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# A Generalized Series Expansion of the Arctangent Function Based on the Enhanced Midpoint Integration

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**Abstract:** In this work we derive a generalized series expansion of the acrtangent function by using the enhanced midpoint integration (EMI). Algorithmic implementation of the generalized series expansion utilizes a simple two-step iteration. This approach significantly improves the convergence and requires no surd numbers in computation of the arctangent function.

**Keywords:** arctangent function; midpoint integration, iterative algorithm, constant pi

## 1. Introduction

In 2010, Adegoke and Layeni published an interesting relation for derivatives of the arctangent function [1]

$$\frac{d^n}{dx^n}\arctan(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x^2)^{n/2}}\sin\left(n\arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)\right), \quad n \in \mathbb{N}^+. \tag{1}$$

Using this relation, they discovered a series expansion

$$\arctan(x) = \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x^2}{1+x^2}\right)^{n/2}\sin\left(n\arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)\right). \tag{2}$$

Equations (1) and (2) have some restrictions. Specifically, when $n$ is even, equation (1) remains valid only at $x \in [0,\infty)$, while equation (2) is valid only at $x \in [0,\infty)$ for $\forall n$.

To resolve this problem, Lampret applied the signum function

$$\mathrm{sgn}(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0, \end{cases}$$

and proved that for complete coverage $x \in (-\infty,\infty)$, the equations (1) and (2) can be modified as [2]

$$\frac{d^n}{dx^n}\arctan(x) = \mathrm{sgn}(-x)^{n-1}\frac{(n-1)!}{(1+x^2)^{n/2}}\sin\left(n\arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)\right) \tag{3}$$

and

$$\arctan(x) = \mathrm{sgn}(x)\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x^2}{1+x^2}\right)^{n/2}\sin\left(n\arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)\right), \tag{4}$$

respectively.

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Equations (3) and (4) represent a theoretical interest. In particular, Lampret noticed that from equation (3) it follows that [2]

$$\operatorname{sgn}(0)^{n-1}\cdot(n-1)!\cdot\sin\left(n\frac{\pi}{2}\right)=\begin{cases}(-1)^{(n-1)/2}(n-1)!, & n \text{ odd},\\ 0, & n \text{ even}.\end{cases} \tag{5}$$

Comparing the following relation (see [3] for detailed derivation procedure by induction)

$$\frac{d^n}{dx^n}\arctan(x)=\frac{(-1)^n(n-1)!}{2i}\left(\frac{1}{(x+i)^n}-\frac{1}{(x-i)^n}\right) \tag{6}$$

with equation (3), we can find the following identity

$$\begin{aligned}\operatorname{sgn}(-x)^{n-1}\frac{(n-1)!}{(1+x^2)^{n/2}}\sin\left(n\arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)\right)=\\ \frac{(-1)^n(n-1)!}{2i}\left(\frac{1}{(x+i)^n}-\frac{1}{(x-i)^n}\right).\end{aligned} \tag{7}$$

It is not difficult to see that the relation (5) immediately follows from the identity (7). Therefore, relation (5) is just a specific case of the identity (7) occurring at $x=0$.

Identity (7) can be rewritten in form

$$\sin\left(n\arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)\right)=-\operatorname{sgn}(x)^{n-1}\frac{(1+x^2)^{n/2}}{2i}\left(\frac{1}{(x+i)^n}-\frac{1}{(x-i)^n}\right).$$

Therefore, from equation (4) it follows that

$$\arctan(x)=\operatorname{sgn}(x)\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x^2}{1+x^2}\right)^{n/2}\left[-\operatorname{sgn}(x)^{n-1}\frac{(1+x^2)^{n/2}}{2i}\left(\frac{1}{(x+i)^n}-\frac{1}{(x-i)^n}\right)\right]$$

or

$$\arctan(x)=\frac{i}{2}\sum_{n=1}^{\infty}\frac{x^n}{n}\left(\frac{1}{(x+i)^n}-\frac{1}{(x-i)^n}\right).$$

As we can see, this series expansion of the arctangent function is just a reformulation of equation (4) that follows due to identity (6).

In our previous publication [3], using the identity (6), we have derived the following series expansion of the arctangent function

$$\arctan(x)=-2\sum_{m=1}^{\infty}\sum_{n=1}^{2m-1}\frac{(-1)^n}{(2m-1)(1+x^2/4)^{2m-1}}\left(\frac{x}{2}\right)^{2(2m-n)-1}\binom{2m-1}{2n-1},$$

from which at $x=1$ we get a formula for $\pi$ expressed in terms of the binomial coefficients

$$\frac{\pi}{4}=-2\sum_{m=1}^{\infty}\sum_{n=1}^{2m-1}\frac{(-1)^n}{(2m-1)(1+1/4)^{2m-1}2^{2(2m-n)-1}}\binom{2m-1}{2n-1}$$

or

$$\frac{\pi}{16}=\sum_{m=1}^{\infty}\sum_{n=1}^{2m-1}\frac{(-4)^{n-1}}{(2m-1)5^{2m-1}}\binom{2m-1}{2n-1}.$$

Later, using again the same identity (6), we have also derived the following series expansion (see [4] and literature therein)

$$\arctan(x)=2\sum_{n=1}^{\infty}\frac{1}{2n-1}\frac{g_n(x)}{g_n^2(x)+h_n^2(x)}, \tag{8}$$

where the expansion coefficients are computed by two-step iteration

$$g_n(x) = g_{n-1}(x)(1 - 4/x^2) + 4h_{n-1}(x)/x,$$
$$h_n(x) = h_{n-1}(x)(1 - 4/x^2) - 4g_{n-1}(x)/x,$$

such that

$$g_1(x) = 2/x,$$
$$h_1(x) = 1.$$

The series expansion (8) requires no surd numbers in computation and it is rapid in convergence.

As further development, in this work we derive a generalized series expansion of the arctangent function. Such an approach may be used to improve further convergence in computation of the arctangent function.

## 2. Derivation

Change of the variable $x \to xt$ in the equation (6) results in

$$\frac{\partial^n}{\partial t^n} \arctan(xt) = \frac{(-1)^n (n-1)! x^n}{2i} \left( \frac{1}{(xt+i)^n} - \frac{1}{(xt-i)^n} \right)$$

or

$$\frac{\partial^n}{\partial t^n} \frac{x}{1+x^2t^2} = \frac{(-1)^{n+1} n! x^{n+1}}{2i} \left( \frac{1}{(xt+i)^{n+1}} - \frac{1}{(xt-i)^{n+1}} \right) \tag{9}$$

since

$$\frac{\partial^n}{\partial t^n} \arctan(xt) = \frac{\partial^{n-1}}{\partial t^{n-1}} \left( \frac{x}{1+x^2t^2} \right).$$

There is an enhanced midpoint integration (EMI) formula (see [5] and literature therein for detailed derivation procedure)

$$\int_0^1 f(t)\,dt = \sum_{m=1}^{M} \sum_{n=0}^{\infty} \frac{(-1)^n + 1}{(2M)^{n+1}(n+1)!} \frac{d^n}{dt^n} f(t) \bigg|_{t=\frac{m-1/2}{M}}. \tag{10}$$

It is interesting to note that if the upper summation bound associated with variable $n$ is an integer $N \geq 1$, then we can also use

$$\int_0^1 f(t)\,dt = \lim_{M\to\infty} \sum_{m=1}^{M} \sum_{n=0}^{N} \frac{(-1)^n + 1}{(2M)^{n+1}(n+1)!} \frac{d^n}{dt^n} f(t) \bigg|_{t=\frac{m-1/2}{M}}.$$

It is easy to show that, excluding all zero terms occurring at odd values of the variable $n$, equation (10) can be rewritten in a more convenient form

$$\int_0^1 f(t)\,dt = 2 \sum_{m=1}^{M} \sum_{n=0}^{\infty} \frac{1}{(2M)^{2n+1}(2n+1)!} \frac{d^{2n}}{dt^{2n}} f(t) \bigg|_{t=\frac{m-1/2}{M}}. \tag{11}$$

Although equation (11) requires even derivatives of the integrand at the points $t = (m - 1/2)/M$, where $m = 1, 2, 3, \ldots M$, its application with help of the Computer Algebra System (CAS) may be very efficient in numerical integration. Specifically, such an approach may be especially useful for numerical integration for the highly oscillating functions. The interested readers can download the MATLAB code based on the integration formula (11) on the MATLAB Central website [6] (file ID #: 71037).

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If an integrand represents a function of two variables $f(x,t)$, then the integration formula (11) reads as

$$\int_0^1 f(x,t)\,dt = 2\sum_{m=1}^{M}\sum_{n=0}^{\infty}\frac{1}{(2M)^{2n+1}(2n+1)!}\frac{\partial^{2n}}{\partial t^{2n}}f(x,t)\bigg|_{t=\frac{m-1/2}{M}}. \tag{12}$$

The arctangent function can be given as an integral

$$\arctan(x) = \int_0^1 \frac{x}{1+x^2t^2}\,dt. \tag{13}$$

Consequently, substituting the integrand from equation (13)

$$\frac{x}{1+x^2t^2} = \frac{x}{2}\left(\frac{1}{1+ixt}+\frac{1}{1-ixt}\right)$$

into equation (12) and using equation (9) for differentiation, we can find that

$$\arctan(x) = i\sum_{m=1}^{M}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2M)^{2n+1}(2n+1)}\left[\frac{1}{(x\frac{m-1/2}{M}+i)^{2n+1}}-\frac{1}{(x\frac{m-1/2}{M}-i)^{2n+1}}\right]. \tag{14}$$

Series expansion (14) is rapid in convergence. However, it requires algebraic manipulations with complex numbers. Therefore, it is very desirable to exclude them. This can be achieved by induction based on two-step iteration

$$\alpha_n(x,t) = \alpha_{n-1}(x,t)(1-1/(xt)^2)+2\beta_{n-1}(x,t)/(xt),$$
$$\beta_n(x,t) = \beta_{n-1}(x,t)\left(1-1/(xt)^2\right)-2\alpha_{n-1}(x,t)/(xt),$$

and

$$\alpha_1(x,t) = 1/(xt),$$
$$\beta_1(x,t) = 1,$$

that transforms equation (14) into the following series expansion

$$\arctan(x) = 2\sum_{m=1}^{M}\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2m-1)^{2n-1}}\frac{\alpha_n(x,\gamma_{m,M})}{\alpha_n^2(x,\gamma_{m,M})+\beta_n^2(x,\gamma_{m,M})}, \tag{15}$$

where the argument is

$$\gamma_{m,M} = \frac{m-1/2}{M}.$$

Equation (15) is a generalization of the equation (8). Consistency between these two equations can be observed by taking $M=1$. In particular, substitution $M=1$ into series expansion (15) of the arctangent function implies that $t=\gamma_{1,1}=1/2$. Consequently, from equation (15) we get equation (8), where the expansion coefficients are

$$g_1(x) = \alpha_1(x,\gamma_{1,1}),$$
$$h_1(x) = \beta_1(x,\gamma_{1,1}),$$
$$g_n(x) = \alpha_n(x,\gamma_{1,1}),$$
$$h_n(x) = \beta_n(x,\gamma_{1,1}).$$

The following is a Mathematica code that generates graphs shown in Figure 1 (this code can be copy-pasted directly to the Mathematica notebook):

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```
Clear[atan,\[Gamma],\[Alpha],\[Beta]];

(* Equation (15) *)
atan[x_,nMax_,M_] := 2*Sum[(1/(2*n - 1))*
  (\[Alpha][x,\[Gamma][m,M],n]/((2*m - 1)^(2*n - 1)*
    (\[Alpha][x,\[Gamma][m, M],n]^2 + \[Beta][x,
      \[Gamma][m,M],n]^2))),{m,1,M},{n,1,nMax}];

(* Argument gamma *)
\[Gamma][m_,M_] := \[Gamma][m,M] = N[(m - 1/2)/M,1000];

(* Expansion coefficients *)
\[Alpha][x_,t_,1] := \[Alpha][x,t,1] = 1/(x*t);
\[Beta][x_,t_,1] := \[Beta][x,t,1] = 1;

\[Alpha][x_,t_,n_] := \[Alpha][x,t,n] =
  \[Alpha][x,t,n - 1]*(1 - 1/(x*t)^2) +
    2*(\[Beta][x, t, n - 1]/(x*t));

\[Beta][x_,t_,n_] := \[Beta][x,t,n] =
  \[Beta][x,t,n - 1]*(1 - 1/(x*t)^2) -
    2*(\[Alpha][x, t, n - 1]/(x*t));

(* Computing data points *)
tabs := {Table[{x,atan[x,10,1]},{x,-20,20,Pi/20}],
  Table[{x,atan[x,10,2]},{x,-20,20,Pi/20}],
    Table[{x,atan[x,10,3]},{x,-20,20,Pi/20}]};

Print["Computing, please wait..."];

(* Plotting graphs *)
ListPlot[tabs,Joined->True,FrameLabel->{"Parameter x",
  "Arctangent approximations"},PlotStyle->{Blue,Red,Green},
    Frame->True,GridLines->Automatic]
```

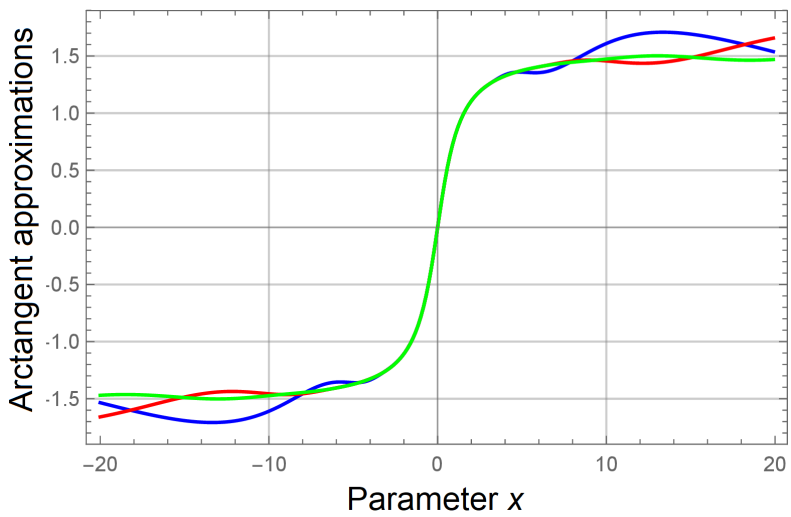


**Figure 1.** Arctangent approximations computed by using series expansion (15) truncated at $n_{\max} = 10$. Blue, red and green curves correspond to $M$ taken to be 1, 2 and 3, respectively.

Graphs in Figure 1 are generated by using series expansion (15) truncated at $n_{\max} = 10$. Blue, red and green curves correspond to integer $M$ taken to be 1, 2 and 3, respectively.

## 3. Convergence

Consider Figure 2 showing approximation curves of the arctangent function $\arctan(x)$ by using equations (15), (16) and (17) truncated at $n_{\max} = 10$. The blue curve corresponding to the Maclaurin expansion series

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \leq 1, \tag{16}$$

diverges beyond $-1$ and 1 due to finite radius of convergence. Although one can resolve this issue by using an elementary relation

$$\arctan(x) = \begin{cases} \dfrac{\pi}{2} - \arctan\left(\dfrac{1}{x}\right), & x > 0, \\ 0, & x = 0, \\ -\dfrac{\pi}{2} - \arctan\left(\dfrac{1}{x}\right), & x < 0, \end{cases}$$

our objective is just to visualize the convergence. The red curve shows the Euler series expansion [7,8]

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!} \frac{x^{2n+1}}{(1+x^2)^{n+1}}. \tag{17}$$

The green curve illustrates the series expansion (15) at $M = 1$. The black dashed curve depicts the original arctangent function for comparison. As we can see from Figure 2, even at smallest $M$ the series expansion (15) provides more rapid convergence as compared to the Euler series expansion (17).

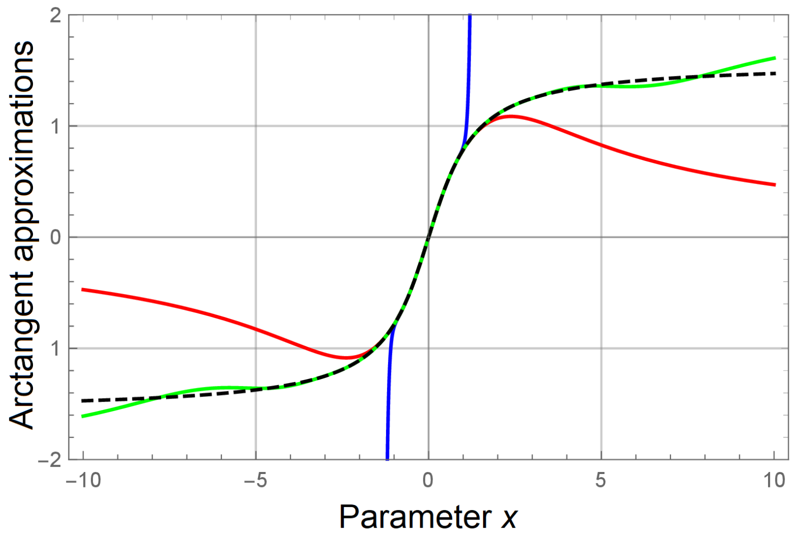


**Figure 2.** Arctangent approximations computed by series expansions (16) (blue), (17) (red) and (15) (green) truncated at $n_{\max} = 10$. Integer $M$ in equation (15) is taken to be equal to 1. The dashed black curve shows the original artangent function.

Figure 3 shows the logarithms of absolute difference $\log_{10} \Delta$ between the arctangent function and its approximations provided by equations (15), (16) and (17). All curves are also computed with truncating integer $n_{\max} = 10$ in all these equations. The blue and red curves correspond to equations (16) and (17) while the green, brown, gray, magenta and black curves correspond to equation (15) when $M$ is equal to 1, 2, 3, 4 and 5, respectively. As we can see from this figure, increase of the integer

$M$ leads to a rapid decrease of the absolute difference $\Delta$ by many orders of the magnitude. These results indicate that the series expansion (15) provides increasing convergence with increasing $M$. Consequently, the series expansion (15) may be promising for efficient computation digits of $\pi$ in the Machin-like formulas [4,9–15] without undesirable surd numbers since computation of any irrational numbers is itself a big challenge.

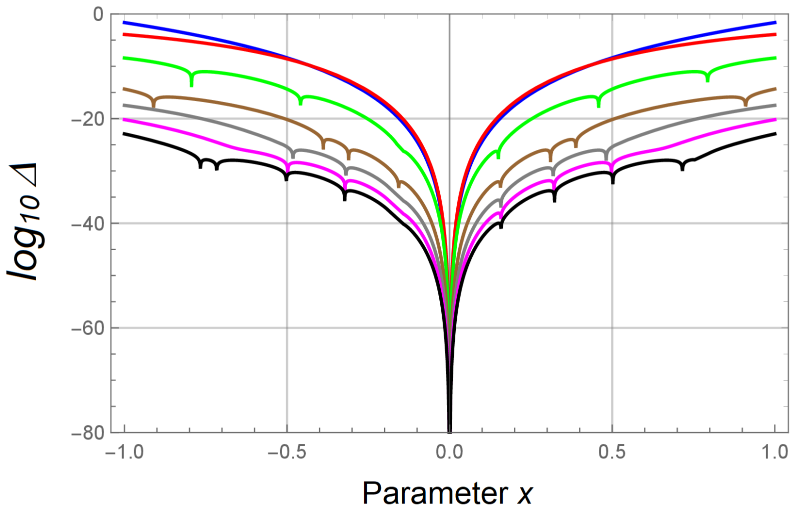


**Figure 3.** Logarithms of absolute difference $log_{10}\Delta$ between original arctangent function and series expansions (16) (blue), (17) (red) and (15) (green-to-black) truncated at $n_{\max} = 10$. Integer $M$ in the series expansion (15) is taken to be 1 (green), 2 (brown), 3 (gray), 4 (magenta) and 5 (black).

Our empirical results show that even using already known Machin-like formulas with sufficiently large integers in actangent arguments, the expansion series (15) at any $M \geq 1$ can provide more than 17 digits of $\pi$ at each increment by 1 of the variable $n$. It is interesting to note that this convergence rate is faster than that of provided by Chudnovsky formula generating 14 to 16 digits of $\pi$ per increment [9,12]. Nowadays, Chudnovsky formula remains most efficient for computing digits of $\pi$ due to its rapid convergence and other advantages in algorithmic implementation. Historically, however, there were several records that appeared due to application of the Machin-like formulas in computing $\pi$ and, in 2002, an algorithm, developed by Kanada on the basis of self-checking pair of the Machin-like formulas, beat the record providing more than a trillion digits of $\pi$ for the first time [10,12]. Therefore, discovery of new Machin-like formulas and rapidly convergent series expansions of the arctangent function may be promising for computing digits of the constant $\pi$.

## 4. Conclusions

We derived a generalized series expansion (15) of the arctangent function by using the EMI formula (12). Computational test we performed reveals that such a generalization significantly improves convergence in computation of the arctangent function.

**Author Contributions:** S.M.A. developed the methodology, wrote the codes and prepared a draft version of the manuscript. R.S., R.K.J and B.M.Q. verified, reviewed and edited the manuscript. All authors have read and agreed to the published version of the manuscript.

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## Abbreviations

The following abbreviations are used in this manuscript:

| | |
|---|---|
| EMI | Enhanced midpoint integration |
| CAS | Computer algebra system |

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