predates Einstein's theory of special relativity by forty-eight years. In practice, relativistic fluid flows are rarely observed under terrestrial conditions.

Here's a polished rephrasing of your paragraph in clear academic English:

The conventional treatment of relativistic flows relies on the energy–momentum tensor [7–9]. However, this approach lacks full rigor, since an energy–momentum tensor can be properly defined only when a Lagrangian density is available [10]. Until recently, no such Lagrangian density existed for relativistic flows. In my recent work [5], I extended Clebsch's formulation to the relativistic regime and thereby filled this gap by deriving a Lagrangian density for relativistic fluids, from which the corresponding energy–momentum tensor for high-velocity flows can be obtained in a consistent manner. Moreover, a link between relativistic fluid dynamics and the Dirac theory of the electron, established through the concept of Fisher information, was discussed in [6]. Nonetheless, earlier studies did not address non-barotropic Eulerian flows, focusing instead on connections to quantum mechanics—a limitation that the present work seeks to resolve.

The formulation of the variational principle for a relativistic charged classical particle interacting through a vector potential, as well as for a system of such particles, has been presented in [5] and will not be repeated here. Nonetheless, we will adopt the notation introduced in [5] whenever required.

We begin with Eckart's [11] Lagrangian variational principle, extended to describe a relativistic charged fluid. This is followed by a presentation of the general theory of the canonical energy–momentum tensor, from which we derive the tensor corresponding to the Eckart Lagrangian. Next, we introduce an Eulerian–Clebsch variational formulation for a non-barotropic relativistic charged fluid and obtain its associated energy–momentum tensor. Finally, we examine the cosmological implications of this tensor.

2. The Lagrangian Description of a Relativistic Charged Fluid

2.1. Action and Lagrangian

The dynamics of a flow are determined by its composition and the forces acting upon it. The fluid is considered to be made up of "fluid elements" [11,12]. Each fluid element may be viewed as a point-like entity with infinitesimal mass $dM_{\vec{\alpha}}$, charge $dQ_{\vec{\alpha}}$, position four-vector $x_{\vec{\alpha}\nu}(\tau_{\vec{\alpha}})$, and four-velocity $u_{\vec{\alpha}\nu}(\tau_{\vec{\alpha}}) \equiv \frac{dx_{\vec{\alpha}\nu}(\tau_{\vec{\alpha}})}{d\tau_{\vec{\alpha}}}$. Here, the continuous label $\vec{\alpha}$ replaces the discrete index n used in the particle-based description (see Section 2 of [5]). However, a fluid element is not a true point particle, as it possesses an infinitesimal volume $dV_{\vec{\alpha}}$, entropy $dS_{\vec{\alpha}}$, and internal energy $dE_{in,\vec{\alpha}}$. The action for each fluid element, following equation (1) of [5], takes the form:

$$\begin{aligned} d\mathcal{A}_{\vec{\alpha}} &= -dM_{\vec{\alpha}}c \int d\tau_{\vec{\alpha}} - dQ_{\vec{\alpha}} \int A^{\mu}(x_{\vec{\alpha}}^{\nu})dx_{\mu\vec{\alpha}} + d\mathcal{A}_{in\vec{\alpha}}, \\ d\mathcal{A}_{in\vec{\alpha}} &\equiv - \int dE_{in\vec{\alpha}}dt. \end{aligned} \quad (1)$$

The Lagrangian corresponding to each "fluid element" can be obtained from the above expression in the following way:

$$\begin{aligned} d\mathcal{A}_{\vec{\alpha}} &= \int_{t1}^{t2} dL_{\vec{\alpha}}dt, \quad dL_{\vec{\alpha}} \equiv dL_{k\vec{\alpha}} + dL_{i\vec{\alpha}} - dE_{in\vec{\alpha}} \\ dL_{k\vec{\alpha}} &\equiv -\frac{dM_{\vec{\alpha}}c^2}{\gamma_{\vec{\alpha}}} \simeq \frac{1}{2}dM_{\vec{\alpha}}v_{\vec{\alpha}}(t)^2 - dM_{\vec{\alpha}}c^2 \\ dL_{i\vec{\alpha}} &\equiv dQ_{\vec{\alpha}}\left(\vec{A}(\vec{x}_{\vec{\alpha}}(t),t) \cdot \vec{v}_{\vec{\alpha}}(t) - \phi(\vec{x}_{\vec{\alpha}}(t),t)\right). \end{aligned} \quad (2)$$

The Action and Lagrangian of the entire fluid are obtained by integrating over all possible values of $\vec{\alpha}$:

$$\begin{aligned} L &= \int_{\vec{\alpha}} dL_{\vec{\alpha}} \\ \mathcal{A} &= \int_{\vec{\alpha}} d\mathcal{A}_{\vec{\alpha}} = \int_{\vec{\alpha}} \int_{t1}^{t2} dL_{\vec{\alpha}}dt = \int_{t1}^{t2} \int_{\vec{\alpha}} dL_{\vec{\alpha}}dt = \int_{t1}^{t2} Ldt. \end{aligned} \quad (3)$$

A density is defined by dividing the quantity of a fluid element by its corresponding volume. This procedure is applied to the Lagrangian, mass, charge, and internal energy of each fluid element, using the following notation:

$$\mathcal{L}_{\vec{\alpha}} \equiv \frac{dL_{\vec{\alpha}}}{dV_{\vec{\alpha}}}, \quad \rho_{\vec{\alpha}} \equiv \frac{dM_{\vec{\alpha}}}{dV_{\vec{\alpha}}}, \quad \rho_{c\vec{\alpha}} \equiv \frac{dQ_{\vec{\alpha}}}{dV_{\vec{\alpha}}}, \quad e_{in\vec{\alpha}} \equiv \frac{dE_{in\vec{\alpha}}}{dV_{\vec{\alpha}}} \quad (4)$$

Each density-type quantity depends on the position $\vec{x}$ of the fluid element labeled $\vec{\alpha}$ at time t . For example:

$$\rho(\vec{x},t) \equiv \rho(\vec{x}_{\vec{\alpha}}(t),t) \equiv \rho_{\vec{\alpha}}(t). \quad (5)$$

We further define specific quantities by dividing a property of a fluid element by its mass. For instance, the specific internal energy $\varepsilon_{\vec{\alpha}}$ is given by:

$$\varepsilon_{\vec{\alpha}} \equiv \frac{dE_{in\vec{\alpha}}}{dM_{\vec{\alpha}}} \Rightarrow \rho_{\vec{\alpha}}\varepsilon_{\vec{\alpha}} = \frac{dM_{\vec{\alpha}}}{dV_{\vec{\alpha}}} \frac{dE_{in\vec{\alpha}}}{dM_{\vec{\alpha}}} = \frac{dE_{in\vec{\alpha}}}{dV_{\vec{\alpha}}} = e_{in\vec{\alpha}} \quad (6)$$

Therefore, the Lagrangian density can be decomposed as follows:

$$\begin{aligned} \mathcal{L}_{\vec{\alpha}} &= \frac{dL_{\vec{\alpha}}}{dV_{\vec{\alpha}}} = \frac{dL_{k\vec{\alpha}}}{dV_{\vec{\alpha}}} + \frac{dL_{i\vec{\alpha}}}{dV_{\vec{\alpha}}} - \frac{dE_{in\vec{\alpha}}}{dV_{\vec{\alpha}}} = \mathcal{L}_{k\vec{\alpha}} + \mathcal{L}_{i\vec{\alpha}} - e_{in\vec{\alpha}} \\ \mathcal{L}_{k\vec{\alpha}} &\equiv -\frac{\rho_{\vec{\alpha}}c^2}{\gamma_{\vec{\alpha}}} \simeq \frac{1}{2}\rho_{\vec{\alpha}}v_{\vec{\alpha}}(t)^2 - \rho_{\vec{\alpha}}c^2, \\ \mathcal{L}_{i\vec{\alpha}} &\equiv \rho_{c\vec{\alpha}}\left(\vec{A}(\vec{x}_{\vec{\alpha}}(t),t) \cdot \vec{v}_{\vec{\alpha}}(t) - \phi(\vec{x}_{\vec{\alpha}}(t),t)\right). \end{aligned} \quad (7)$$

The fluid Lagrangian can therefore be expressed as a spatial integral:

$$L = \int_{\vec{\alpha}} dL_{\vec{\alpha}} = \int_{\vec{\alpha}} \mathcal{L}_{\vec{\alpha}} dV_{\vec{\alpha}} = \int \mathcal{L}(\vec{x}, t) d^3x. \quad (8)$$

This expression will be utilized in a later section dealing with the Eulerian representation of the fluid.

2.2. Variational Analysis

We now define the notation $\Delta\vec{x}(\vec{\alpha}) \equiv \vec{\xi}(\vec{\alpha})$ to represent a variation in the trajectory $\vec{x}_{\vec{\alpha}}(t)$. Thus:

$$\Delta\vec{v}_{\vec{\alpha}}(t) = \Delta\frac{d\vec{x}_{\vec{\alpha}}(t)}{dt} = \frac{d\Delta\vec{x}_{\vec{\alpha}}(t)}{dt} = \frac{d\vec{\xi}_{\vec{\alpha}}(t)}{dt}. \quad (9)$$

And thus according to equation (9) of [5]:

$$\Delta\left(\frac{1}{\gamma_{\vec{\alpha}}}\right) = -\frac{\gamma_{\vec{\alpha}}\vec{v}_{\vec{\alpha}}(t)}{c^2} \frac{d\vec{\xi}_{\vec{\alpha}}(t)}{dt}, \quad \Delta\gamma_{\vec{\alpha}} = \frac{\gamma_{\vec{\alpha}}^3\vec{v}_{\vec{\alpha}}(t)}{c^2} \frac{d\vec{\xi}_{\vec{\alpha}}(t)}{dt}. \quad (10)$$

An ideal fluid is characterized by the property that each fluid element does not exchange mass, electric charge, or heat with neighboring elements. In variational form, this condition can be expressed as:

$$\Delta dM_{\vec{\alpha}} = \Delta dQ_{\vec{\alpha}} = \Delta dS_{\vec{\alpha}} = 0. \quad (11)$$

From the perspective of thermodynamics, a change in the internal energy of a fluid element obeys the following relation in the rest frame of the element:

$$\Delta dE_{in \vec{\alpha}0} = T_{\vec{\alpha}0}\Delta dS_{\vec{\alpha}0} - P_{\vec{\alpha}0}\Delta dV_{\vec{\alpha}0}. \quad (12)$$

In this expression, the first term represents the heating of the fluid element, while the second term corresponds to the work performed by the element on its neighbors. Here, $T_{\vec{\alpha}0}$ is the temperature of the fluid element in its rest frame, and $P_{\vec{\alpha}0}$ is its pressure. Since the rest mass of the fluid element remains constant and is independent of the reference frame, we can divide the equation by $dM_{\vec{\alpha}}$ to obtain the variation of the specific energy:

$$\begin{aligned} \Delta\varepsilon_{\vec{\alpha}0} &= \Delta\frac{dE_{in \vec{\alpha}0}}{dM_{\vec{\alpha}}} = T_{\vec{\alpha}0}\Delta\frac{dS_{\vec{\alpha}0}}{dM_{\vec{\alpha}}} - P_{\vec{\alpha}0}\Delta\frac{dV_{\vec{\alpha}0}}{dM_{\vec{\alpha}}} \\ &= T_{\vec{\alpha}0}\Delta s_{\vec{\alpha}0} - P_{\vec{\alpha}0}\Delta\frac{1}{\rho_{\vec{\alpha}0}} = T_{\vec{\alpha}0}\Delta s_{\vec{\alpha}0} + \frac{P_{\vec{\alpha}0}}{\rho_{\vec{\alpha}0}^2}\Delta\rho_{\vec{\alpha}0}. \quad s_{\vec{\alpha}0} \equiv \frac{dS_{\vec{\alpha}0}}{dM_{\vec{\alpha}}} \end{aligned} \quad (13)$$

$s_{\vec{\alpha}0}$ is the specific entropy of the fluid element in its rest frame. It then follows that (we omit the index $\vec{\alpha}$ below):

$$\frac{\partial\varepsilon_0}{\partial s_0} = T_0, \quad \frac{\partial\varepsilon_0}{\partial\rho_0} = \frac{P_0}{\rho_0^2}. \quad (14)$$

Another important thermodynamic quantity is the enthalpy, defined for a fluid element in its rest frame as:

$$dW_{\vec{\alpha}0} = dE_{in \vec{\alpha}0} + P_{\vec{\alpha}0}dV_{\vec{\alpha}0}. \quad (15)$$

and the specific enthalpy:

$$w_{\vec{\alpha}0} = \frac{dW_{\vec{\alpha}0}}{dM_{\vec{\alpha}}} = \frac{dE_{in \vec{\alpha}0}}{dM_{\vec{\alpha}}} + P_{\vec{\alpha}0}\frac{dV_{\vec{\alpha}0}}{dM_{\vec{\alpha}}} = \varepsilon_{\vec{\alpha}0} + \frac{P_{\vec{\alpha}0}}{\rho_{\vec{\alpha}0}}. \quad (16)$$

By combining the above with Equation (14), we arrive at the following useful relations:

$$w_0 = \varepsilon_0 + \frac{P_0}{\rho_0} = \varepsilon_0 + \rho_0 \frac{\partial\varepsilon_0}{\partial\rho_0} = \frac{\partial(\rho_0\varepsilon_0)}{\partial\rho_0}. \quad (17)$$

$$\frac{\partial w_0}{\partial \rho_0} = \frac{\partial(\varepsilon_0 + \frac{P_0}{\rho_0})}{\partial \rho_0} = -\frac{P_0}{\rho_0^2} + \frac{1}{\rho_0} \frac{\partial P_0}{\partial \rho_0} + \frac{\partial \varepsilon_0}{\partial \rho_0} = -\frac{P_0}{\rho_0^2} + \frac{1}{\rho_0} \frac{\partial P_0}{\partial \rho_0} + \frac{P_0}{\rho_0^2} = \frac{1}{\rho_0} \frac{\partial P_0}{\partial \rho_0}. \quad (18)$$

$$\vec{\nabla} w_0 = T_0 \vec{\nabla} s_0 + \frac{1}{\rho_0} \vec{\nabla} P_0. \quad (19)$$

For an ideal fluid, heat conduction and radiation are neglected, so only convection is taken into account. Therefore, $\Delta dS_{\vec{\alpha}0} = 0$ and it follows that:

$$\Delta dE_{in \vec{\alpha}0} = -P_0 \Delta dV_{\vec{\alpha}0}. \quad (20)$$

We now establish some relationships between the rest frame of a fluid element and any other frame in which the element is moving (often referred to as the "laboratory" frame). First, we note that in the rest frame the velocity is zero (by definition), so according to equation (9) of [5]:

$$d\tau = c dt_0 = c dt \sqrt{1 - \frac{v^2}{c^2}} = \frac{c dt}{\gamma} \Rightarrow dt_0 = \frac{dt}{\gamma}. \quad (21)$$

It is a well-known fact that the four-volume is Lorentz invariant, so that:

$$dV_0 dt_0 = dV dt = dV dt_0 \gamma, \quad \Rightarrow dV_0 = \gamma dV. \quad (22)$$

Thus:

$$\rho_0 = \frac{dM}{dV_0} = \frac{1}{\gamma} \frac{dM}{dV} = \frac{\rho}{\gamma}, \quad \Rightarrow \rho = \gamma \rho_0. \quad (23)$$

Furthermore, the action given in Equation (1) is Lorentz invariant, which implies:

$$\begin{aligned} dE_{in \vec{\alpha}0} dt_0 &= dE_{in \vec{\alpha}} dt = dE_{in \vec{\alpha}} dt_0 \gamma \\ \Rightarrow dE_{in \vec{\alpha}0} &= \gamma dE_{in \vec{\alpha}}, \quad dE_{in \vec{\alpha}} = \frac{dE_{in \vec{\alpha}0}}{\gamma} \end{aligned} \quad (24)$$

We can now consider a variation of the internal energy of a fluid element:

$$\Delta dE_{in \vec{\alpha}} = \Delta \left(\frac{1}{\gamma} \right) dE_{in \vec{\alpha}0} + \frac{1}{\gamma} \Delta dE_{in \vec{\alpha}0}. \quad (25)$$

Thus according to Equation (20) and Equation (22) it follows that:

$$\Delta dE_{in \vec{\alpha}} = \Delta \left(\frac{1}{\gamma} \right) dE_{in \vec{\alpha}0} - \frac{1}{\gamma} P_0 \Delta dV_{\vec{\alpha}0} = \Delta \left(\frac{1}{\gamma} \right) dE_{in \vec{\alpha}0} - \frac{1}{\gamma} P_0 \Delta (\gamma dV_{\vec{\alpha}}). \quad (26)$$

Taking into account the enthalpy quantity defined in Equation (15) it follows that:

$$\Delta dE_{in \vec{\alpha}} = \Delta \left(\frac{1}{\gamma} \right) (dE_{in \vec{\alpha}0} + P_0 dV_{\vec{\alpha}0}) - P_0 \Delta dV_{\vec{\alpha}} = \Delta \left(\frac{1}{\gamma} \right) dW_{\vec{\alpha}0} - P_0 \Delta dV_{\vec{\alpha}}. \quad (27)$$

We shall now vary the fluid element's volume. At some specific time t the fluid elements volume is:

$$dV_{\vec{\alpha},t} = d^3 x(\vec{\alpha}, t) \quad (28)$$

The Jacobian relates the fluid element current volume to its volume at $t = 0$:

$$d^3 x(\vec{\alpha}, t) = J d^3 x(\vec{\alpha}, 0), \quad J \equiv \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) \quad (29)$$

$\vec{\nabla}_0$ is calculated with respect to the fluid element's coordinates at $t = 0$:

$$\vec{\nabla}_0 \equiv \left(\frac{\partial}{\partial x(\vec{\alpha}, 0)_1}, \frac{\partial}{\partial x(\vec{\alpha}, 0)_2}, \frac{\partial}{\partial x(\vec{\alpha}, 0)_3} \right). \quad (30)$$

An interesting observation is that building a spatial volume element from three vectors, using a Jacobian as in Equation (29), closely resembles the construction of a four-dimensional volume element from four scalars, where one also considers the Jacobian of the transformation from space-time coordinates to scalar field space, as was done long ago in: [17–20]. Thus:

$$\begin{aligned} \Delta dV_{\vec{\alpha}, t} &= \Delta d^3 x(\vec{\alpha}, t) = \Delta J d^3 x(\vec{\alpha}, 0) = \frac{\Delta J}{J} d^3 x(\vec{\alpha}, t) = \frac{\Delta J}{J} dV_{\vec{\alpha}, t}, \\ (\Delta d^3 x(\vec{\alpha}, 0) &= 0). \end{aligned} \quad (31)$$

We calculate the variation of J as follows:

$$\begin{aligned} \Delta J &= \vec{\nabla}_0 \Delta x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) + \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 \Delta x_2 \times \vec{\nabla}_0 x_3) \\ &+ \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 \Delta x_3), \end{aligned} \quad (32)$$

Thus:

$$\begin{aligned} \vec{\nabla}_0 \Delta x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) &= \vec{\nabla}_0 \xi_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) \\ &= \partial_k \xi_1 \vec{\nabla}_0 x_k \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) = \partial_1 \xi_1 \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) = \partial_1 \xi_1 J, \\ \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 \Delta x_2 \times \vec{\nabla}_0 x_3) &= \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 \xi_2 \times \vec{\nabla}_0 x_3) \\ &= \partial_k \xi_2 \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_k \times \vec{\nabla}_0 x_3) = \partial_2 \xi_2 \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) = \partial_2 \xi_2 J, \\ \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 \Delta x_3) &= \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 \xi_3) \\ &= \partial_k \xi_3 \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_k) = \partial_3 \xi_3 \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) = \partial_3 \xi_3 J. \end{aligned} \quad (33)$$

Which leads to:

$$\Delta J = \partial_1 \xi_1 J + \partial_2 \xi_2 J + \partial_3 \xi_3 J = \vec{\nabla} \cdot \vec{\xi} J, \quad \Delta dV_{\vec{\alpha}, t} = \vec{\nabla} \cdot \vec{\xi} dV_{\vec{\alpha}, t}. \quad (34)$$

So we derive the complete variation of the internal energy given in Equation (27) in the form:

$$\Delta dE_{in \vec{\alpha}} = \Delta \left(\frac{1}{\gamma} \right) dW_{\vec{\alpha}0} - P_0 \vec{\nabla} \cdot \vec{\xi} dV_{\vec{\alpha}, t}. \quad (35)$$

If we consider also Equation (10) the following result follows:

$$\Delta dE_{in \vec{\alpha}} = -P_{\vec{\alpha}0} \vec{\nabla} \cdot \vec{\xi}_{\vec{\alpha}} dV_{\vec{\alpha}, t} - \frac{\gamma_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t)}{c^2} dW_{\vec{\alpha}0} \cdot \frac{d\vec{\xi}_{\vec{\alpha}}(t)}{dt}. \quad (36)$$

The variation of internal energy is the only new aspect compared to the calculation for a previously described system of particles; the remainder of the analysis is straightforward. By varying Equation (1), we therefore obtain:

$$\begin{aligned} \Delta d\mathcal{A}_{\vec{\alpha}} &= \int_{t1}^{t2} \Delta dL_{\vec{\alpha}} dt, \quad \Delta dL_{\vec{\alpha}} = \Delta dL_{k\vec{\alpha}} + \Delta dL_{i\vec{\alpha}} - \Delta dE_{in \vec{\alpha}} \\ \Delta dL_{k\vec{\alpha}} &= -dM_{\vec{\alpha}} c^2 \Delta \left(\frac{1}{\gamma_{\vec{\alpha}}} \right) = dM_{\vec{\alpha}} \gamma_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t) \cdot \frac{d\vec{\xi}_{\vec{\alpha}}(t)}{dt}, \\ \Delta dL_{i\vec{\alpha}} &= dQ_{\vec{\alpha}} \left(\Delta \vec{A}(\vec{x}_{\vec{\alpha}}(t), t) \cdot \vec{v}_{\vec{\alpha}}(t) + \vec{A}(\vec{x}_{\vec{\alpha}}(t), t) \cdot \Delta \vec{v}_{\vec{\alpha}}(t) \right. \\ &\quad \left. - \Delta \phi(\vec{x}_{\vec{\alpha}}(t), t) \right). \end{aligned} \quad (37)$$

Subtracting the internal and kinetic parts of the varied Lagrangian while taking into account the specific enthalpy defined in Equation (16), the following result is derived:

$$\Delta dL_{k\vec{\alpha}} - \Delta dE_{in \vec{\alpha}} = dM_{\vec{\alpha}} \gamma_{\vec{\alpha}} \left(1 + \frac{w_0}{c^2}\right) \vec{v}_{\vec{\alpha}}(t) \cdot \frac{d\vec{\zeta}_{\vec{\alpha}}(t)}{dt} + P_{\vec{\alpha}0} \vec{\nabla} \cdot \vec{\zeta}_{\vec{\alpha}} dV_{\vec{\alpha},t}. \quad (38)$$

The electromagnetic related variation terms are not different from equations A47 and A48 of [1], and their derivation will not be repeated here. Denoting the Lorentz force that is suffered by particle $\vec{\alpha}$ as follows:

$$d\vec{F}_{L\vec{\alpha}} \equiv dQ_{\vec{\alpha}} \left[\vec{v}_{\vec{\alpha}} \times \vec{B}(\vec{x}_{\vec{\alpha}}(t), t) + \vec{E}(\vec{x}_{\vec{\alpha}}(t), t) \right], \quad (39)$$

we obtain:

$$\Delta dL_{i\vec{\alpha}} = \frac{d(dQ_{\vec{\alpha}} \vec{A}(\vec{x}_{\vec{\alpha}}(t), t) \cdot \vec{\zeta}_{\vec{\alpha}})}{dt} + d\vec{F}_{L\vec{\alpha}} \cdot \vec{\zeta}_{\vec{\alpha}}. \quad (40)$$

Let us introduce the shorthand notation:

$$\bar{\lambda} \equiv 1 + \frac{w_0}{c^2}, \quad \lambda \equiv \gamma \bar{\lambda} = \gamma \left(1 + \frac{w_0}{c^2}\right). \quad (41)$$

Provided that the enthalpy of a fluid element in its frame of rest is minute with respect to its rest energy, that is:

$$dW_{\vec{\alpha}0} \ll dM_{\vec{\alpha}} c^2 \Rightarrow 1 \gg \frac{dW_{\vec{\alpha}0}}{dM_{\vec{\alpha}} c^2} = \frac{w_{\vec{\alpha}0}}{c^2}. \quad (42)$$

It follows that in the classical limit (with limitations on **both** the enthalpy of the fluid element and its speed) we will have:

$$\bar{\lambda} \simeq 1, \quad \lambda \simeq 1. \quad (43)$$

The varied Action of a single relativistic fluid element is thus:

$$\begin{aligned} \Delta d\mathcal{A}_{\vec{\alpha}} &= \int_{t1}^{t2} \Delta dL_{\vec{\alpha}} dt = (dM_{\vec{\alpha}} \lambda_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t) + dQ_{\vec{\alpha}} \vec{A}(\vec{x}_{\vec{\alpha}}(t), t)) \cdot \vec{\zeta}_{\vec{\alpha}} \Big|_{t1}^{t2} \\ &- \int_{t1}^{t2} (dM_{\vec{\alpha}} \frac{d(\lambda_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t))}{dt} \cdot \vec{\zeta}_{\vec{\alpha}} - d\vec{F}_{L\vec{\alpha}} \cdot \vec{\zeta}_{\vec{\alpha}} - P_{\vec{\alpha}0} \vec{\nabla} \cdot \vec{\zeta}_{\vec{\alpha}} dV_{\vec{\alpha},t}) dt. \end{aligned} \quad (44)$$

It follows that the variation of the entire relativistic fluid action can be integrated as follows:

$$\begin{aligned} \Delta \mathcal{A} &= \Delta \int_{\vec{\alpha}} d\mathcal{A}_{\vec{\alpha}} = \int_{\vec{\alpha}} (dM_{\vec{\alpha}} \lambda_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t) + dQ_{\vec{\alpha}} \vec{A}(\vec{x}(\vec{\alpha}, t), t)) \cdot \vec{\zeta}_{\vec{\alpha}} \Big|_{t1}^{t2} \\ &- \int_{t1}^{t2} \int_{\vec{\alpha}} (dM_{\vec{\alpha}} \frac{d(\lambda_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t))}{dt} \cdot \vec{\zeta}_{\vec{\alpha}} - d\vec{F}_{L\vec{\alpha}} \cdot \vec{\zeta}_{\vec{\alpha}} - P_{\vec{\alpha}0} \vec{\nabla} \cdot \vec{\zeta}_{\vec{\alpha}} dV_{\vec{\alpha}}) dt. \end{aligned} \quad (45)$$

From Equation (4) we obtain:

$$dM_{\vec{\alpha}} = \rho_{\vec{\alpha}} dV_{\vec{\alpha}}, \quad dQ_{\vec{\alpha}} = \rho_{c\vec{\alpha}} dV_{\vec{\alpha}} \quad (46)$$

using the above equality relations the $\vec{\alpha}$ integral becomes a volume integral and we are able to write the variation of the fluid action in the form (we omit $\vec{\alpha}$):

$$\Delta \mathcal{A} = \int (\rho \lambda \vec{v} + \rho_c \vec{A}) \cdot \vec{\zeta} dV \Big|_{t1}^{t2} - \int_{t1}^{t2} \int (\rho \frac{d(\lambda \vec{v})}{dt} \cdot \vec{\zeta} - \vec{f}_L \cdot \vec{\zeta} - P_0 \vec{\nabla} \cdot \vec{\zeta}) dV dt. \quad (47)$$

in which we use the following Lorentz force density definition:

$$\vec{f}_{L\vec{\alpha}} \equiv \frac{d\vec{F}_{L\vec{\alpha}}}{dV_{\vec{\alpha}}} = \rho_{c\vec{\alpha}} \left[\vec{v}_{\vec{\alpha}} \times \vec{B}(\vec{x}_{\vec{\alpha}}(t), t) + \vec{E}(\vec{x}_{\vec{\alpha}}(t), t) \right]. \quad (48)$$

Taking into account that:

$$P_0 \vec{\nabla} \cdot \vec{\xi} = \vec{\nabla} \cdot (P_0 \vec{\xi}) - \vec{\xi} \cdot \vec{\nabla} P_0, \quad (49)$$

and taking advantage of the theorem of Gauss, the action's variation takes the form:

$$\begin{aligned} \Delta \mathcal{A} &= \int (\rho \lambda \vec{v} + \rho_c \vec{A}) \cdot \vec{\xi} dV \Big|_{t_1}^{t_2} \\ &- \int_{t_1}^{t_2} \left[\int (\rho \frac{d(\lambda \vec{v})}{dt} - \vec{f}_L + \vec{\nabla} P_0) \cdot \vec{\xi} dV - \oint P_0 \vec{\xi} \cdot d\vec{\Sigma} \right] dt. \end{aligned} \quad (50)$$

It follows that if the variation of the action is zero for a displacement field $\vec{\xi}$ satisfying $\vec{\xi}(t_1) = \vec{\xi}(t_2) = 0$ and vanishing on the boundary surface enclosing the fluid, the Euler equation for a relativistic charged fluid holds, namely:

$$\frac{d(\lambda \vec{v})}{dt} = -\frac{\vec{\nabla} P_0}{\rho} + \frac{\vec{f}_L}{\rho}, \quad \frac{d\vec{v}}{dt} \simeq -\frac{\vec{\nabla} P_0}{\rho} + \frac{\vec{f}_L}{\rho}, \quad (51)$$

In the specific case where each fluid element consists of identical microscopic particles with mass m and charge e , the mass and charge densities are directly proportional to the particle number density n :

$$\rho = m n, \quad \rho_c = e n \Rightarrow \frac{\vec{f}_L}{\rho} = k [\vec{v} \times \vec{B} + \vec{E}], k \equiv \frac{e}{m} \quad (52)$$

Therefore, aside from the contributions associated with the internal energy, the equation has the same structure as that of a point particle. In the case of a neutral fluid, it takes the form:

$$\frac{d(\lambda \vec{v})}{dt} = -\frac{\vec{\nabla} P_0}{\rho}, \quad \frac{d\vec{v}}{dt} \simeq -\frac{\vec{\nabla} P_0}{\rho}. \quad (53)$$

Certain authors choose to express the preceding equation using the energy of each fluid element per unit volume in the rest frame, which combines the internal energy and the rest mass contributions:

$$e_0 \equiv \rho_0 c^2 + \rho_0 \varepsilon_0. \quad (54)$$

It is easy to show that:

$$\bar{\lambda} = 1 + \frac{w_0}{c^2} = \frac{e_0 + P_0}{\rho_0 c^2}. \quad (55)$$

By applying the above relation and performing a few algebraic steps, we can express Equation (53) in a form that some authors find more convenient:

$$(e_0 + P_0) \frac{\gamma}{c^2} \frac{d(\gamma \vec{v})}{dt} = -\vec{\nabla} P_0 - \frac{\gamma^2}{c^2} \frac{dP_0}{dt} \vec{v}. \quad (56)$$

In practical fluid dynamics, the fluid is characterized using local, measurable quantities rather than those associated with invisible infinitesimal "fluid elements." This approach is known as the Eulerian description, where the formulation is based on flow fields instead of following individual fluid elements, as will be explained later. Before introducing the Eulerian framework, it is useful to briefly discuss the canonical energy-momentum tensor in general, and its specific form within the relativistic Lagrangian description of fluid flow.

3. Canonical Energy-Momentum Tensor

Let us consider a general field of space-time of K components $\eta_a(x^\mu)$ and let us consider a Lagrangian density of the form:

$$\mathcal{L} = \mathcal{L}(\eta_a, \partial_\mu \eta_a), \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \quad (57)$$

Raising and lowering of indices will be done as usual with the Lorentz-Minkowski metric $\eta_{\mu\nu} \equiv \text{diag}(+1, -1, -1, -1)$. Provided we are given the action:

$$\mathcal{A} \equiv \int d^4x \mathcal{L} \quad (58)$$

it will take an extremum: $\delta\mathcal{A} = 0$ if $\eta_a(x^\mu)$ satisfies the Euler-Lagrange equations:

$$\frac{d}{dx_\nu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \eta_a)} \right) = \frac{\partial \mathcal{L}}{\partial \eta_a}, \quad (59)$$

in the above $\frac{d}{dx_\nu}$ is the total derivative with respect to the coordinate x_ν , that is one takes into account the dependence of η_a on the coordinates. Also Einstein summation convention is assumed. The above is correct provided that the appropriate initial and final time conditions as well as boundary conditions are satisfied.

Then one can define an energy momentum tensor of the form [10]:

$$T_{\mu\nu} \equiv \partial_\mu \eta_a \frac{\partial \mathcal{L}}{\partial(\partial_\nu \eta_a)} - \eta_{\mu\nu} \mathcal{L}. \quad (60)$$

In the above we assume Einstein summation convention with respect to the scalar field index a . This quantity will satisfy due to Euler-Lagrange equations (59) the following conservation laws:

$$\frac{d}{dx_\nu} T_{\mu\nu} = 0 \quad (61)$$

The quantities:

$$\bar{P}_\mu = \int d^3x T_{\mu 0} \quad (62)$$

are conserved in time, that is:

$$\frac{d\bar{P}_\mu}{dt} = 0. \quad (63)$$

$\bar{P}_0$ denotes the system's total energy and $\bar{P}_i$ denotes the components of the total linear momentum of the system $i \in [1, 2, 3]$.

For a Lagrangian fluid there is dependence of one coordinate t ($x^0 = ct$), however, the field indices are continuous (through the $\vec{\alpha}$ label) and therefore infinite. It follows that:

$$T_{L00} = \int_{\vec{\alpha}} \left[\frac{d\vec{x}_{\vec{\alpha}}(t)}{dt} \cdot \frac{\partial L}{\partial(\frac{d\vec{x}_{\vec{\alpha}}(t)}{dt})} \right] - L = \int_{\vec{\alpha}} \left[\vec{v}_{\vec{\alpha}} \cdot \frac{\partial L}{\partial \vec{v}_{\vec{\alpha}}} \right] - L. \quad (64)$$

Now according to Equation (3):

$$\frac{\partial L}{\partial \vec{v}_{\vec{\alpha}}} = \frac{\partial \int_{\vec{\alpha}'} dL_{\vec{\alpha}'}}{\partial \vec{v}_{\vec{\alpha}}} = \frac{\partial dL_{\vec{\alpha}}}{\partial \vec{v}_{\vec{\alpha}}}. \quad (65)$$

Thus it follows from Equation (2) that:

$$\frac{\partial L}{\partial \vec{v}_{\vec{\alpha}}} = \frac{\partial dL_{k\vec{\alpha}}}{\partial \vec{v}_{\vec{\alpha}}} + \frac{\partial dL_{i\vec{\alpha}}}{\partial \vec{v}_{\vec{\alpha}}} - \frac{\partial dE_{in \vec{\alpha}}}{\partial \vec{v}_{\vec{\alpha}}}. \quad (66)$$

Using the definitions of $dL_{k\vec{\alpha}}, dL_{i\vec{\alpha}}, dE_{in \vec{\alpha}}$ given in Equation (2), and the fact that $dE_{in \vec{\alpha}}$ is velocity dependent according to Equation (24) it follows that:

$$\frac{\partial L}{\partial \vec{v}_{\vec{\alpha}}} = dM_{\vec{\alpha}} \gamma_{\vec{\alpha}} \vec{v}_{\vec{\alpha}} + dQ_{\vec{\alpha}} \vec{A}_{\vec{\alpha}} + dE_{in \vec{\alpha}0} \frac{\gamma_{\vec{\alpha}}}{c^2} \vec{v}_{\vec{\alpha}}. \quad (67)$$

in which we have used the identity:

$$\frac{\partial \gamma_{\vec{\alpha}}^{-1}}{\partial \vec{v}_{\vec{\alpha}}} = -\frac{\gamma_{\vec{\alpha}}}{c^2} \vec{v}_{\vec{\alpha}}. \quad (68)$$

Plugging Equation (68) into Equation (64) will lead after some simplification to the form:

$$T_{L00} = \int_{\vec{\alpha}} \left[dM_{\vec{\alpha}} c^2 \gamma_{\vec{\alpha}} + dQ_{\vec{\alpha}} \phi(\vec{x}_{\vec{\alpha}}) + dE_{in \vec{\alpha}0} \gamma_{\vec{\alpha}} \right]. \quad (69)$$

This can be written as a volume integral:

$$T_{L00} = \int d^3x \left[\rho(c^2 + \varepsilon_0) \gamma + \rho_c \phi \right]. \quad (70)$$

The above expression denotes the fluid's total energy which is constant in time.

4. Action Principle of Eulerian Relativistic Charged Fluids

In what follows, we adopt the derivation presented in [1,2,13], but now we include the relativistic character of the flow. This requires constructing an action that remains invariant under Lorentz transformations. Let us therefore consider the action:

$$\begin{aligned} \mathcal{A} &\equiv \int \mathcal{L} d^3x dt, & \mathcal{L} &\equiv \mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_i \\ \mathcal{L}_0 &\equiv -\rho \left(\frac{c^2}{\gamma} + \varepsilon \right) = -\rho_0 (c^2 + \varepsilon_0) = -e_0, \\ \mathcal{L}_2 &\equiv v \partial^\nu (\rho_0 u_\nu) - \rho_0 \alpha u_\nu \partial^\nu \beta - \rho_0 \sigma u_\nu \partial^\nu s, \\ \mathcal{L}_i &\equiv -\rho_c A^\nu v_\nu, & v_\nu &\equiv \frac{dx_\nu}{dt}. \end{aligned} \quad (71)$$

We thus obtain the limit of small velocities as follows:

$$\mathcal{L}_0 \simeq \rho \left(\frac{1}{2} v^2 - \varepsilon - c^2 \right) \quad (72)$$

It is easy to show that:

$$u_\mu = \gamma(c, \vec{v}). \quad (73)$$

Taking into account Equation (23), we write the relativistic Lagrangian densities in a space-time formalism:

$$\mathcal{L}_2 = v \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) \right] - \rho \alpha \frac{d\beta}{dt} - \rho \sigma \frac{ds}{dt} \quad \mathcal{L}_i = \rho_c (\vec{A} \cdot \vec{v} - \phi). \quad (74)$$

In this formulation, the variational quantities are treated as space and time-dependent fields. These consist of the velocity vector field $\vec{v}(\vec{x}, t)$ and the scalar density field $\rho(\vec{x}, t)$. In this context, we employ the material derivative, which can be expressed as:

$$\frac{dg(\vec{\alpha}, t)}{dt} = \frac{dg(\vec{x}(\vec{\alpha}, t), t)}{dt} = \frac{\partial g}{\partial t} + \frac{d\vec{x}}{dt} \cdot \vec{\nabla} g = \frac{\partial g}{\partial t} + \vec{v} \cdot \vec{\nabla} g \quad (75)$$

Once g is treated as a function of $\vec{x}$ and t instead of a function of $\vec{\alpha}$ and t . Quantities that are straightforwardly conserved in the Lagrangian approach—such as labels, mass, charge, and entropy—can be incorporated into this formalism by introducing Lagrange multipliers ν , α , and σ to enforce the corresponding constraints:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) &= 0 \\ \frac{d\beta}{dt} &= 0 \\ \frac{ds}{dt} &= 0. \end{aligned} \quad (76)$$

Assuming $\rho \neq 0$, these conditions include the continuity equation, which guarantees mass conservation, the requirement that β acts as a label (i.e., a comoving quantity), and the conservation of the specific entropy s . Varying the action with respect to β and using the continuity equation (lagmulb) leads to an equation analogous to the derivation of Eqs. (67–69) in [2]:

$$\frac{d\alpha}{dt} = 0 \quad (77)$$

Thus, for $\rho \neq 0$, both α and β serve as labels. The specific internal energy ε_0 , defined in Equation (6), depends on the fluid's thermodynamic properties and is specified through an equation of state as a function of density and specific entropy. We assume that the entropy of a fluid element is Lorentz invariant, meaning $dS_{\vec{\alpha}0} = dS_{\vec{\alpha}}$, and therefore $s_{\vec{\alpha}0} = s_{\vec{\alpha}}$. Varying the action with respect to the specific entropy s then gives:

$$\frac{d\sigma}{dt} = \frac{T_0}{\gamma}, \quad (78)$$

assuming that the relevant boundary conditions are fulfilled. The electromagnetic potentials $\vec{A}$ and ϕ are treated as fixed functions of the coordinates and are not varied. We also assume that each fluid element consists of microscopic particles with mass m and charge e , so it follows from Equation (52) that:

$$\rho_c = k\rho. \quad (79)$$

Taking a variational derivative with respect to ρ , we derive:

$$\begin{aligned} \delta_\rho A &= \int d^3x dt \delta\rho \left[-\frac{c^2}{\gamma} - w_0 \frac{\delta\rho_0}{\delta\rho} - \frac{\partial v}{\partial t} - \vec{v} \cdot \vec{\nabla} v + k(\vec{A} \cdot \vec{v} - \phi) \right] \\ &+ \oint d\vec{S} \cdot \vec{v} \delta\rho v + \int d\vec{\Sigma} \cdot \vec{v} \delta\rho [v] + \int d^3x v \delta\rho \Big|_{t_0}^{t_1} \end{aligned} \quad (80)$$

Or as:

$$\begin{aligned} \delta_\rho A &= \int d^3x dt \delta\rho \left[-\frac{c^2 + w_0}{\gamma} - \frac{\partial v}{\partial t} - \vec{v} \cdot \vec{\nabla} v + k(\vec{A} \cdot \vec{v} - \phi) \right] \\ &+ \oint d\vec{S} \cdot \vec{v} \delta\rho v + \int d\vec{\Sigma} \cdot \vec{v} \delta\rho [v] + \int d^3x v \delta\rho \Big|_{t_0}^{t_1} \end{aligned} \quad (81)$$

Provided that $\delta\rho$ disappears on the spatial and temporal boundaries and the cut, we derive:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \vec{v} \cdot \vec{\nabla} v = -\frac{c^2 + w_0}{\gamma} + k(\vec{A} \cdot \vec{v} - \phi) \Rightarrow \frac{dv}{dt} \simeq \frac{1}{2}v^2 - w_0 - c^2 + k(\vec{A} \cdot \vec{v} - \phi). \quad (82)$$

Equation (82) can be written compactly as:

$$\frac{dv}{dt} = -c^2 \frac{\bar{\lambda}}{\gamma} + k(\vec{A} \cdot \vec{v} - \phi) = -c^2 \frac{\lambda}{\gamma^2} + k(\vec{A} \cdot \vec{v} - \phi), \quad (83)$$

$\bar{\lambda}$ is introduced in Equation (41). As a final step the action is varied with respect to the velocity $\vec{v}$, taking into account the identity:

$$\delta_{\vec{v}} \frac{1}{\gamma} = -\gamma \frac{\vec{v} \cdot \delta\vec{v}}{c^2} \quad (84)$$

We obtain the following result:

$$\begin{aligned} \delta_{\vec{v}} A &= \int d^3x dt \rho \delta\vec{v} \cdot \left[\gamma \vec{v} - \frac{w_0}{\rho} \frac{\delta\rho_0}{\delta\vec{v}} - \vec{\nabla} v - \alpha \vec{\nabla} \beta - \sigma \vec{\nabla} s + k\vec{A} \right] \\ &+ \oint d\vec{S} \cdot \delta\vec{v} \rho v + \int d\vec{\Sigma} \cdot \delta\vec{v} \rho [v]. \end{aligned} \quad (85)$$

However:

$$\frac{\delta \rho_0}{\delta \vec{v}} = \rho \frac{\delta \frac{1}{\gamma}}{\delta \vec{v}} = -\rho \gamma \frac{\vec{v}}{c^2} \quad (86)$$

Using λ defined in Equation (41), we may write:

$$\begin{aligned} \delta_{\vec{v}} A &= \int d^3x dt \rho \delta \vec{v} \cdot [\lambda \vec{v} - \vec{\nabla} \nu - \alpha \vec{\nabla} \beta - \sigma \vec{\nabla} s + k \vec{A}] \\ &+ \oint d\vec{S} \cdot \delta \vec{v} \rho \nu + \int d\vec{\Sigma} \cdot \delta \vec{v} \rho [\nu]. \end{aligned} \quad (87)$$

The boundary terms above include an integral over the external boundary, $\oint d\vec{S}$, and an integral over a cut, $\int d\vec{\Sigma}$, which is needed if ν is not single-valued. The external boundary term vanishes in situations such as astrophysical flows where $\rho = 0$ at the free-flow boundary, or when the fluid is contained in a vessel imposing a no-flux condition, $\delta \vec{v} \cdot \hat{n} = 0$, with $\hat{n}$ the unit normal to the boundary. The cut term vanishes if the velocity variations are parallel to the cut, satisfying a Kutta-type condition. When the boundary terms vanish, the velocity field $\vec{v}$ must take the following form:

$$\lambda \vec{v} = \alpha \vec{\nabla} \beta + \sigma \vec{\nabla} s + \vec{\nabla} \nu - k \vec{A}, \Rightarrow \vec{v} \simeq \alpha \vec{\nabla} \beta + \sigma \vec{\nabla} s + \vec{\nabla} \nu - k \vec{A}. \quad (88)$$

This represents a generalization of the Clebsch representation for a non-barotropic flow field (see, e.g., [11] and [14, page 248]) extended to a relativistic charged flow. We note that the function ν contributes to the velocity field only through its gradient, which implies that adding an arbitrary function of time to ν does not change the physical velocity field. By redefining:

$$\bar{\nu} \equiv \nu + c^2 t \quad (89)$$

we derive the more classical form of Equation (82) and Equation (88).

$$\frac{d\bar{\nu}}{dt} \simeq \frac{1}{2} v^2 - w_0 + k(\vec{A} \cdot \vec{v} - \phi), \quad \vec{v} \simeq \alpha \vec{\nabla} \beta + \sigma \vec{\nabla} s + \vec{\nabla} \bar{\nu} - k \vec{A}. \quad (90)$$

4.1. Euler's equations

We will now demonstrate that a velocity field of the form Equation (88), where the functions α , β , and ν satisfy the corresponding equations (76, 82, 77), necessarily obeys Euler's equations. To do this, we first compute the material derivative of $\lambda \vec{v}$:

$$\frac{d(\lambda \vec{v})}{dt} = \frac{d\vec{\nabla} \nu}{dt} + \frac{d\alpha}{dt} \vec{\nabla} \beta + \alpha \frac{d\vec{\nabla} \beta}{dt} + \frac{d\sigma}{dt} \vec{\nabla} s + \sigma \frac{d\vec{\nabla} s}{dt} - k \frac{d\vec{A}}{dt} \quad (91)$$

Without much effort we derive:

$$\begin{aligned} \frac{d\vec{\nabla} \nu}{dt} &= \vec{\nabla} \frac{d\nu}{dt} - \vec{\nabla} v_n \frac{\partial \nu}{\partial x_n} = \vec{\nabla} \left(-\frac{c^2 + w_0}{\gamma} + k \vec{A} \cdot \vec{v} - k\phi \right) - \vec{\nabla} v_n \frac{\partial \nu}{\partial x_n} \\ \frac{d\vec{\nabla} \beta}{dt} &= \vec{\nabla} \frac{d\beta}{dt} - \vec{\nabla} v_n \frac{\partial \beta}{\partial x_n} = -\vec{\nabla} v_n \frac{\partial \beta}{\partial x_n} \\ \frac{d\vec{\nabla} s}{dt} &= \vec{\nabla} \frac{ds}{dt} - \vec{\nabla} v_n \frac{\partial s}{\partial x_n} = -\vec{\nabla} v_n \frac{\partial s}{\partial x_n} \end{aligned} \quad (92)$$

In which we use the Cartesian coordinates x_n and the summation convention. Inserting equations (92) into Equation (91) it follows that:

$$\begin{aligned}
 \frac{d(\lambda \vec{v})}{dt} &= -\vec{\nabla} v_n \left(\frac{\partial v}{\partial x_n} + \alpha \frac{\partial \beta}{\partial x_n} + \sigma \frac{\partial s}{\partial x_n} \right) + \vec{\nabla} \left(-\frac{c^2 + w_0}{\gamma} + k \vec{A} \cdot \vec{v} - k\phi \right) \\
 &\quad - k \frac{d\vec{A}}{dt} + \frac{T_0}{\gamma} \vec{\nabla} s \\
 &= -\vec{\nabla} v_n (\lambda v_n + k A_n) + \vec{\nabla} \left(-\frac{c^2 + w_0}{\gamma} + k \vec{A} \cdot \vec{v} - k\phi \right) \\
 &\quad - k \partial_t \vec{A} - k(\vec{v} \cdot \vec{\nabla}) \vec{A} + \frac{T_0}{\gamma} \vec{\nabla} s \\
 &= -\frac{1}{\gamma} \vec{\nabla} w_0 + k \vec{E} + k(v_n \vec{\nabla} A_n - v_n \partial_n \vec{A}) + \frac{T_0}{\gamma} \vec{\nabla} s,
 \end{aligned} \tag{93}$$

we use equation (8) of [5] as the definition of the electric field. Also by equation (7) of [5] the following equation is derived:

$$(v_n \vec{\nabla} A_n - v_n \partial_n \vec{A})_l = v_n (\partial_l A_n - \partial_n A_l) = \epsilon_{lnj} v_n B_j = (\vec{v} \times \vec{B})_l, \tag{94}$$

The Euler equation of a charged relativistic fluid are thus obtained:

$$\frac{d(\lambda \vec{v})}{dt} = -\frac{1}{\gamma} \vec{\nabla} w_0 + \frac{T_0}{\gamma} \vec{\nabla} s + k [\vec{v} \times \vec{B} + \vec{E}] = -\frac{1}{\rho} \vec{\nabla} P_0 + k [\vec{v} \times \vec{B} + \vec{E}], \tag{95}$$

in which we have used Equation (19). Equation (95) is the same as Equation (51). We have thus proved that the Euler equations can be derived from the action principle (74). To summarize, all equations of relativistic charged fluid dynamics can be derived from the action principle (74).

4.2. Simplified action

One might argue that the previous formulation complicates relativistic fluid dynamics by introducing four additional scalar fields, α , β , ν , and σ , beyond the physical variables $\vec{v}$, ρ , and s . However, this is only an apparent complication. We will show that the expression in Equation (71), written in a pedagogical form, can be simplified. To this end, we define a four-dimensional non-barotropic Clebsch four-vector:

$$\begin{aligned}
 v_C^\mu &\equiv \alpha \partial^\mu \beta + \sigma \partial^\mu s + \partial^\mu \nu = \left(\frac{1}{c} (\alpha \partial_t \beta + \sigma \partial_t s + \partial_t \nu), \alpha \vec{\nabla} \beta + \sigma \vec{\nabla} s + \vec{\nabla} \nu \right) \\
 &= \left(\frac{1}{c} (\alpha \partial_t \beta + \sigma \partial_t s + \partial_t \nu), \vec{v}_C \right).
 \end{aligned} \tag{96}$$

In addition we define a non-barotropic electromagnetic Clebsch four vector:

$$v_E^\mu \equiv v_C^\mu + k A^\mu = (v_E^0, \vec{v}_E) = \left(\frac{1}{c} (\alpha \partial_t \beta + \sigma \partial_t s + \partial_t \nu + k\phi), \vec{v}_C - k \vec{A} \right). \tag{97}$$

Equation (76), Equation (83) and Equation (88) lead to the result:

$$v_\mu = -\frac{v_{E\mu}}{\lambda} \Rightarrow \quad \vec{v} = \frac{\vec{v}_E}{\lambda}, \quad \lambda = -\frac{1}{c^2} (\alpha \partial_t \beta + \sigma \partial_t s + \partial_t \nu + k\phi). \tag{98}$$

The low velocity (classical) limit of the above equations is:

$$\vec{v} \simeq \vec{v}_E, \quad \lambda \simeq 1. \tag{99}$$

$\vec{v}$ can now be eliminated from Equation (74) which (up to surface terms) takes the compact form:

$$\hat{\mathcal{L}}[\rho_0, s, \alpha, \beta, \nu, \sigma] = \rho_0 \left[c \sqrt{v_{E\mu} v_E^\mu} - \varepsilon_0 - c^2 \right] \quad (100)$$

This Lagrangian density produces the six equations (76, 77, 78, 82). Once these equations are solved, the scalar fields α , β , s , σ , and ν can be substituted into Equation (88) to determine $\vec{v}$. Consequently, the general problem of relativistic, charged, non-barotropic fluid dynamics is reformulated: instead of directly solving the Euler and continuity equations, one solves an equivalent set derived from the Lagrangian density $\hat{\mathcal{L}}$. The classical limit of Equation (100) can then be found as follows. First, we observe that:

$$\sqrt{v_{E\mu} v_E^\mu} = |v_{E0}| \sqrt{1 - \frac{\vec{v}_E^2}{v_{E0}^2}} = |v_{E0}| \sqrt{1 - \frac{\vec{v}^2}{c^2}} = \frac{|v_{E0}|}{\gamma} = \frac{c|\lambda|}{\gamma} = c|\bar{\lambda}| \quad (101)$$

using Equation (98). According to the same equation $v_{E0} < 0$ as $\lambda > 0$, and we obtain:

$$|v_{E0}| = -v_{E0} = -\frac{1}{c}(\alpha \partial_t \beta + \sigma \partial_t s + \partial_t \nu + k\phi). \quad (102)$$

Equation (101) and Equation (102) are combined to derive:

$$c \sqrt{v_{E\mu} v_E^\mu} = -\frac{1}{\gamma}(\alpha \partial_t \beta + \sigma \partial_t s + \partial_t \nu + k\phi) \quad (103)$$

Which can be expressed in terms of $\bar{v}$ (Equation (89)) in the following form:

$$c \sqrt{v_{E\mu} v_E^\mu} = -\frac{1}{\gamma}(\alpha \partial_t \beta + \sigma \partial_t s + \partial_t \bar{v} + k\phi - c^2) \quad (104)$$

We can now formulate Equation (100) as follows:

$$\begin{aligned} \hat{\mathcal{L}} &= \rho_0 \left[-\frac{1}{\gamma}(\alpha \partial_t \beta + \sigma \partial_t s + \partial_t \bar{v} + k\phi - c^2) - \varepsilon_0 - c^2 \right] \\ &= -\frac{\rho_0}{\gamma} [\alpha \partial_t \beta + \sigma \partial_t s + \partial_t \bar{v} + k\phi + \gamma \varepsilon_0 + (\gamma - 1)c^2]. \end{aligned} \quad (105)$$

The classical limit is straight forward: $\frac{v}{c} \rightarrow 0, \gamma \rightarrow 1$. Notice, however that the final term should be handled with caution:

$$(\gamma - 1)c^2 = \left(\frac{\gamma - 1}{\left(\frac{v}{c}\right)^2} \right) v^2 \simeq \frac{1}{2} v^2 \simeq \frac{1}{2} v_E^2. \quad (106)$$

Using Equation (106) it follows that:

$$\hat{\mathcal{L}} \simeq -\rho_0 \left[\frac{\partial \bar{v}}{\partial t} + \alpha \frac{\partial \beta}{\partial t} + \sigma \frac{\partial s}{\partial t} + \varepsilon(\rho_0, s_0) + k\phi + \frac{1}{2}(\vec{\nabla} \bar{v} + \alpha \vec{\nabla} \beta + \sigma \vec{\nabla} s - k \vec{A})^2 \right] \quad (107)$$

However, from Equation (24) it follows that:

$$\varepsilon_0 = \frac{dE_{in \vec{a}0}}{dM_{\vec{a}}} = \gamma \frac{dE_{in \vec{a}}}{dM_{\vec{a}}} = \gamma \varepsilon. \quad (108)$$

We conclude that in the (low velocity) classical limit there is no difference between ρ and ρ_0 , and ε_0 and ε , since they differ by a term of order $\left(\frac{v}{c}\right)^2$. We thus obtain:

$$\hat{\mathcal{L}} \simeq -\rho \left[\frac{\partial \bar{v}}{\partial t} + \alpha \frac{\partial \beta}{\partial t} + \sigma \frac{\partial s}{\partial t} + \varepsilon(\rho, s) + k\phi + \frac{1}{2}(\vec{\nabla} \bar{v} + \alpha \vec{\nabla} \beta + \sigma \vec{\nabla} s - k \vec{A})^2 \right] \quad (109)$$

5. Energy Momentum Tensor

We shall now calculate the canonical energy momentum tensor of the Lagrangian density given in Equation (100), this is done through the definition given in Equation (60). However, as $\hat{\mathcal{L}}$ is dependent only on the derivatives: $\partial^\mu \beta, \partial^\mu s, \partial^\mu \nu$ we obtain:

$$T_{E\mu\nu} = \partial_\mu \beta \frac{\partial \hat{\mathcal{L}}}{\partial(\partial^\nu \beta)} + \partial_\mu s \frac{\partial \hat{\mathcal{L}}}{\partial(\partial^\nu s)} + \partial_\mu \nu \frac{\partial \hat{\mathcal{L}}}{\partial(\partial^\nu \nu)} - \eta_{\mu\nu} \hat{\mathcal{L}}. \quad (110)$$

Taking into account Equation (101) it is easy to see that:

$$\frac{\partial \hat{\mathcal{L}}}{\partial(\partial^\nu \beta)} = \frac{\rho\alpha}{\lambda} v_{Ev}, \quad \frac{\partial \hat{\mathcal{L}}}{\partial(\partial^\nu s)} = \frac{\rho\sigma}{\lambda} v_{Ev}, \quad \frac{\partial \hat{\mathcal{L}}}{\partial(\partial^\nu \nu)} = \frac{\rho}{\lambda} v_{Ev}. \quad (111)$$

Plugging Equation (111) into Equation (110) we have:

$$T_{E\mu\nu} = \frac{\rho}{\lambda} (\alpha \partial_\mu \beta + \sigma \partial_\mu s + \partial_\mu \nu) v_{Ev} - \eta_{\mu\nu} \hat{\mathcal{L}} = \frac{\rho}{\lambda} v_{C\mu} v_{Ev} - \eta_{\mu\nu} \hat{\mathcal{L}}, \quad (112)$$

in which we have used the definition given in Equation (96). The above formula can be expressed in terms of more physical velocity fields using Equation (97) and Equation (98) as:

$$T_{E\mu\nu} = \rho(\lambda v_\mu + k A_\mu) v_\nu - \eta_{\mu\nu} \hat{\mathcal{L}}. \quad (113)$$

By using Equation (101) we notice that Equation (100) can be formulated as:

$$\hat{\mathcal{L}} = \rho_0 [c^2 \bar{\lambda} - \varepsilon_0 - c^2]. \quad (114)$$

Taking into account Equation (41) we see that $\hat{\mathcal{L}}$ is numerically equal to the pressure at the rest frame:

$$\hat{\mathcal{L}} = \rho_0 \left[c^2 \left(1 + \frac{w_0}{c^2} \right) - \varepsilon_0 - c^2 \right] = \rho_0 [w_0 - \varepsilon_0] = P_0. \quad (115)$$

in which for the last equality sign we have used Equation (17). Thus:

$$T_{E\mu\nu} = \rho(\lambda v_\mu + k A_\mu) v_\nu - \eta_{\mu\nu} P_0. \quad (116)$$

It is interesting to compare this expression to the one suggested by Weinberg [7], to do this we look at uncharged fluids for which $k = 0$. Using Equation (41) and Equation (73) we derive:

$$\rho \lambda v_\mu v_\nu = \rho \lambda \frac{u_\mu u_\nu}{\gamma^2} = \rho_0 \bar{\lambda} u_\mu u_\nu = \rho_0 \left(1 + \frac{w_0}{c^2} \right) u_\mu u_\nu = \rho_0 \left(1 + \frac{\varepsilon_0 + \frac{P_0}{\rho_0}}{c^2} \right) u_\mu u_\nu \quad (117)$$

Thus:

$$\rho \lambda v_\mu v_\nu = \frac{1}{c^2} (\rho_0 c^2 + \rho_0 \varepsilon_0 + P_0) u_\mu u_\nu = \frac{1}{c^2} (e_0 + P_0) u_\mu u_\nu \quad (118)$$

in which we have used the definition given in Equation (54). Thus we may rewrite the energy momentum tensor of an uncharged flow in the form:

$$T_{E\mu\nu} = \frac{1}{c^2} (e_0 + P_0) u_\mu u_\nu - \eta_{\mu\nu} P_0. \quad (119)$$

This should be compared to the Weinberg [7] form:

$$T_{W\mu\nu} = (\rho_W + P_W) u_{W\mu} u_{W\nu} + \eta_{W\mu\nu} P_W. \quad (120)$$

In which we need to take into account that Weinberg does not use practical MKS units as we do, but natural units in which $c = 1$, thus $u_{W\mu} = u_\mu$. The choice of metric that Weinberg uses is also opposite

to our own hence $\eta_{W\mu\nu} = -\eta_{\mu\nu}$. It also follows that Weinberg's pressure is the pressure at the rest frame $P_w = P_0$ and that Weinberg's proper energy density is the total energy density in the rest frame: $\rho_w = \rho_0$.

6. Conclusions

Current cosmological models in particular Λ CDM postulate an unconfirmed component (the Λ in Λ CDM) which is denoted "dark energy". This is needed to fit the accelerating expansion of the universe as obtained through distant supernovae observations, the CMB spectrum and the large scale structure which developed through baryonic acoustic oscillations in the primordial matter. The energy momentum tensor obtain in Equation (116) allows us two alternative routes. One is to assume that primordial fluid was not electrically neutral and thus one must take into account the electromagnetic contributions to the energy-momentum tensor. The other one is to take into account a non barotropic fluid in which the quantity $\rho_0\epsilon_0$ is not just a simple multiplication of P_0 , which may be either 3 or $\frac{3}{2}$ depending on whether the fluid is ultra relativistic or non-relativistic. But rather a more realistic form in which $\rho_0\epsilon_0$ is dependent on two thermodynamical quantities which can be either the density, specific entropy, pressure and temperature all defined in the fluid element's rest frame. Of course more detailed analysis is needed which will be hopefully published in future papers.

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