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Periodic Solutions of the 4-Body Electromagnetic Problem and Application to *Li* Atom

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Abstract

The 4-body equations of motion are derived in our previously published paper. Here we prove the existence and uniqueness of a periodic solution by applying the fixed point method for a suitable introduced operator. A natural example of such a problem is the Lithium atom, which has three electrons orbiting the nucleus. The inequalities that ensure the existence of a periodic solution allow us to estimate the minimal distances between the electrons on the first and second Bohr-Sommerfeld stationary states.

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MSC: 78A35; 34K40; 47H10

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1. Introduction

The main purpose of the present paper is to prove the existence-uniqueness of a periodic solution of the equations of motion for 4-body problem of classical electrodynamics derived in a recent paper [1], where a system of 16 equations of motion in Minkowski space is introduced:

$$m_k \frac{d\lambda_r^{(k)}}{ds_k} = \frac{e_k}{c^2}\left(\sum_{n=1,n\neq k}^{4} F_{rl}^{(kn)}\lambda_l^{(k)} + F_{rl}^{(k)rad}\lambda_l^{(k)}\right) \; (k=1,2,3,4;\ r=1,2,3,4)\,,$$

(1)

where there is a summation on repeating l, c is the vacuum speed of light, $m_k\,(k=1,2,3,4)$ are the masses, $e_k\,(k=1,2,3,4)$ – the charges of the moving particles, $\lambda_l^{(k)}$ – the unit tangent vectors to the world lines; $\langle .,.\rangle_4$ is the dot product in the Minkowski space, $\langle .,.\rangle$ – the usual dot product in the 3-dimensional subspace. The elements of the electromagnetic tensors $F_{rl}^{(kn)} = \frac{\partial\Phi_l^{(n)}}{\partial x_r^{(k)}} - \frac{\partial\Phi_r^{(n)}}{\partial x_l^{(k)}}$ can be calculated by the retarded Lienard-Wiechert potentials $\Phi_r^{(n)} = -\frac{e_n\lambda_r^{(n)}}{\left\langle \lambda^{(n)}, \xi^{(kn)}\right\rangle_4}$ (cf. [2] - [4]), while the radiation terms $F_{rl}^{(k)rad}$ – as a half a difference of retarded and advanced potentials in accordance with the Dirac assumption [5]:

$$F_{mn}^{(k)rad} = \frac{1}{2}\left[\left(\frac{\partial A_n^{(k)ret}}{\partial x_m^{(k)ret}} - \frac{\partial A_m^{(n)ret}}{\partial x_n^{(k)ret}}\right) - \left(\frac{\partial A_n^{(k)adv}}{\partial x_m^{(k)adv}} - \frac{\partial A_m^{(n)adv}}{\partial x_n^{(k)adv}}\right)\right] \quad , \quad A_n^{(k)ret} = -\frac{e_k\lambda_n^{(k)ret}}{\left\langle \lambda^{(k)ret}, \xi^{(k)ret}\right\rangle_4} \quad ,$$

$$A_n^{(k)adv} = -\frac{e_k\lambda_n^{(k)adv}}{\left\langle \lambda^{(k)adv}, \xi^{(k)adv}\right\rangle_4}\,.$$

In [1] we have proved that every fourth equation of (1) is a consequence of the first three ones. In this way we obtain 12 equations for unknown velocities.

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The paper consists of six sections and four appendices. Section 1 is an Introduction. Section 2 contains the system of equations of motion simplified for non-relativistic cases. In Section 3 the operator formulation for periodic solution is given. We introduce a suitable space of periodic functions and recall some properties of the operator functions. Section 4 begins with the Main lemma – the system of equations of motion has a unique solution iff the operator defined has a unique fixed point. Then we formulate the Main theorem guaranteeing an existence-uniqueness of smooth periodic solution which implies an existence of orbits for the 4-body problem. The proof is based on the fixed-point theorem proved in a previous paper [14]. In Section 5 we apply the result obtained to *Li* atom and calculate the minimal distance between moving electrons. Section 6 is a Conclusion, where we confirm the results from the 2- and 3- body problem showing the existence of periodic orbits.

2. Equations of Motion

Compared to [1] here we extend the assumption **(C0)**: $\left\|\vec{u}^{(k)}\right\| = \sqrt{\left\langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t)\right\rangle} \le \bar{c} < c \ (k = 1,2,3,4)$ by the following one

$$\textbf{(C)}: \sqrt{\left\langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t)\right\rangle} \le c_k < c \ \ (k = 1,2,3,4).$$

Following Sommerfeld [6] we denote by $\beta_k = c_k / c < 1 (k = 1,2,3,4)$ and rewrite the system of equations of motion from [1] in the form:

$$\dot{u}_\alpha^{(k)}(t) = \sum_{n=1, n\ne k}^{4} G_\alpha^{(kn)} - \beta_k^2 \sum_{\gamma=1,\gamma\ne\alpha}^{3} \sum_{n=1,n\ne k}^{4} G_\gamma^{(kn)} + G_\alpha^{(k)rad} - \beta_k^2 \sum_{\gamma=1,\gamma\ne\alpha}^{3} G_\gamma^{(k)rad} \equiv U_\alpha^{(k)}(t) \ (k = 1,2,3,4;\ \alpha = 1,2,3), \qquad \textbf{(BS0)}$$

where $u_\alpha^{(k)}(t) = u_{\alpha 0}^{(k)}(t)$, $\dot{u}_\alpha^{(k)}(t) = \dot{u}_{\alpha 0}^{(k)}(t)$, $t \le 0$ $(k = 1,2,3,4; \alpha = 1,2,3)$, $u_{\alpha 0}^{(k)}(t)$ are prescribed initial functions.

The above 4-body system of equations of motion consists of neutral differential equations with retarded arguments with second-order derivatives generated by the radiation terms. The delays depend on the unknown trajectories (cf. [7]). Such type of equations generates specific difficulties.

Remark 2.1. *Our considerations include moving electrons in the first and second N. Bohr-A. Sommerfeld stationary states, therefore, following Sommerfeld's notation, we introduce the notation for the dependencies* $\beta_1 = c_1 / c = 1/137$, $\beta_2 = \frac{\beta_1}{2} = \frac{1}{2 \times 137}; \beta_3 = \frac{\beta_1}{4} = \frac{1}{4 \times 137}, \ldots$ *(cf. [8]-[10]), where* c_1 *is the velocity of the electron in the first stationary state,* c_2 *- in the second stationary state and so on. Obviously one can disregard the values*

$$\beta_1^2 = 1/\left(137^2\right) \approx 0; \beta_2^2 = 1/\left(2^2 \times 137^2\right) \approx 0; \beta_3^2 = 1/\left(4^2 \times 137^2\right) \approx 0. \qquad (2)$$

In view of $U_0 e^{\mu_k T^{(k)}} \le \bar{c}_k < c$, ($\bar{c}_k / c = \beta_k < 1$) the inequalities (cf. (7) in Appendix 1)

$$\left|G_\alpha^{(kn)}(t)\right| \le \frac{|e_k e_n|(1+\beta_n)}{m_k}\left(\frac{1}{c^2 \tau_{kn}^2} + \frac{\omega_n U_0 e^{\mu_n T^{(n)}}}{c^3 \tau_{kn}}\right) \frac{|e_k e_n|(1+\beta_n)}{m_k c^2}\left(\frac{1}{\tau_{kn}^2} + \frac{\omega_n \beta_n}{\tau_{kn}}\right);$$

$$\left|G_\alpha^{(k)rad}\right| \le \frac{e_k^2}{m_k} \frac{\omega_k^2 U_0 e^{\mu_k T^{(k)}}}{c^3} \le \frac{e_k^2}{m_k} \frac{\omega_k^2 c_k}{c^3} = \frac{e_k^2}{m_k} \frac{\omega_k^2 \beta_k}{c^2}$$

show that all terms $G_\alpha^{(kn)}$ are of the same order. Consequently, we can neglect the terms containing multipliers $\beta_1^2, \beta_2^2, \beta_3^2 \approx 0$. In this way we consider the following simplified system of equations of motion (BS)**:**

$$\dot{u}_\alpha^{(k)}(t) = \sum_{n=1,n\ne k}^{4} G_\alpha^{(kn)} + G_\alpha^{(k)rad} \equiv U_\alpha^{(k)}(t) \ (k = 1,2,3,4;\ \alpha = 1,2,3)$$

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or in details

$$\dot{u}_{\alpha}^{(k)}(t)=\sum_{n=1,n\neq k}^{4}\frac{e_k e_n\left(A_{kn}\xi_{\alpha}^{(kn)}-B_{kn}u_{\alpha}^{(n)}+C_{kn}\dot{u}_{\alpha}^{(n)}\right)}{m_k c}-\frac{e_k^2}{m_k}\frac{u_{\alpha}^{(k)}\left\langle \vec{u}^{(k)},\ddot{\vec{u}}^{(k)}\right\rangle+c^2\ddot{u}_{\alpha}^{(k)}}{c^5}\equiv U_{\alpha}^{(k)}(t)\;(k=1,2,3,4;\;\alpha=1,2,3),$$

(BS)

where A_{kn}, B_{kn}, C_{kn} are defined in Appendix 1.

Recalling (cf. [1]) that $\vec{u}^{(k)}=\vec{u}^{(k)}(t)$ and $u_{\alpha}^{(n)}=u_{\alpha}^{(n)}(t-\tau_{kn})$, $\dot{u}_{\alpha}^{(n)}=\dot{u}_{\alpha}^{(n)}(t-\tau_{kn})$ we conclude that (BS) is a neutral system with state dependent delays τ_{kn} which are defined as solutions of the functional equations $\tau_{kn}(t)=\frac{1}{c}\sqrt{\left\langle \vec{\xi}^{(kn)},\vec{\xi}^{(kn)}\right\rangle}\equiv\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_{\gamma}^{(k)}(t)-x_{\gamma}^{(n)}(t-\tau_{kn}(t))\right]^2}$,

(3)

where $\vec{\xi}^{(kn)}=\left(\xi_1^{(kn)},\xi_2^{(kn)},\xi_3^{(kn)}\right)=\left(x_1^{(k)}(t)-x_1^{(n)}(t-\tau_{kn}),x_2^{(k)}(t)-x_2^{(n)}(t-\tau_{kn}),x_3^{(k)}(t)-x_3^{(n)}(t-\tau_{kn})\right)$.

3. Operator Formulation of the Periodic Problem in Suitable Function Spaces and Preliminary Results

We prove the existence-uniqueness of $T^{(k)}$-periodic solution of the system (BS) jointly with the functional equations for the delays (3).

Further on we assume that the following compatibility condition is satisfied:

(CC) $\dot{u}_{\alpha0}^{(k)}(0)=U_{\alpha}^{(k)}(0);\ddot{u}_{\alpha0}^{(k)}(0)=\dot{U}_{\alpha}^{(k)}(0),\dddot{u}_{\alpha0}^{(k)}(0)=\ddot{U}_{\alpha}^{(k)}(0),\ldots\;(\alpha=1,2,3;k=1,2,3,4)$.

By $C_{T^{(k)}}^{\infty}[0,\infty),(k=1,2,3,4)$ we denote the set of all infinite differentiable $T^{(k)}$-periodic functions. Denoting by $T_p^{(k)}=pT^{(k)}(p=0,1,2,\ldots)$ we introduce the sets of functions:

$$M_{\alpha}^{k}=\left\{u_{\alpha}^{(k)}\in C_{T^{(k)}}^{\infty}[-T^{(k)},\infty):\left|\frac{d^m u_{\alpha}^{(k)}(t)}{dt^m}\right|\leq U_0^{(k)}\omega_k{}^m e^{\mu_k(t-T_p{}^{(k)})},\;t\in[T_p{}^{(k)},T_{p+1}{}^{(k)}];\right.$$
$$\left.\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}u_{\alpha}^{(k)}(t)dt=0\;,\frac{d^m u_{\alpha}^{(k)}(0)}{dt^m}=0;u_{\alpha}^{(k)}(t)=u_{\alpha0}^{(k)}(t),\frac{d^m u_{\alpha0}^{(k)}(0)}{dt^m}=\frac{d^m u_{\alpha0}^{(k)}(-T^{(k)})}{dt^m}=0,t\in[-T^{(k)},0]\right\}$$

$(k=1,2,3,4;\;\alpha=1,2,3;\;p=0,1,\ldots;\;m=1,2,\ldots$, where $U_0^{(k)}$, ω_k, $T^{(k)}$, $\mu_k>\omega_k$ are positive constants and $u_{\alpha0}^{(k)}(t)$ are prescribed initial functions satisfying **(CC)**.

Introduce a family of pseudometrics

$$\rho_{(p,m)}\left(u^{(k)},\bar{u}^{(k)}\right)=ess\sup\left\{e^{-\mu_k(t-T_p{}^{(k)})}\omega_k{}^{-m}\left|\frac{d^m u^{(k)}(t)}{dt^m}-\frac{d^m\bar{u}^{(k)}(t)}{dt^m}\right|:t\in[T_p{}^{(k)},T_{p+1}{}^{(k)}]\right\},(p=0,1,2,\ldots;\;m=0,1,2,\ldots)$$

.

It is easy to see that the following inequalities

$$e^{-\mu_k(t-T_p{}^{(k)})}\omega_k{}^{-m}\left|\frac{d^m u^{(k)}(t)}{dt^m}-\frac{d^m\bar{u}^{(k)}(t)}{dt^m}\right|\leq e^{-\mu_k(t-T_p{}^{(k)})}\omega_k{}^{-m}\,2\,\omega_k{}^m e^{\mu_k(t-T_p{}^{(k)})}U_0^{(k)}=2U_0^{(k)}<\infty$$

(4)

are satisfied for every (p,m) and $t\in[T_p{}^{(k)},T_{p+1}{}^{(k)}]$. Therefore $\sup\left\{\rho_{(p,m)}\left(u,\bar{u}\right):m=0,1,2,\ldots;\;p=1,2,\ldots\right\}<\infty$.

Remark 3.1. *Let $T^{(k)}$ be the period of the solution. We notice that all arguments of the unknown functions are $t-\tau_{km}$, that is, $\vec{u}=\vec{u}^{(m)}(t-\tau_{km})$. Therefore, we must look for a solution on the initial set, that is, for $t-\tau_{km}(t)\in[\tau_0;0]$, where $\tau_0=\min\{t-\tau_{km}(t):t\in[0,T^{(k)}]\}$. Since $1-d\tau_{km}(t)/dt>0$, then $t-\tau_{km}(t)$ is an* *increasing function. We have proved that if the trajectories are $T^{(k)}$-periodic, then $\tau_{km}(t)$ is $T^{(k)}$-periodic, too. It follows that $-T^{(k)}-\tau_{km}(-T^{(k)})\le 0-\tau_{km}(0)\Leftrightarrow \tau_{km}(0)\le T^{(k)}-\tau_{km}(0)$, $-T^{(k)}-\tau_{km}(0)\le -\tau_{km}(0)$ and then*

$0-\tau_{km}(0)\le t-\tau_{km}(t)\le -\tau_{km}(T^{(k)})=T^{(k)}-\tau_{km}(0)=0$. Consequently, $-T^{(k)}\le t-\tau_{km}(t)\le 0$. Therefore, $t-\tau_{km}(t):[0,T^{(k)}]\to[-T^{(k)},0]$, that is, $\tau_0=-T^{(k)}$ and $t-\tau_{km}(t)\in[-T^{(k)},0]$.

Recall that the condition $\int\limits_{T_p^{(k)}}^{T_{p+1}^{(k)}} u(t)dt=0$ $(p=0,1,2,3,...)$ implies $x(t)=\int\limits_0^t u(s)ds=0$ is $T^{(k)}$-periodic function. We have, however, already proved in [11] that every solution of the functional equation for τ_{kn} has a unique continuous solution for all continuous Lipschitz trajectories and $\tau_{km}\ge r_{km}(t)/2c$, and if $\left(x_1^{(k)}(t),x_2^{(k)}(t),x_3^{(k)}(t)\right),(k=1,2,3,4)$ are $T^{(k)}$-periodic functions, then $\tau_{kn}(t)$ are $T^{(k)}$-periodic functions, too.

Introduce operator B as a 12-tuple $B=\left(\vec{B}^{(1)}(u)(t),\vec{B}^{(2)}(u)(t),\vec{B}^{(3)}(u)(t),\vec{B}^{(4)}(u)(t)\right)$,

$$B_\alpha^{(k)}(p,u)(t):=\begin{cases}\int\limits_0^t U_\alpha^{(k)}(s)ds-\left(\dfrac{t}{T^{(k)}}-\dfrac{1}{2}\right)\int\limits_0^{T^{(k)}} U_\alpha^{(k)}(s)ds-\dfrac{1}{T^{(k)}}\int\limits_0^{T^{(k)}}\int\limits_0^{\theta} U_\alpha^{(k)}(s)dsd\theta, t\in[0,T^{(k)}];\\ \int\limits_{T_p^{(k)}}^{t} U_\alpha^{(k)}(s)ds-\left(\dfrac{t-T_p^{(k)}}{T^{(k)}}-\dfrac{1}{2}\right)\int\limits_{T_p^{(k)}}^{T_{p+1}^{(k)}} U_\alpha^{(k)}(s)ds-\dfrac{1}{T^{(k)}}\int\limits_{T_p^{(k)}}^{T_{p+1}^{(k)}}\int\limits_{T_p^{(k)}}^{t} U_\alpha^{(k)}(s)dsdt, t\in[T_p^{(k)},T_{p+1}^{(k)}], p=1,2,...\\ u_{\alpha 0}^{(k)}(t), t\in[-T^{(k)},0]\end{cases}$$

$(k=1,2,3,4;\ \alpha=1,2,3)$, where $u_{\alpha 0}^{(k)}(t)$ are prescribed infinite differentiable initial functions defined on $[-T^{(k)},0]$.

We use the assertions:

1) $u_\alpha^{(k)}\in M_\alpha^k\Rightarrow\int\limits_{T_p^{(k)}}^{T_{p+1}^{(k)}}\int\limits_{T_p^{(k)}}^{s} U_\alpha^{(k)}(\theta)d\theta ds=\int\limits_{T_{p+1}^{(k)}}^{T_{p+2}^{(k)}}\int\limits_{T_{p+1}^{(k)}}^{s} U_\alpha^{(k)}(\theta)d\theta ds$ $(p=0,1,2,...)$ (cf. [12]),

2) $B_\alpha^{(k)}(.)\in M_\alpha^k$ (cf. [12]).

4. Existence-Uniqueness of a Periodic Solution of the Equations of Motion

Lemma 4.1 (Main lemma) *The* $\left(T^{(1)},T^{(2)},T^{(3)},T^{(4)}\right)$*-periodic problem for* **(BS)** *has a unique solution from*

$$M=M_1^1\times M_2^1\times M_3^1\times M_1^2\times M_2^2\times M_3^2\times M_1^3\times M_2^3\times M_3^3\times M_1^4\times M_2^4\times M_3^4$$

iff the operator $\left(\vec{B}^{(1)}(u)(t),\vec{B}^{(2)}(u)(t),\vec{B}^{(3)}(u)(t),\vec{B}^{(4)}(u)(t)\right)$ has a fixed point, belonging to M.

The proof is like the case of 2- and 3-body problems (cf. [12], [13]).

Theorem 4.1 *Let the following conditions be fulfilled:*

1) **(IN)** the initial trajectories $x_{\alpha 0}^{(k)}(t)$ and velocities $u_{\alpha 0}^{(k)}(t), t\in[-T^{(k)};0]$ are infinitely differentiable functions such that $r_{kn}(t)=\sqrt{\sum\limits_{\gamma=1}\left(x_\gamma^{(k)}(t)-x_\gamma^{(n)}(t)\right)}\ge \widehat{r}_{kn}>0$, $t\in[-T^{(k)};0]$ and satisfy **(CC)**.

2) The following (infinitely in number) inequalities are satisfied for $k=1,2,3,4$:

$$2\left[\sum_{n=1,n\ne k}^{4}\frac{|e_k e_n|(1+\beta_n)}{m_k}\left(\frac{4}{\left(r_{kn}^{(0)}\right)^2}+\frac{2\omega_n\beta_n}{cr_{kn}^{(0)}}\right)+\frac{e_k^2}{m_k}\frac{\omega_k{}^2\beta_k}{c^2}\right]\frac{e^{\mu_k T^{(k)}}-1}{\mu_k}\le U_0^{(k)}$$

$$2\left(1+\frac{e^{\mu_k T^{(k)}}-1}{\mu_k T^{(k)}}\right)\left[\sum_{n=1,n\neq k}^{4}\frac{|e_k e_n|(1+\beta_n)}{m_k}\left(\frac{4}{\left(r_{kn}^{(0)}\right)^2}+\frac{2\omega_n\beta_n}{cr_{kn}^{(0)}}\right)+\frac{e_k^2}{m_k}\frac{\omega_k^2\beta_k}{c^2}\right]\le\omega_k U_0^{(k)}$$

$$\sum_{n=1,n\neq k}^{4}\frac{|e_k e_n|}{m_k}\left(\frac{32c(\beta_k+\beta_n)}{\hat{r}_{kn}^{3}}+\frac{8\omega_n\beta_n}{\hat{r}_{kn}^{2}}+\frac{8\omega_n^2\beta_n}{c\hat{r}_{kn}}\right)+\frac{e_k^2}{m_k}\frac{\beta_k\omega_k^3}{c^2}\le\omega_k^2 U_0^{(k)}$$

and so on.

Then there is a unique $\left(T^{(1)},T^{(2)},T^{(3)},T^{(4)}\right)$ -periodic solution $(u_1^{(1)},u_2^{(1)},u_3^{(1)},u_1^{(2)},u_2^{(2)},u_3^{(2)},u_1^{(3)},u_2^{(3)},u_3^{(3)},u_1^{(4)},u_2^{(4)},u_3^{(4)})$ of **(BS)** for $t\ge 0$.

Proof: *In accordance with the Main lemma we must prove that the operator* B *possesses a unique fixed point, which means that the 4-body problem has a unique periodic solution.*

First, we show that operator B maps the set M into itself.

The set M can be considered as a uniform space with saturated family of pseudo-metrics formed by $\left\{\rho_{(p,m)}\left(u_\alpha^{(k)},\bar{u}_\alpha^{(k)}\right): p=0,1,\ldots\,;\, m=0,1,\ldots\right\}$ in the following way

$$\left\{\rho_{(p,m)}\left((u_1,u_2,\ldots,u_{12}),(\bar{u}_1,\bar{u}_2,\ldots,\bar{u}_{12})\right)=\sum_{q=1}^{12}\rho_{(p,m)}\left(u_q,\bar{u}_q\right):\ p=0,1,\ldots\,;\ m=0,1,\ldots\right\},$$

where $(u_1,u_2,\ldots,u_{12})=(u_1^{(1)},u_2^{(1)},u_3^{(1)},u_1^{(2)},u_2^{(2)},u_3^{(2)},u_1^{(3)},u_2^{(3)},u_3^{(3)},u_1^{(4)},u_2^{(4)},u_3^{(4)})$.

The operator functions $B_\alpha^{(k)}(u_1^{(1)},\ldots,u_3^{(4)})(.)\in M_\alpha^k$, $(k=1,2,3,4)$, $(\alpha=1,2,3)$.

Indeed, recall that $T^{(k)}=T_{p+1}^{(k)}-T_p^{(k)}(p=0,1,2,\ldots)$. One obtains:

$$B_\alpha^{(k)}(u_1^{(1)},\ldots,u_3^{(4)})(0)=\int_0^0 U_\alpha^{(k)}(s)ds-\left(\frac{0}{T^{(k)}}-\frac{1}{2}\right)\int_0^{T^{(k)}}U_\alpha^{(k)}(s)ds-\frac{1}{T^{(k)}}\int_0^{T^{(k)}}\int_0^{\theta}U_\alpha^{(k)}(s)dsd\theta=\frac{1}{2}\int_0^{T^{(k)}}U_\alpha^{(k)}(s)ds-\frac{1}{T^{(k)}}\int_0^{T^{(k)}}\int_0^{\theta}U_\alpha^{(k)}(s)dsd\theta$$

,

$$B_\alpha^{k}(u_1^{(1)},\ldots,u_3^{(4)})(T^{(k)})=\int_0^{T^{(k)}}U_\alpha^{(k)}(s)ds-\left(\frac{T^{(k)}}{T^{(k)}}-\frac{1}{2}\right)\int_0^{T^{(k)}}U_\alpha^{(k)}(s)ds-\frac{1}{T^{(k)}}\int_0^{T^{(k)}}\int_0^{\theta}U_\alpha^{(k)}(s)dsd\theta=\frac{1}{2}\int_0^{T^{(k)}}U_\alpha^{(k)}(s)ds-\frac{1}{T^{(k)}}\int_0^{T^{(k)}}\int_0^{\theta}U_\alpha^{(k)}(s)dsd\theta$$

,

that is $B_\alpha^{(k)}(u_1^{(1)},\ldots,u_3^{(4)})(0)=B_\alpha^{k}(u_1^{(1)},\ldots,u_3^{(4)})(T^{(k)})$.

In view of the **(CC)** we have $\frac{dB_\alpha^{(k)}(u_1^{(1)},\ldots,u_3^{(3)})(0)}{dt}=U_\alpha^{(k)}(0)=\dot{u}_\alpha^{(k)}(0)$, $\frac{d^2B_\alpha^{(k)}(u_1^{(1)},\ldots,u_3^{(3)})(0)}{dt^2}=\dot{U}_\alpha^{(k)}(0)=\ddot{u}_\alpha^{(k)}(0)$ and so on. Since $\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\left(\frac{t-T_p^{(k)}}{T^{(k)}}-\frac{1}{2}\right)dt=0$ we obtain

$$\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}B_\alpha^{(k)}(p)(u_1^{(1)},\ldots,u_3^{(4)})(t)dt=\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\int_{T_p}^{\theta}U_\alpha^{(k)}(s)dsd\theta-\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\left(\frac{t}{T^{(k)}}-\frac{1}{2}\right)dt\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}U_\alpha^{(k)}(s)ds-T^{(k)}\frac{1}{T^{(k)}}\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\int_{T_p^{(k)}}^{\theta}U_\alpha^{(k)}(s)dsd\theta=0$$

,

that is, $B_\alpha^{(k)}(u_1^{(1)},\ldots,u_3^{(4)})(.)\in M_\alpha^k$.

The inequalities from Appendix 1 imply:

$$\left|B_\alpha^{(k)}(p,u)(t)\right| \le \int_{T_p^{(k)}}^{t}\left|U_\alpha^{(k)}(s)\right|ds + \frac{1}{2}\left|\int_{T_p^{(k)}}^{T_{p+1}^{(k)}} U_\alpha^{(k)}(s)ds\right| + \frac{1}{2}\left|\int_{T_p^{(k)}}^{T_{p+1}^{(k)}} U_\alpha^{(k)}(s)ds\right| \le 2\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\left|U_\alpha^{(k)}(s)\right|ds \le$$

$$\le 2\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\left|\sum_{n=1,n\neq k}^{4} G_\alpha^{(kn)} + G_\alpha^{(k)rad}\right|ds \le 2\left(\sum_{n=1,n\neq k}^{4}\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\left|G_\alpha^{(kn)}\right|ds + \int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\left|G_\alpha^{(k)rad}\right|ds\right) \le$$

$$\le 2\left[\sum_{n=1,n\neq k}^{4}\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\frac{|e_k e_n|(1+\beta_n)}{m_k}\left(\frac{1}{c^2\tau_{kn}^{\,2}} + \frac{\omega_n U_0^k e^{\mu_n T^{(n)}}}{c^3\tau_{kn}}\right)ds + \int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\frac{e_k^2}{m_k}\frac{\omega_k^{\,2} U_0^k e^{\mu_k T^{(k)}}}{c^3}ds\right] \le$$

$$\le 2\left[\sum_{n=1,n\neq k}^{4}\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\frac{|e_k e_n|(1+\beta_n)}{m_k}\left(\frac{1}{c^2\tau_{kn}^{\,2}} + \frac{\omega_n\beta_n}{c^2\tau_{kn}}\right)ds + \int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\frac{e_k^2}{m_k}\frac{\omega_k^{\,2}\beta_k}{c^2}ds\right] \le$$

$$\le 2\left[\sum_{n=1,n\neq k}^{4}\frac{|e_k e_n|(1+\beta_n)}{m_k}\left(\frac{4}{\left(r_{kn}^{\,(0)}\right)^2} + \frac{2\omega_n\beta_n}{c r_{kn}^{\,(0)}}\right) + \frac{e_k^2}{m_k}\frac{\omega_k^{\,2}\beta_k}{c^2}\right]\frac{e^{\mu_k T^{(k)}}-1}{\mu_k} \le U_0^{(k)} e^{\mu_k (t-T_p^{(k)})}.$$

For the first derivatives we have

$$\left|\frac{dB_\alpha^{(k)}(t)}{dt}\right| \le \left|U_\alpha^k(t)\right| + \left|\frac{1}{T^{(k)}}\int_{T_p^{(k)}}^{T_{p+1}^{(k)}} U_\alpha^{(k)}(s)ds\right| \le 2\left[\sum_{n=1,n\neq k}^{4}\frac{|e_k e_n|(1+\beta_n)}{m_k}\left(\frac{4}{\hat{r}_{kn}^{\,2}} + \frac{2\omega_n\beta_n}{c\hat{r}_{kn}}\right) + \frac{e_k^2}{m_k}\frac{\omega_k^{\,2}\beta_k}{c^2}\right] +$$

$$+\frac{2(e^{\mu_k T^{(k)}}-1)}{\mu_k T^{(k)}}\left[\sum_{n=1,n\neq k}^{4}\frac{|e_k e_n|(1+\beta_n)}{m_k}\left(\frac{4}{\hat{r}_{kn}^{\,2}} + \frac{2\omega_n\beta_n}{c\hat{r}_{kn}}\right) + \frac{e_k^2}{m_k}\frac{\omega_k^{\,2}\beta_k}{c^2}\right] \le 2\left(1 + \frac{e^{\mu_k T^{(k)}}-1}{\mu_k T^{(k)}}\right)\left[\sum_{n=1,n\neq k}^{4}\frac{|e_k e_n|(1+\beta_n)}{m_k}\left(\frac{4}{\hat{r}_{kn}^{\,2}} + \frac{2\omega_n\beta_n}{c\hat{r}_{kn}}\right) + \frac{e_k^2}{m_k}\frac{\omega_k^{\,2}\beta_k}{c^2}\right] \le \omega_k U_0^{(k)} e^{\mu_k (t-T_p^{(k)})}$$

For the second derivative we have

$$\left|\frac{d^2 B_\alpha^{(k)}(t)}{dt^2}\right| = \left|\frac{dU_\alpha^{(k)}(t)}{dt}\right| \le \sum_{n=1,n\neq k}^{4}\frac{|e_k e_n|}{m_k}\left(\frac{24c}{\hat{r}_{kn}^{\,3}} + \frac{20\beta_n\omega_n}{\hat{r}_{kn}^{\,2}} + \frac{4\omega_n^{\,2}\beta_n}{c\hat{r}_{kn}}\right) + \frac{e_k^2}{m_k}\frac{\beta_k\omega_k^{\,3}}{c^2} \le \omega_k^{\,2} U_0^{(k)} e^{\mu_k (t-T_p^{(k)})}.$$

The last inequality is satisfied because on the right-hand side appears $\omega_k^{\,2}$ and the same is true for higher-order derivatives.

Consequently, B maps $M = M_1^1\times M_2^1\times M_3^1\times M_1^2\times M_2^2\times M_3^2\times M_1^3\times M_2^3\times M_3^3\times M_1^4\times M_2^4\times M_3^4$ into itself.

It remains to show that B is a contractive operator.

First, we define mappings of the index set into itself. They are generated by delay functions τ_{kn}, that is, $j_{kn} = t-\tau_{kn}(t):[T_p^{(k)},T_{p+1}^{(k)}]\to[T_{p-1}^{(k)},T_p^{(k)}]$; $(k\,n) = (12),(13),(14),(21),(23),(24),(31),(32),(34),(41),(42),(43)$.

It is easy to see that every interval $[T_p^{(k)},T_{p+1}^{(k)}]$ after finite number of iterations of j_{kn} coincides with $[-T^{(k)},0]$,

$$\left|B_\alpha^{(k)}(p,u)(t) - B_\alpha^{(k)}(p,\bar{u})(t)\right| \le \int_{T_p^{(k)}}^{t}\left|U_\alpha^{(k)}(u) - U_\alpha^{(k)}(\bar{u})\right|ds + \left|\frac{t-T_p^{(k)}}{T^{(k)}} - \frac{1}{2}\right|\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\left|U_\alpha^{(k)}(u) - U_\alpha^{(k)}(\bar{u})\right|ds +$$

$$+\frac{1}{T^{(k)}}\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\int_{T_p^{(k)}}^{\theta}\left|U_\alpha^{(k)}(u) - U_\alpha^{(k)}(\bar{u})\right|ds d\theta \le \int_{T_p^{(k)}}^{t}\left|U_\alpha^{(k)}(u) - U_\alpha^{(k)}(\bar{u})\right|ds + \int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\left|U_\alpha^{(k)}(u) - U_\alpha^{(k)}(\bar{u})\right|ds \le 2\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\left|U_\alpha^{(k)}(u) - U_\alpha^{(k)}(\bar{u})\right|ds \le$$

$$\le 2\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\sum_{n\neq k,n=1}^{4}\left|G_\alpha^{(kn)}(u) - G_\alpha^{(kn)}(\bar{u})\right|ds + 2\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\left|G_\alpha^{(k)rad}(u) - G_\alpha^{(k)rad}(\bar{u})\right|ds \le$$

$$\begin{aligned}
&\le 2\int_{T_p^{(k)}}^{T_{p+1}^{(k)}} e^{\mu_k(s-T_p^{(k)})} \sum_{n=1,n\ne k}^{4} \frac{|e_k e_n|}{m_k}\left[\left(\frac{8}{\hat r_{kn}^{3}} + \frac{8(1+\beta_n)\omega_n U_0^k e^{\mu_k T^{(k)}}}{c^2\hat r_{kn}^2}\right)\sum_{\gamma=1}^3\left(\frac{\omega_k^h}{\mu_k^{h+1}}\rho_{(p,h)}(u_\alpha^{(k)},\bar u_\alpha^{(k)})\right)+\right.\\
&+\frac{2\omega_n U_0^k e^{\mu_k T^{(k)}}}{c^3 \hat r_{kn}}\sum_{\gamma=1}^3\left(\frac{\omega_k^h}{\mu_k^h}\rho_{(p,h)}(u_\alpha^{(k)},\bar u_\alpha^{(k)})\right)+\left(\frac{8}{\hat r_{kn}^3}+\frac{8\omega_n U_0^k e^{\mu_k T^{(k)}}}{c^2\hat r_{kn}^2}\right)e^{\mu_k T^{(k)}}\sum_{\gamma=1}^3\left(\frac{\omega_k^h}{\mu_k^{h+1}}\rho_{(p-1,h)}(u_\alpha^{(k)},\bar u_\alpha^{(k)})\right)+\\
&\left.+4\frac{c^2+c\tau_{kn}\omega_n U_0^k e^{\mu_k T^{(k)}}}{c^3\hat r_{kn}^2} e^{\mu_k T^{(k)}}\sum_{\gamma=1}^3\left(\frac{\omega_k^h}{\mu_k^h}\rho_{(p-1,h)}(u_\gamma^{(n)},\bar u_\gamma^{(n)})\right)+\frac{4(1+\beta_k+\beta_n)e^{\mu_k T^{(k)}}\frac{\omega_k^h}{\mu_k^{h+1}}\sum_{\gamma=1}^3\rho_{(p-1,h)}(u_\gamma^{(k)},\bar u_\gamma^{(k)})}{c^2\hat r_{kn}}\right]ds+\\
&+\int_{T_p^{(k)}}^{T_{p+1}^{(k)}} e^{\mu_k(s-T_p^{(k)})}\frac{e_k^2\omega_k^2}{m_k c^3}e^{\mu_k T_p^{(k)}}\left(\frac{2\beta_k U_0^k}{c}\sum_{\gamma=1}^3\left(\frac{\omega_k}{\mu_k}\right)^h\rho_{(p,h)}(u_\gamma^{(k)},\bar u_\gamma^{(k)})+\sum_{\gamma=1}^3\frac{\omega_k^{h+2}}{\mu_k^h}\rho_{(p,h+2)}(u_\gamma^{(k)},\bar u_\gamma^{(k)})\right)ds\le\\
&\le K_{(p-1,h)}\rho_{(p-1,h)}\left((u_1^{(1)},u_2^{(1)},...,u_3^{(4)}),(\bar u_1^{(1)},\bar u_2^{(1)},...,\bar u_3^{(4)})\right)+K_{(p,h+2)}\rho_{(p,h+2)}\left((u_1^{(1)},u_2^{(1)},...,u_3^{(4)}),(\bar u_1^{(1)},\bar u_2^{(1)},...,\bar u_3^{(4)})\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\rho_{(p,0)}\left(B_1^{(1)},B_2^{(1)},...,B_3^{(4)}\right),\left(\bar B_1^{(1)},\bar B_2^{(1)},...,\bar B_3^{(4)}\right)\le\\
&\le K_{(p-1,h)}\rho_{(p-1,h)}\left((u_1^{(1)},u_2^{(1)},...,u_3^{(4)}),(\bar u_1^{(1)},\bar u_2^{(1)},...,\bar u_3^{(4)})\right)+K_{(p,h+2)}\rho_{(p,h+2)}\left((u_1^{(1)},u_2^{(1)},...,u_3^{(4)}),(\bar u_1^{(1)},\bar u_2^{(1)},...,\bar u_3^{(4)})\right),
\end{aligned}$$

where $K_{(p-1,h)}+K_{(p,h+2)}<1$ for sufficiently large $h\in N$ and $\mu_k>\omega_k$.

Define the map of the index set into itself $j_1(p,0)\to(p-1,h), j_2(p,0)\to(p,h+2)$. It is easy to see that the maps j_1,j_2 commute, and in view of inequality (4) the space

$$M_1^1\times M_2^1\times M_3^1\times M_1^2\times M_2^2\times M_3^2\times M_1^3\times M_2^3\times M_3^3\times M_1^4\times M_2^4\times M_3^4$$

is (j_1,j_2) – bounded in the sense introduced in [14]. Consequently, the operator B is contracting and has a unique fixed point. It is a periodic solution to the 4-body problem in view of the Main lemma.

The Theorem 4.1 is thus proved.

Remark 4.1. *From the above inequalities of the Theorem 4.1, we obtain*

$$\begin{aligned}
&2\left(1+\frac{e^{\mu_k T^{(k)}}-1}{\mu_k T^{(k)}}\right)\left[\sum_{n=1,n\ne k}^4\frac{|e_k e_n|(1+\beta_n)}{m_k}\left(\frac{4}{\hat r_{kn}^2}+\frac{2\omega_n\beta_n}{c\hat r_{kn}}\right)+\frac{e_k^2}{m_k}\frac{\omega_k^2\beta_k}{c^2}\right]\le\\
&\le\left(1+\frac{e^{\mu_kT^{(k)}}-1}{\mu_kT^{(k)}}\right)\frac{\mu_k}{e^{\mu_kT^{(k)}}-1}U_0^{(k)}=\left(1+\frac{e^{\mu_kT^{(k)}}-1}{\mu_kT^{(k)}}\right)\frac{\mu_kT^{(k)}}{e^{\mu_kT^{(k)}}-1}\frac{U_0^{(k)}}{T^{(k)}}=\left(\frac{\mu_kT^{(k)}}{e^{\mu_kT^{(k)}}-1}+1\right)\frac{U_0^{(k)}}{T^{(k)}}\le\omega_kU_0^{(k)}\Leftrightarrow\frac{\mu_kT^{(k)}}{e^{\mu_kT^{(k)}}-1}+1\le\omega_kT^{(k)}=2\pi.
\end{aligned}$$

Indeed, $f(u)=\frac{u}{e^u-1}+1\le 2\pi\Rightarrow f'(u)=\frac{e^u-1-ue^u}{\left(e^u-1\right)^2}<0\Leftrightarrow g(u)=1+ue^u-e^u\Rightarrow g'(u)=ue^u>0$,

$$g(u)>g(0)\Leftrightarrow 1+ue^u-e^u>0\Rightarrow f'(u)<0\Rightarrow f(0)=2\Rightarrow f(u)=\frac{u}{e^u-1}+1\le 2\pi, u>0.$$

This means that the inequalities for the operator functions imply the inequalities for their derivatives.

5. Conditions of the Main Theorem Applied to the Lithium Atom

Let us consider the inequalities from Theorem 4,1, implying the existence-uniqueness of periodic solution for *Li* atom. First, we put the first particle (the nuclei) at the origin:

$$\left(x_1^{(1)}(t)=0,x_2^{(1)}(t)=0,x_3^{(1)}(t)=0\right)\Rightarrow\left(u_1^{(1)}(t)=0,u_2^{(1)}(t)=0,u_3^{(1)}(t)=0\right)\Rightarrow T^{(1)}=0,\omega_1=0.$$

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This means that the equation of motion for the first particle becomes $0=0$. Then the second, third and fourth particles are electrons – the second and third one move along the first stationary state, while the fourth – moves along the second stationary state. In view of [8]-[9] $\beta_2,\beta_3=\beta=1/137;\beta_4=\beta/2=1/274$; $c=3\times10^8\,m/\sec$. For the charges we have $e_1=3e_k=3\times1,6\times10^{-19}C=4.8\times10^{-19};e_2=e_3=e_4=-1,6.10^{-19}C$ and then $|e_1e_k|=4.8\times10^{-19}\times1,6\times10^{-19}=7.68\times10^{-38}$.

The masses are $m_1=m\times3\times1836=9,11.10^{-31}\times5508kg\simeq5.02\times10^{-27}kg$, $m=m_2=m_3=m_4=9,11\times10^{-31}kg$.

For the frequencies on the first Bohr-Sommerfeld orbit we take

$\omega_2=\omega_.=\omega=4.1\times10^{16}\Rightarrow T=T^{(2)}=T^{(3)}=2\pi/\omega\approx1.53\times10^{-16};\omega_4=\omega/8\simeq5.03\times10^{15}$.

Then $T_4=2\pi/\omega_4=(2\pi/\omega)\times8=8T\approx12.26\times10^{-16}$ and $\dfrac{m}{4|e_2e_3|}=\dfrac{9.11\times10^{-31}}{4\times2,56\times10^{-38}}\approx8.89\times10^6$; $\dfrac{5508m}{4|e_1e_2|}\approx4.9\times10^{10}$.

Let us choose $\mu_2=\mu_3=\mu=4.5\times10^{16}$. Then $\Rightarrow\mu T=4.5\times10^{16}\times1,53.10^{-16}=6.885\Rightarrow e^{\mu T}\simeq977.5$ and

$\dfrac{e^{\mu T}-1}{\mu T}=\dfrac{977.5-1}{6.885}=\dfrac{976.5}{6.885}\approx141.8\,;\dfrac{\mu}{e^{\mu T}-1}=\dfrac{4.5\times10^{16}}{977.5-1}\simeq5\times10^{13}$. For the second stationary state (cf. [8]-[9]) we take μ_4 such that $\mu_4T_4=\mu T$. This implies $\mu_4T_4=8\mu_4T=\mu T\Rightarrow\mu_4=\mu/8$ and then $\mu_4T_4=\mu T=688.5\Rightarrow e^{\mu_4T_4}\simeq977.5\,;\ \dfrac{e^{\mu_4T^{(4)}}-1}{\mu_4}=8\dfrac{e^{\mu T}-1}{\mu}=8\dfrac{977.5-1}{10^{15}}\simeq7.812\times10^{-12}$.

We recall $r_{kn}=r_{nk}$. The constants $\mu,\mu_4>0$ and $U_0^{(k)}>0\ (k=1,2,3,4)$ we choose as control parameters with restrictions $\mu,\mu_4\in(0;\infty);\mu T=const.$ and $U_0^{(k)}e^{\mu_kT^{(k)}}\le c_k\Leftrightarrow U_0^{(2)},U_0^{(3)}\le\dfrac{ce^{-\mu T}}{137}\le\dfrac{3\times10^8}{137}\simeq2.19\times10^6;U_0^{(4)}\le\dfrac{ce^{-\mu_4T^{(4)}}}{2\times137}\le\dfrac{3\times10^8}{274}\simeq10^6$.

Assuming $1+\beta_n\approx1$ we obtain the first group of inequalities for the distances between the particles of the lithium atom:

$$\frac{4}{\hat{r}_{23}^{\,2}}+\frac{2\omega}{137c\hat{r}_{23}}+\frac{4}{\hat{r}_{24}^{\,2}}+\frac{\omega\beta}{8c\hat{r}_{24}}\le\frac{U_0^{(2)}}{2}\frac{m}{|e_2e_3|}\frac{\mu}{e^{\mu T}-1}-\frac{12}{\hat{r}_{12}^{\,2}}-\frac{\omega^2\beta}{c^2};$$

$$\frac{4}{\hat{r}_{32}^{\,2}}+\frac{2\omega\beta}{c\hat{r}_{32}}+\frac{4}{\hat{r}_{34}^{\,2}}+\frac{\omega\beta}{8c\hat{r}_{34}}\le\frac{U_0^{(3)}}{2}\frac{m}{|e_3e_2|}\frac{\mu}{e^{\mu T}-1}-\frac{12}{\hat{r}_{13}^{\,2}}-\frac{\omega^2\beta}{c^2};;$$

$$\frac{4}{\hat{r}_{42}^{\,2}}+\frac{2\omega\beta}{c\hat{r}_{42}}+\frac{4}{\hat{r}_{43}^{\,2}}+\frac{2\omega\beta}{c\hat{r}_{43}}\le\frac{U_0^{(4)}}{2}\frac{m}{|e_4e_2|}\frac{\mu_4}{e^{\mu T}-1}-\frac{12}{\hat{r}_{14}^{\,2}}-\frac{\omega^2\beta}{128c^2}.$$

Since $\dfrac{e_2^{\,2}}{m_2}=\dfrac{e_3^{\,2}}{m_3}=\dfrac{e_4^{\,2}}{m_4}$ we denote by $A_k=\dfrac{m_k}{2e_k^{\,2}}\dfrac{\mu}{e^{\mu T}-1}(k=2,3,4)\Rightarrow A_2=A_3=A_4;A_k=\dfrac{m_k}{2e_k^{\,2}T}\dfrac{\mu T}{e^{\mu T}-1}\le\dfrac{m_k(2\pi-1)}{2e_k^{\,2}T}$;

$r_{12}=r_{31}=5.29\times10^{-11};r_{41}=4r_{14}=4\times5.29\times10^{-11}=21.16\times10^{-11}$; $\dfrac{e_k^2}{m_k}=\dfrac{1,6^2.10^{-38}}{9,11\times10^{-31}}=2.8\times10^{-8}$ and

$\dfrac{|e_1e_k|}{m}=\dfrac{3|e_3e_2|}{m}=3\times2.8\times10^{-8}=8.4\times10^{-8}(k=2,3,4)$. Then we obtain:

$$\frac{4}{\hat{r}_{23}^{\,2}}+\frac{2\times10^6}{\hat{r}_{23}}+\frac{4}{\hat{r}_{24}^{\,2}}+\frac{1.25\times10^5}{\hat{r}_{24}}\le\left(8.92U_0^{(2)}-42.9\right)\times10^{20}-1.36\times10^{14};$$

$$\frac{4}{\hat{r}_{23}^{\,2}}+\frac{2\times10^6}{\hat{r}_{23}}+\frac{4}{\hat{r}_{34}^{\,2}}+\frac{1.25\times10^5}{\hat{r}_{34}}\le\left(8.92U_0^{(3)}-42.9\right)\times10^{20}-1.36\times10^{14};$$

$$\frac{4}{\hat{r}_{24}^{\,2}}+\frac{2\times10^6}{\hat{r}_{24}}+\frac{4}{\hat{r}_{34}^{\,2}}+\frac{2\times10^6}{\hat{r}_{34}}\le(2.28U_0^{(4)}-268)10^{18}-10^{12}.$$

Disregard the terms $1.36\times 10^{14}, 10^{12}$ we obtain

(i) $\frac{4}{\hat{r}_{23}^{\,2}}+\frac{2\times 10^{6}}{\hat{r}_{23}}+\frac{4}{\hat{r}_{24}^{\,2}}+\frac{1.25\times 10^{5}}{\hat{r}_{24}}\le\left(8.92U_0^{(2)}-42.9\right)10^{20}$;

(ii) $\frac{4}{\hat{r}_{23}^{\,2}}+\frac{2\times 10^{6}}{\hat{r}_{23}}+\frac{4}{\hat{r}_{34}^{\,2}}+\frac{1.25\times 10^{5}}{\hat{r}_{34}}\le\left(8.92U_0^{(3)}-42.9\right)10^{20}$;

(iii)33 $\frac{4}{\hat{r}_{24}^{\,2}}+\frac{2\times 10^{6}}{\hat{r}_{24}}+\frac{4}{\hat{r}_{34}^{\,2}}+\frac{2\times 10^{6}}{\hat{r}_{34}}\le(2.28U_0^{(4)}-268)10^{18}$.

Let us assume $U_0^{(2)}>\frac{42.9}{8.92}\approx 4.81$; $U_0^{(3)}>4.81$; $U_0^{(4)}>\frac{268}{2.28}\approx 117.55$. From inequality (i) we have:

$$0<\frac{4}{\hat{r}_{24}^{\,2}}+\frac{1.25\times 10^{5}}{\hat{r}_{24}}\le\left(8.92U_0^{(2)}-42.9\right)\times 10^{20}-\frac{4}{\hat{r}_{23}^{\,2}}-\frac{2\times 10^{6}}{\hat{r}_{23}}\Rightarrow\left(8.92U_0^{(2)}-42.9\right)\times 10^{20}\hat{r}_{23}^{\,2}-2\times 10^{6}\hat{r}_{23}-4>0\Rightarrow$$

$$\hat{r}_{23}\ge\frac{10^{6}+\sqrt{10^{12}+4\left(8.92U_0^{(2)}-42.9\right)\times 10^{20}}}{\left(8.92U_0^{(2)}-42.9\right)\times 10^{20}}\approx\frac{2}{\sqrt{8.92U_0^{(2)}-42.9}}\times 10^{-10}.$$

In a similar way we have

$$0<\frac{4}{\hat{r}_{23}^{\,2}}+\frac{2\times 10^{6}}{\hat{r}_{23}}\le\left(8.92U_0^{(2)}-42.9\right)\times 10^{20}-\frac{4}{\hat{r}_{24}^{\,2}}-\frac{1.25\times 10^{5}}{\hat{r}_{24}}\Rightarrow\left(8.92U_0^{(2)}-42.9\right)\times 10^{20}\hat{r}_{24}^{\,2}-1.25\times 10^{5}\hat{r}_{24}-4>0\Rightarrow$$

$$\hat{r}_{24}\ge\frac{1.25\times 10^{5}+\sqrt{1.25^{2}\times 10^{10}+16\left(8.92U_0^{(2)}-42.9\right)\times 10^{20}}}{2\times\left(8.92U_0^{(2)}-42.9\right)\times 10^{20}}\approx\frac{2}{\sqrt{8.92U_0^{(2)}-42.9}}\times 10^{-10}.$$

From inequality (ii) we obtain $\hat{r}_{23}\ge\frac{2}{\sqrt{8.92U_0^{(3)}-42.9}}\times 10^{-10}$, $\hat{r}_{34}\ge\frac{2}{\sqrt{8.92U_0^{(3)}-42.9}}\times 10^{-10}$.

From inequality (iii) we get:

$$0<\frac{4}{\hat{r}_{34}^{\,2}}+\frac{2\times 10^{6}}{\hat{r}_{34}}\le(2.28U_0^{(4)}-268)10^{18}-\frac{4}{\hat{r}_{24}^{\,2}}-\frac{2\times 10^{6}}{\hat{r}_{24}}\Rightarrow(2.28U_0^{(4)}-268)10^{18}\hat{r}_{24}^{\,2}-2\times 10^{6}\hat{r}_{24}-4>0\Rightarrow$$

$$\hat{r}_{24}\ge\frac{10^{6}+\sqrt{10^{12}+4(2.28U_0^{(4)}-268)10^{18}}}{(2.28U_0^{(4)}-268)10^{18}}\approx\frac{2}{\sqrt{2.28U_0^{(4)}-268}}\times 10^{-9}$$

and

$$0<\frac{4}{\hat{r}_{24}^{\,2}}+\frac{2\times 10^{6}}{\hat{r}_{24}}\le(2.28U_0^{(4)}-268)10^{18}-\frac{4}{\hat{r}_{34}^{\,2}}-\frac{2\times 10^{6}}{\hat{r}_{34}}\Rightarrow\hat{r}_{34}\ge\frac{2}{\sqrt{2.28U_0^{(4)}-268}}\times 10^{-9}.$$

Consequently,

$$\hat{r}_{23}\ge\max\left\{\frac{2\times 10^{-10}}{\sqrt{8.92U_0^{(2)}-42.9}};\frac{2\times 10^{-10}}{\sqrt{8.92U_0^{(3)}-42.9}}\right\};$$

$$\hat{r}_{24}\ge\max\left\{\frac{2\times 10^{-10}}{\sqrt{8.92U_0^{(2)}-42.9}},\frac{2\times 10^{-9}}{\sqrt{2.28U_0^{(4)}-268}}\right\};\tag{5}$$

$$\hat{r}_{34}\ge\max\left\{\frac{2\times 10^{-10}}{\sqrt{8.92U_0^{(3)}-42.9}},\frac{2\times 10^{-9}}{\sqrt{2.28U_0^{(4)}-268}}\right\}.\tag{6}$$

Finally, we check the inequalities from the Main theorem for the second derivatives:

$$\sum_{n=1,n\neq k}^{4}\frac{|e_ke_n|}{m_k}\left(\frac{32c(\beta_k+\beta_n)}{\hat{r}_{kn}^{\,3}}+\frac{8\omega_n\beta_n}{\hat{r}_{kn}^{\,2}}+\frac{8\omega_n^{\,2}\beta_n}{c\hat{r}_{kn}}\right)+\frac{e_k^2}{m_k}\frac{\beta_k\omega_k^{\,3}}{c^2}\le\omega_k^{\,2}U_0^{(k)}(k=2,3,4).$$

Indeed, the first one is satisfied

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$$\frac{|e_2e_3|}{137m_2}\left(\frac{96c}{\omega^2\hat{r}_{21}{}^3}+\frac{64c}{\omega^2\hat{r}_{23}{}^3}+\frac{8}{\omega\hat{r}_{23}{}^2}+\frac{8}{c\hat{r}_{23}}+\frac{48c}{\omega^2\hat{r}_{24}{}^3}+\frac{1}{2\hat{r}_{24}{}^2\omega}+\frac{1}{16c\hat{r}_{24}}+\frac{\omega}{c^2}\right)\le U_0^{(2)}$$

because

$$10^{-10}\left(1.16\times10^8+3.19\times10^6+0.85\times10^4+1.74\times10^2+1.4\times10^4+1.7\times10+0.24+0.456\right)\le 5\,.$$

The second one can be checked in a similar way. The last one becomes

$$10^{-4}\left(1.12+0.0167+0.00034+0.00004+0.189+0.0017+0.000089+0.00000000056\right)\le 1.246\,.$$

To assure the inequalities of higher order derivatives we notice that for the third derivative we have

$$\frac{1}{\omega^3\hat{r}_{21}{}^4}\approx\frac{1}{4.1^3\times10^{48}}\frac{1}{5.29^4\times10^{-44}},$$

for the n-th derivative we obtain the inequality

$$\frac{1}{\omega^n\hat{r}_{21}{}^{n+1}}\approx\frac{1}{4.1^n\times10^{16n}}\frac{1}{5.29^{n+1}\times10^{-11(n+1)}}=\frac{1}{5.29\times(21.68)^n\times10^{5n-11}}\le\frac{1}{5.29\times(2)^n\times10^{5n-10}}(n=2,3,...)\,.$$

Remark 5.1. *To illustrate the role of the parameters we check the distance relation* $\hat{r}_{14}=\hat{r}_{12}+\hat{r}_{24}=4\hat{r}_{12}\Rightarrow\hat{r}_{24}=3\hat{r}_{12}$ *known from quantum mechanics (cf. [9], [10]).*

Inequality (5) implies

$$\hat{r}_{24}=\frac{2}{\sqrt{2.28U_0^{(4)}-268}}\times10^{-9}=3\times5.29\times10^{-11}=3\hat{r}_{12}\Leftrightarrow U_0^{(4)}=\frac{12.6^2+268}{2.28}\approx187.18>117.55\quad\text{or}$$

$$\frac{2\times10^{-10}}{\sqrt{8.92U_0^{(2)}-42.9}}=3\times5.29\times10^{-11}\Leftrightarrow U_0^{(2)}=4.987>4.81\,.$$

Inequality (6) yields $\hat{r}_{34}=\dfrac{2\times10^{-9}}{\sqrt{2.28U_0^{(4)}-268}}=3\times5.29\times10^{-11}=3\hat{r}_{13}\Rightarrow U_0^{(4)}=187.2$ or

$$\hat{r}_{34}=\frac{2\times10^{-10}}{\sqrt{8.92U_0^{(3)}-42.9}}=3\times5.29\times10^{-11}=3\hat{r}_{13}\Leftrightarrow U_0^{(3)}=4.987>4.809\,.$$

6. Conclusion

The interest in the topic is evident from the articles [15]-[28]. We have obtained the existence-uniqueness of general $\left(T^{(1)},T^{(2)},T^{(3)},T^{(4)}\right)-$periodic solution of the 4-body problem extending the approach from the case of the 2- and 3-body problems. In fact, we apply the same form of radiation terms following the Dirac's physical assumption justified by nonstandard analysis (cf. [1]). Our advantage is that we obtain the estimates of distances between the moving charged particles which include as a particular case the values from quantum mechanics. Consequently, classical electrodynamics as a theory of the structure of elementary charged particle and its interaction with other elementary particles is more general than quantum mechanics.

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Appendix 1. Estimates of the Right-Hand Sides of Equations of Motion and Their Derivatives

$$\sqrt{\left\langle \vec{u}^{(k)},\vec{u}^{(k)}\right\rangle} \le c_k < c; \left|\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle\right| \le \sqrt{\left\langle \vec{\xi}^{(kn)},\vec{\xi}^{(kn)}\right\rangle}\sqrt{\left\langle \vec{u}^{(n)},\vec{u}^{(n)}\right\rangle} \le c\tau_{kn}c_n; \tau_{kn} \ge \frac{r_{kn}(t)}{2c}\ (k=1,2,3,4)$$

(cf. [11]),

where $r_{kn}(t)$ is the distance between the k-th and n-th particle at time t.

Differentiating $\tau_{kn}(t)=\frac{1}{c}\sqrt{\left\langle \vec{\xi}^{(kn)},\vec{\xi}^{(kn)}\right\rangle}$ and solving with respect to $\frac{d\tau_{kn}}{dt}$ we obtain $\frac{d\tau_{kn}(t)}{dt}=\frac{1}{c}\frac{\left\langle \vec{\xi}^{(kn)},\vec{u}^{(k)}(t)\right\rangle-\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}(t-\tau_{kn})\right\rangle}{\sqrt{\left\langle \vec{\xi}^{(kn)},\vec{\xi}^{(kn)}\right\rangle}-\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}(t-\tau_{kn})\right\rangle}$. Using that $\tau_{kn}(t)=\frac{1}{c}\sqrt{\left\langle \vec{\xi}^{(kn)},\vec{\xi}^{(kn)}\right\rangle}$ has a unique solution $\tau_{kn}(t)$ we obtain

$$1-\frac{d\tau_{kn}(t)}{dt}=1-\frac{\left\langle \vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle-\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle}{c^2\tau_{kn}(t)-\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle}=\frac{c^2\tau_{kn}(t)-\left\langle \vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle}{c^2\tau_{kn}(t)-\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle}\ge\frac{c^2\tau_{kn}(t)-c\tau_{kn}(t)c_k}{c^2\tau_{kn}(t)+c\tau_{kn}(t)c_n}=\frac{1-\beta_k}{1+\beta_n}>0 .$$

Consequently $1-\dot{\tau}_{kn}(t)>0$ and $\frac{1}{1-\dot{\tau}_{kn}(t)}\le\frac{1+\beta_n}{1-\beta_k}$ and $\frac{d\tau_{kn}(t)}{dt}\le\frac{\beta_k+\beta_n}{1+\beta_n}$

Since the velocities on the first and second steady states are "small" we can simplify the relativistic terms:

$$\Delta_k=\sqrt{c^2-\left\langle \vec{u}^{(k)}(t),\vec{u}^{(k)}(t)\right\rangle}=c\sqrt{1-\left(\left\|\vec{u}^{(k)}\right\|^2/c^2\right)}\simeq c\sqrt{1-c_k^{\ 2}/c^2}=c\sqrt{1-\beta_k^{\ 2}}\simeq c ;$$

$$\tau_{kn}\simeq(1-\beta_n)\tau_{kn}\le\frac{c^2\tau_{kn}-\left\langle \xi^{(kn)},u^{(n)}\right\rangle}{c^2}\le(1+\beta_n)\tau_{kn}\simeq\tau_{kn}\Leftrightarrow c^2\tau_{kn}-\left\langle \xi^{(kn)},u^{(n)}\right\rangle\simeq c^2\tau_{kn} .$$

Then $D_{kn}=\frac{c^2\tau_{kn}-\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle}{c^2\tau_{kn}-\left\langle \vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle}\simeq 1$ and using denotations from [1] we obtain:

$$H_{kn}=\Delta_{kn}^2+D_{kn}\left(\left\langle \vec{\xi}^{(kn)},\dot{\vec{u}}^{(n)}\right\rangle+\frac{\left(\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle-c^2\tau_{kn}\right)\left\langle \vec{u}^{(n)},\dot{\vec{u}}^{(n)}\right\rangle}{\Delta_{kn}^2}\right)\simeq c^2+\left\langle \vec{\xi}^{(kn)},\dot{\vec{u}}^{(n)}\right\rangle-\tau_{kn}\left\langle \vec{u}^{(n)},\dot{\vec{u}}^{(n)}\right\rangle;$$

$$A_{kn}=\frac{H_{kn}\left(c^2-\left\langle \vec{u}^{(k)},\vec{u}^{(n)}\right\rangle\right)}{\left(c^2\tau_{kn}-\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle\right)^3}-D_{kn}\frac{\Delta_{kn}^2\left\langle \vec{u}^{(k)},\dot{\vec{u}}^{(n)}\right\rangle+\left(\left\langle \vec{u}^{(k)},\vec{u}^{(n)}\right\rangle-c^2\right)\left\langle \vec{u}^{(n)},\dot{\vec{u}}^{(n)}\right\rangle}{\Delta_{kn}^2\left(c^2\tau_{kn}-\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle\right)^2}\simeq\frac{c^2+\left\langle \vec{\xi}^{(kn)},\dot{\vec{u}}^{(n)}\right\rangle}{c^4\tau_{kn}^{\ 3}}-\frac{\left\langle \vec{u}^{(k)},\dot{\vec{u}}^{(n)}\right\rangle}{c^4\tau_{kn}^{\ 2}}$$

;

$$B_{kn}=\frac{H_{kn}\left(c^2\tau_{kn}-\left\langle \vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle\right)}{\left(c^2\tau_{kn}-\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle\right)^3}-\frac{D_{kn}\left(\left\langle \vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle-c^2\tau_{kn}\right)\left\langle \vec{u}^{(n)},\dot{\vec{u}}^{(n)}\right\rangle}{\Delta_{kn}^2\left(\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle-c^2\tau_{kn}\right)^2}\simeq\frac{c^2+\left\langle \vec{\xi}^{(kn)},\dot{\vec{u}}^{(n)}\right\rangle}{c^4\tau_{kn}^{\ 2}};$$

$$C_{kn}=\frac{D_{kn}\left(\left\langle \vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle-c^2\tau_{kn}\right)}{\left(\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle-c^2\tau_{kn}\right)^2}\simeq\frac{\left\langle \vec{\xi}^{(kn)},\vec{u}^{(k)}\right\rangle-c^2\tau_{kn}}{\left(c^2\tau_{kn}-\left\langle \vec{\xi}^{(kn)},\vec{u}^{(n)}\right\rangle\right)^2}\simeq-\frac{1}{c^2\tau_{kn}} .$$

Further on, if $r_{kn}(t)\ge\hat{r}_{kn}=const.>0$ and $U_0^{(k)}e^{\mu T}\le\overline{c}_k<c$ then:

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$$\left|H_{kn}\right| \le \frac{\left|c^2 + \left\langle \vec{\xi}^{(kn)}, \dot{\vec{u}}^{(n)} \right\rangle - \tau_{kn} \left\langle \vec{u}^{(n)}, \dot{\vec{u}}^{(n)} \right\rangle\right|}{c^2} \le \frac{c^2 + (c + c_n)\tau_{kn} \left\|\dot{\vec{u}}^{(n)}\right\|}{c^2} \le \frac{c^2 + (c + c_n)\tau_{kn}\omega_n U_0 e^{\mu_n T^{(n)}}}{c^2} \le \frac{c^2 + (c + c_n)\tau_{kn}\omega_n c_n}{c^2} =$$
$$= 1 + (\beta_n + \beta_n{}^2)\tau_{kn}\omega_n \approx 1 + \tau_{kn}\omega_n\beta_n;$$

$$\left|A_{kn}\right| = \frac{1}{c^4}\left|\frac{c^2 + \left\langle \vec{\xi}^{(kn)}, \dot{\vec{u}}^{(n)} \right\rangle}{\tau_{kn}{}^3} - \frac{\left\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(n)} \right\rangle}{\tau_{kn}{}^2}\right| \le \frac{1}{c^2\tau_{kn}{}^3} + \frac{(1+\beta_k)\left\|\dot{\vec{u}}^{(n)}\right\|}{c^3\tau_{kn}{}^2} \le \frac{1}{c^2\tau_{kn}{}^3} + \frac{(1+\beta_k)\beta_n\omega_n}{c^2\tau_{kn}{}^2} \simeq \frac{1}{c^2\tau_{kn}{}^3} + \frac{\beta_n\omega_n}{c^2\tau_{kn}{}^2};$$

$$\left|B_{kn}\right| = \frac{1}{c^4}\left|\frac{c^2 + \left\langle \vec{\xi}^{(kn)}, \dot{\vec{u}}^{(n)} \right\rangle}{\tau_{kn}{}^2}\right| \le \frac{1}{c^2\tau_{kn}{}^2} + \frac{\left\|\dot{\vec{u}}^{(n)}\right\|}{c^3\tau_{kn}} \le \frac{1}{c^2\tau_{kn}{}^2} + \frac{\omega_n\beta_n}{c^2\tau_{kn}}; \quad \left|C_{kn}\right| = \frac{1}{c^2\tau_{kn}};$$

$$\left|G_\alpha^{(kn)}(t)\right| = \left|\frac{e_k e_n\left(A_{kn}\xi_\alpha^{(kn)} - B_{kn}u_\alpha^{(n)} + C_{kn}\dot{u}_\alpha^{(n)}\right)}{m_k c}\right| \le \frac{\left|e_k e_n\right|}{m_k c}\left(\left|A_{kn}\right|\left|\xi_\alpha^{(kn)}\right| + \left|B_{kn}\right|\left|u_\alpha^{(n)}\right| + \left|C_{kn}\right|\left|\dot{u}_\alpha^{(n)}\right|\right) \le$$
$$\le \frac{\left|e_k e_n\right|}{m_k c}\left[\left(\frac{1}{c^2\tau_{kn}{}^3} + \frac{\beta_n\omega_n}{c^2\tau_{kn}{}^2}\right)c\tau_{kn} + \left(\frac{1}{c^2\tau_{kn}{}^2} + \frac{\omega_n\beta_n}{c^2\tau_{kn}}\right)c_n + \frac{\omega_n U_0 e^{\mu_n T^{(n)}}}{c^2\tau_{kn}}\right] \cong \frac{\left|e_k e_n\right|}{m_k c^2}\left(\frac{1+\beta_n}{\tau_{kn}{}^2} + \frac{2\omega_n\beta_n}{\tau_{kn}}\right);$$

(7)

$$\left|G_\alpha^{(k)rad}\right| \le \left|-\frac{e_k^2}{m_k}\frac{u_\alpha^{(k)}\left\langle \vec{u}^{(k)}, \ddot{\vec{u}}^{(k)} \right\rangle + c^2\ddot{u}_\alpha{}^{(k)}}{c^5}\right| \le \frac{e_k^2}{m_k}\frac{c_k{}^2\omega_k{}^2U_0e^{\mu_k T^{(k)}} + c^2\omega_k{}^2U_0e^{\mu_k T^{(k)}}}{c^5} \simeq \frac{e_k^2}{m_k}\frac{\omega_k{}^2\beta_k}{c^2}.$$

Appendix 2. Estimates of the First Derivatives of the Right-Hand Sides

$$\dot{\vec{\xi}}^{(kn)} = \left(u_1^{(k)}(t) - u_1^{(n)}(t-\tau_{kn}) \times (1-\dot{\tau}_{kn}), u_2^{(k)}(t) - u_2^{(n)}(t-\tau_{kn}) \times (1-\dot{\tau}_{kn}), u_3^{(k)}(t) - u_3^{(n)}(t-\tau_{kn})(1-\dot{\tau}_{kn})\right);$$

$$\dot{\xi}_\alpha{}^{(kn)} = u_\alpha^{(k)}(t) - u_\alpha^{(n)}(t-\tau_{kn}) \times (1-\dot{\tau}_{kn}) \Rightarrow \left|\dot{\xi}_\alpha{}^{(kn)}\right| \le \left|u_\alpha^{(k)}(t)\right| + \left|u_\alpha^{(n)}(t-\tau_{kn})\right| \le c_k + c_n;$$

$$\left|\frac{dA_{kn}}{dt}\right| = \frac{1}{c^4}\left|\frac{\left\langle \dot{\vec{\xi}}^{(kn)}, \dot{\vec{u}}^{(n)} \right\rangle + \left\langle \vec{\xi}^{(kn)}, \ddot{\vec{u}}^{(n)} \right\rangle}{\tau_{kn}{}^3} + \frac{(-3)\left(c^2 + \left\langle \vec{\xi}^{(kn)}, \dot{\vec{u}}^{(n)} \right\rangle\right)\dot{\tau}_{kn}}{\tau_{kn}{}^4} - \frac{\left\langle \dot{\vec{u}}^{(k)}, \dot{\vec{u}}^{(n)} \right\rangle + \left\langle \vec{u}^{(k)}, \ddot{\vec{u}}^{(n)} \right\rangle}{\tau_{kn}{}^2} - \frac{(-2)\left\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(n)} \right\rangle\dot{\tau}_{kn}}{\tau_{kn}{}^3}\right| \le$$

$$\le \frac{1}{c^4}\left|\frac{\left\|\dot{\vec{\xi}}^{(kn)}\right\|\left\|\dot{\vec{u}}^{(n)}\right\| + \left\|\vec{\xi}^{(kn)}\right\|\left\|\ddot{\vec{u}}^{(n)}\right\|}{\tau_{kn}{}^3} + \frac{3\left(c^2 + \left\|\vec{\xi}^{(kn)}\right\|\left\|\dot{\vec{u}}^{(n)}\right\|\right)}{\tau_{kn}{}^4}\frac{\beta_k+\beta_n}{1+\beta_n} + \frac{\left\|\dot{\vec{u}}^{(k)}\right\|\left\|\dot{\vec{u}}^{(n)}\right\| + \left\|\vec{u}^{(k)}\right\|\left\|\ddot{\vec{u}}^{(n)}\right\|}{\tau_{kn}{}^2} + \frac{2\left\|\vec{u}^{(k)}\right\|\left\|\dot{\vec{u}}^{(n)}\right\|}{\tau_{kn}{}^3}\frac{\beta_k+\beta_n}{1+\beta_n}\right| \le$$

$$\le \frac{1}{c^4}\left(\frac{\sqrt{3}\left(c_k+c_n\right)\omega_n c_n + c\tau_{kn}\sqrt{3}\omega_n{}^2c_n}{\tau_{kn}{}^3} + \frac{3\left(c^2 + c\tau_{kn}\sqrt{3}\omega_n c_n\right)}{\tau_{kn}{}^4}\frac{\beta_k+\beta_n}{1+\beta_n} + \frac{3\omega_k\omega_n c_k c_n + \sqrt{3}\omega_n{}^2c_k c_n}{\tau_{kn}{}^2} + \frac{2\sqrt{3}\omega_n c_k c_n}{\tau_{kn}{}^3}\frac{\beta_k+\beta_n}{1+\beta_n}\right) \le$$
$$\le \frac{1}{c^4}\left(\frac{\sqrt{3}\left(c_k c_n + c_n{}^2\right)\omega_n}{\tau_{kn}{}^3} + \frac{\sqrt{3}\omega_n{}^2cc_n}{\tau_{kn}{}^2} + \frac{3c^2}{\tau_{kn}{}^4}\frac{\beta_k+\beta_n}{1+\beta_n} + \frac{3\sqrt{3}cc_n + 2\sqrt{3}c_k c_n}{\tau_{kn}{}^3}\omega_n\frac{\beta_k+\beta_n}{1+\beta_n} + \frac{3\omega_k\omega_n c_k c_n + \sqrt{3}\omega_n{}^2c_k c_n}{\tau_{kn}{}^2}\right)$$
$$\approx \frac{3(\beta_k+\beta_n)}{c^2\tau_{kn}{}^4} + \frac{4\sqrt{3}\beta_n\omega_n(\beta_k+\beta_n)}{c^2\tau_{kn}{}^3} + \frac{3\omega_n{}^2\beta_n}{c^2\tau_{kn}{}^2};$$

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$$\left|\dot{A}_{kn}\vec{\xi}^{(kn)}\right| \le \left(\frac{3}{c^2\tau_{kn}^{\ 4}}\frac{\beta_k+\beta_n}{1+\beta_n} + \frac{4\sqrt{3}\beta_n\omega_n}{c^2\tau_{kn}^{\ 3}}\frac{\beta_k+\beta_n}{1+\beta_n} + 3\frac{\omega_n^{\ 2}\beta_n+\omega_n^{\ 2}\beta_k\beta_n+\omega_k\omega_n\beta_k\beta_n}{c^2\tau_{kn}^{\ 2}} \right) c\tau_{kn} \le$$

$$\approx \frac{3(\beta_k+\beta_n)}{c\tau_{kn}^{\ 3}} + \frac{4\sqrt{3}\beta_n\omega_n(\beta_k+\beta_n)}{c\tau_{kn}^{\ 2}} + \frac{3\omega_n^{\ 2}\beta_n}{c\tau_{kn}};$$

$$\left|\frac{dB_{kn}}{dt}\right| = \frac{1}{c^4}\left| \frac{\left\langle \dot{\vec{\xi}}^{(kn)}, \dot{\vec{u}}^{(n)} \right\rangle + \left\langle \vec{\xi}^{(kn)}, \ddot{\vec{u}}^{(n)} \right\rangle}{\tau_{kn}^{\ 2}} + 2\frac{c^2+\left\langle \vec{\xi}^{(kn)}, \dot{\vec{u}}^{(n)} \right\rangle}{\tau_{kn}^{\ 3}}\dot{\tau}_{kn} \right| \le \frac{1}{c^4}\left| \frac{\sqrt{3}(c_k+c_n)\omega_n c_n + c\tau_{kn}\omega_n^{\ 2}c_n}{\tau_{kn}^{\ 2}} + 2\frac{c^2+c\tau_{kn}\omega_n c_n}{\tau_{kn}^{\ 3}}\frac{\beta_k+\beta_n}{1+\beta_n} \right| \le$$

$$\le \frac{2(\beta_k+\beta_n)}{c^2\tau_{kn}^{\ 3}(1+\beta_n)} + \frac{2(\beta_k+\beta_n)(2+\beta_n)\omega_n\beta_n}{c^2\tau_{kn}^{\ 2}(1+\beta_n)} + \frac{\omega_n^{\ 2}\beta_n}{c^2\tau_{kn}} \approx \frac{2(\beta_k+\beta_n)}{c^2\tau_{kn}^{\ 3}} + \frac{4\omega_n\beta_n(\beta_k+\beta_n)}{c^2\tau_{kn}^{\ 2}} + \frac{\omega_n^{\ 2}\beta_n}{c^2\tau_{kn}};$$

$$\left|\frac{dC_{kn}}{dt}\right| = \left| -\frac{1}{c^2}\left(-\frac{\dot{\tau}_{kn}}{\tau_{kn}^{\ 2}} \right) \right| \le \frac{1}{c^2\tau_{kn}^{\ 2}}.$$

Appendix 3. Estimates of the Second Derivatives of the Right-Hand Sides

$$\left|\ddot{B}_{\alpha}^{(k)}(t)\right| = \left|\dot{U}_{\alpha}^{k}(t)\right| \le \left| \sum_{n=1,n\ne k}^{4} \frac{e_k e_n}{m_k c}\left(\dot{A}_{kn}\xi_{\alpha}^{(kn)} + A_{kn}\dot{\xi}_{\alpha}^{(kn)} - \dot{B}_{kn}u_{\alpha}^{(n)} - B_{kn}\dot{u}_{\alpha}^{(n)} + \dot{C}_{kn}\dot{u}_{\alpha}^{(n)} + C_{kn}\ddot{u}_{\alpha}^{(n)} \right) \right| + \frac{e_k^2}{m_k c^5}\left| \dot{u}_{\alpha}^{(k)}\left\langle \vec{u}^{(k)}, \ddot{\vec{u}}^{(k)} \right\rangle + u_{\alpha}^{(k)}\left(\left\langle \dot{\vec{u}}^{(k)}, \ddot{\vec{u}}^{(k)} \right\rangle + \left\langle \vec{u}^{(k)}, \dddot{\vec{u}}^{(k)} \right\rangle \right) + c^2\ddot{u}_{\alpha}^{(k)} \right| \le$$

$$\le \sum_{n=1,n\ne k}^{4} \frac{|e_k e_n|}{m_k c}\left(\frac{3(\beta_k+\beta_n)}{c\tau_{kn}^{\ 3}} + \frac{4\sqrt{3}\beta_n\omega_n(\beta_k+\beta_n)}{c\tau_{kn}^{\ 2}} + \frac{3\omega_n^{\ 2}\beta_n}{c\tau_{kn}} + \frac{\beta_n+\beta_k}{c\tau_{kn}^{\ 3}} + \frac{(\beta_n+\beta_k)\beta_n\omega_n}{c^2\tau_{kn}^{\ 2}} \right) +$$
$$+ \sum_{n=1,n\ne k}^{4} \frac{|e_k e_n|}{m_k c}\left(\frac{2(\beta_k+\beta_n)\beta_n}{c\tau_{kn}^{\ 3}} + \frac{4\omega_n\beta_n^{\ 2}(\beta_k+\beta_n)}{c\tau_{kn}^{\ 2}} + \frac{\omega_n^{\ 2}\beta_n^{\ 2}}{c\tau_{kn}} + \frac{\omega_n\beta_n}{c\tau_{kn}^{\ 2}} + \frac{\omega_n^{\ 2}\beta_n^{\ 2}}{c\tau_{kn}} \right) +$$
$$+ \sum_{n=1,n\ne k}^{4} \frac{|e_k e_n|}{m_k c}\left(\frac{\omega_n\beta_n}{c\tau_{kn}^{\ 2}} + \frac{\omega_n^{\ 2}\beta_n}{c\tau_{kn}} \right) + \frac{e_k^2}{m_k}\frac{\beta_k\omega_k^{\ 3}}{c^2} \approx \sum_{n=1,n\ne k}^{4} \frac{|e_k e_n|}{m_k c}\left(\frac{4(\beta_k+\beta_n)}{c\tau_{kn}^{\ 3}} + \frac{2\omega_n\beta_n}{c\tau_{kn}^{\ 2}} + \frac{4\omega_n^{\ 2}\beta_n}{c\tau_{kn}} \right) + \frac{e_k^2}{m_k}\frac{\beta_k\omega_k^{\ 3}}{c^2} \le \omega_k^{\ 2}U_0^{(k)}.$$

Appendix 4. Lipschitz Estimates of the Right-Hand Sides

We note that for $t \in [T_p, T_{p+1}] \Rightarrow t - \tau_{kn}(t) \in [T_{p-1}, T_p]$ and then

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$$\left|u_\alpha^{(k)}(t)-\overline{u}_\alpha^{(k)}(t)\right|\le\left|u_\alpha^{(k)}(T_p^{(k)})-\overline{u}_\alpha^{(k)}(T_p^{(k)})\right|+\left|\int_{T_p^{(k)}}^{t}(\dot{u}_\alpha^{(k)}(s)-\dot{\overline{u}}_\alpha^{(k)}(s))ds\right|\le\rho_{(p,0)}(u_\alpha^{(k)},\overline{u}_\alpha^{(k)})e^{\mu_k(t-T_p^{(k)})}+\frac{\omega_k}{\mu_k}\rho_{(p,1)}(u_\alpha^{(k)},\overline{u}_\alpha^{(k)})e^{\mu_k(t-T_p^{(k)})}\le\ldots$$

$$\ldots\le\frac{\omega_k^{\,h}}{\mu_k^{\,h}}\rho_{(p,h)}(u_\alpha^{(k)},\overline{u}_\alpha^{(k)})e^{\mu_k(t-T_p^{(k)})}+\frac{\omega_k^{\,h}}{\mu_k^{\,h}}\rho_{(p,h)}(u_\alpha^{(k)},\overline{u}_\alpha^{(k)})e^{\mu_k(t-T_p^{(k)})}=2\frac{\omega_k^{\,h}}{\mu_k^{\,h}}\rho_{(p,h)}(u_\alpha^{(k)},\overline{u}_\alpha^{(k)})e^{\mu_k(t-T_p^{(k)})};$$

$$\left|x_\alpha^{(k)}(t)-\overline{x}_\alpha^{(k)}(t)\right|\le\left|x_\alpha^{(k)}(-T_p^{(k)})-\overline{x}_\alpha^{(k)}(-T_p^{(k)})\right|e^{\mu_k(t-T_p^{(k)})}+\left|\int_{T_p^{(k)}}^{t}\left(u_\alpha^{(k)}(s)-\overline{u}_\alpha^{(k)}(s)\right)ds\right|\le$$

$$\le\sup\left\{\left|x_\alpha^{(k)}(t)-\overline{x}_\alpha^{(k)}(t)\right|:t\in[T_p^{(k)},T_{p+1}^{(k)}]\right\}+\left|\int_{T_p^{(k)}}^{t}\left(u_{\alpha0}^{(k)}-\overline{u}_{\alpha0}^{(k)}\right)ds\right|+\left|\int_{T_p^{(k)}}^{t}\int_{T_p^{(k)}}^{s}\left(\dot{u}_\alpha^{(k)}(\theta)-\dot{\overline{u}}_\alpha^{(k)}(\theta)\right)d\theta ds\right|\le$$

$$\le\sup\left\{\int_{T_p^{(k)}}^{t}\left|u_\alpha^{(k)}(s)-\overline{u}_\alpha^{(k)}(s)\right|ds:t\in[T_p^{(k)},T_{p+1}^{(k)}]\right\}+\rho_{(p,0)}(u_\alpha^{(k)},\overline{u}_\alpha^{(k)})\frac{e^{\mu_k(t-T_p^{(k)})}}{\mu_k}+\rho_{(p,1)}(u_\alpha^{(k)},\overline{u}_\alpha^{(k)})\frac{\omega_k e^{\mu_k(t-T_p^{(k)})}}{\mu_k^{\,2}}\le$$

$$\le\rho_{(p,0)}(u_\alpha^{(k)},\overline{u}_\alpha^{(k)})\frac{e^{\mu_k(T_{p+1}^{(k)}-T_p^{(k)})}}{\mu_k}+\rho_{(p,0)}(u_\alpha^{(k)},\overline{u}_\alpha^{(k)})\frac{e^{\mu_k(t-T_p^{(k)})}}{\mu_k}+\rho_{(p,1)}(u_\alpha^{(k)},\overline{u}_\alpha^{(k)})\frac{\omega_k e^{\mu_k(t-T_p^{(k)})}}{\mu_k^{\,2}}\le$$

$$\le\left(2\frac{\omega_k^{\,h}e^{\mu_kT^{(k)}}}{\mu_k^{\,h+1}}+\frac{\omega_k^{\,h-1}}{\mu_k^{\,h+1}}\right)\rho_{(p,h)}(u_\alpha^{(k)},\overline{u}_\alpha^{(k)})e^{\mu_k(t-T_p^{(k)})};$$

$$\left|x_\alpha^{(n)}(t-\tau_{kn})-\overline{x}_\alpha^{(n)}(t-\tau_{kn})\right|\le\left|x_{\alpha0}^{(n)}-\overline{x}_{\alpha0}^{(n)}\right|+\left|\int_{T_p^{(n)}}^{t-\tau_{kn}}\left(u_\alpha^{(n)}(s)-\overline{u}_\alpha^{(n)}(s)\right)ds\right|\le$$

$$\le\sup\left\{\left|x_\alpha^{(n)}(t-\tau_{kn})-\overline{x}_\alpha^{(n)}(t-\tau_{kn})\right|:t-\tau_{kn}\in[T_{p-1}^{(k)},T_p^{(k)}]\right\}+\left|\int_{T_p^{(n)}}^{t}\left(u_\alpha^{(n)}(s)-\overline{u}_\alpha^{(n)}(s)\right)ds\right|+\left|\int_{t}^{t-\tau_{kn}}\left(u_\alpha^{(n)}(s)-\overline{u}_\alpha^{(n)}(s)\right)ds\right|\le$$

$$\le\sup\left\{\left|\int_{T_{p-1}^{(n)}}^{t-\tau_{kn}}\left(u_\alpha^{(n)}(s)-\overline{u}_\alpha^{(n)}(s)\right)ds\right|:s\in[T_{p-1}^{(k)},T_p^{(k)}]\right\}+\left|\int_{T_p^{(n)}}^{t}\left(u_\alpha^{(n)}(s)-\overline{u}_\alpha^{(n)}(s)\right)ds\right|+\left|\int_{t-\tau_{kn}}^{t}\left(u_\alpha^{(n)}(s)-\overline{u}_\alpha^{(n)}(s)\right)ds\right|\le$$

$$\le\rho_{(p-1,0)}(u_\alpha^{(n)},\overline{u}_\alpha^{(n)})\frac{e^{\mu_n(t-T_{p-1}^{(n)})}e^{-\mu_n\tau_{kn}}}{\mu_n}+\left|\int_{T_p^{(n)}}^{t}e^{\mu_n(s-T_p^{(n)})}ds\right|\rho_{(p,0)}(u_\alpha^{(n)},\overline{u}_\alpha^{(n)})+\left|\int_{t-\tau_{kn}}^{t}\left(u_\alpha^{(n)}(s)-\overline{u}_\alpha^{(n)}(s)\right)ds\right|\le$$

$$\le\rho_{(p-1,0)}(u_\alpha^{(n)},\overline{u}_\alpha^{(n)})\frac{e^{\mu_n(t-T_{p-1}^{(n)})}e^{-\mu_n\frac{2c}{\overline{r}_{kn}}}}{\mu_n}+\frac{e^{\mu_n(t-T_p^{(n)})}}{\mu_n}\rho_{(p,0)}(u_\alpha^{(n)},\overline{u}_\alpha^{(n)})+\int_{T_{p-1}^{(n)}}^{t}\left|u_\alpha^{(n)}(s)-\overline{u}_\alpha^{(n)}(s)\right|ds\le$$

$$\le\rho_{(p-1,0)}(u_\alpha^{(n)},\overline{u}_\alpha^{(n)})\frac{e^{\mu_n(t-T_{p-1}^{(n)})}e^{-\mu_n\frac{2c}{\overline{r}_{kn}}}}{\mu_n}+\frac{e^{\mu_n(t-T_p^{(n)})}}{\mu_n}\rho_{(p,0)}(u_\alpha^{(n)},\overline{u}_\alpha^{(n)})+\rho_{(p-1,0)}(u_\alpha^{(n)},\overline{u}_\alpha^{(n)})\frac{e^{\mu_n(t-T_{p-1}^{(n)})}}{\mu_n}+\frac{e^{\mu_n(t-T_p^{(n)})}}{\mu_n}\rho_{(p,0)}(u_\alpha^{(n)},\overline{u}_\alpha^{(n)})\le$$

$$\le\left(\frac{\omega_n^{\,h}}{\mu_n^{\,h}}\frac{e^{-2c\mu_n/\overline{r}_{kn}}+1}{\mu_n}e^{\mu_nT^{(n)}}\rho_{(p-1,h)}(u_\alpha^{(n)},\overline{u}_\alpha^{(n)})+\frac{2}{\mu_n}\frac{\omega_n^{\,h}}{\mu_n^{\,h}}\rho_{(p,h)}(u_\alpha^{(n)},\overline{u}_\alpha^{(n)})\right)e^{\mu_n(t-T_p^{(n)})};$$

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$$\left|\dot{u}_\alpha^{(k)}(t)-\dot{\bar{u}}_\alpha^{(k)}(t)\right|\le\ldots\le\frac{\omega_k^{\,h+1}}{\mu_k^{\,h}}\rho_{(p,h)}(u_\alpha^{(k)},\bar{u}_\alpha^{(k)})e^{\mu_k(t-T^{(k)})}\qquad;$$

$$\left|u_\gamma^{(n)}(t-\tau_{kn})-\bar{u}_\gamma^{(n)}(t-\tau_{kn})\right|\le\left|u_\gamma^{(n)}(-\tau_{kn}(0))-\bar{u}_\gamma^{(n)}(-\tau_{kn}(0))\right|+\rho_{(p-1,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})e^{\mu_n(t-\tau_{kn}-T_{p-1}{}^{(n)})}\le$$
$$\le\rho_{(p-1,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})e^{-2c\mu_n/r_{kn}{}^m}e^{\mu_n(t-T_{p-1}{}^{(n)})}+\rho_{(p-1,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})e^{-2c\mu_n/r_{kn}{}^m}e^{\mu_n(t-T_{p-1}{}^{(n)})}=2\rho_{(p-1,0)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})e^{-2c\mu_n/r_{kn}{}^m}e^{\mu_n(t-T_{p-1}{}^{(n)})}\le\ldots$$
$$\le2\frac{\omega_k^{\,h}}{\mu_k^{\,h}}\rho_{(p-1,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})e^{-2c\mu_n/r_{kn}{}^m}e^{\mu_n(t-T_{p-1}{}^{(n)})};$$

$$\left|A_{kn}-\bar{A}_{kn}\right|\le\frac{\sum_{\gamma=1}^{3}\left|\xi_\gamma^{(kn)}\dot{u}_\gamma^{(n)}-\bar{\xi}_\gamma^{(kn)}\dot{u}_\gamma^{(n)}\right|+\sum_{\gamma=1}^{3}\left|\bar{\xi}_\gamma^{(kn)}\dot{u}_\gamma^{(n)}-\bar{\xi}_\gamma^{(kn)}\dot{\bar{u}}_\gamma^{(n)}\right|}{c^4\tau_{kn}{}^3}+\frac{\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}\dot{u}_\gamma^{(n)}-\bar{u}_\gamma^{(k)}\dot{u}_\gamma^{(n)}\right|+\sum_{\gamma=1}^{3}\left|\bar{u}_\gamma^{(k)}\dot{u}_\gamma^{(n)}-\bar{u}_\gamma^{(k)}\dot{\bar{u}}_\gamma^{(n)}\right|}{c^4\tau_{kn}{}^2}\le$$
$$\le\frac{\omega_nU_0e^{\mu_nT^{(n)}}\sum_{\gamma=1}^{3}\left|\xi_\gamma^{(kn)}-\bar{\xi}_\gamma^{(kn)}\right|}{c^4\tau_{kn}{}^3}+\frac{c\tau_{kn}\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{(n)}-\dot{\bar{u}}_\gamma^{(n)}\right|}{c^4\tau_{kn}{}^3}+\frac{\omega_n\beta_n\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}-\bar{u}_\gamma^{(k)}\right|}{c^3\tau_{kn}{}^2}+\frac{\beta_k\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{(n)}-\dot{\bar{u}}_\gamma^{(n)}\right|}{c^3\tau_{kn}{}^2}\le$$
$$\le\frac{\omega_n\beta_n\sum_{\gamma=1}^{3}\left|x_\gamma^{(k)}-\bar{x}_\gamma^{(k)}\right|+\omega_n\beta_n\sum_{\gamma=1}^{3}\left|x_\gamma^{(n)}-\bar{x}_\gamma^{(n)}\right|}{c^3\tau_{kn}{}^3}+\frac{\omega_n\beta_n\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}-\bar{u}_\gamma^{(k)}\right|}{c^3\tau_{kn}{}^2}+\frac{(1+\beta_k)\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{(n)}-\dot{\bar{u}}_\gamma^{(n)}\right|}{c^3\tau_{kn}{}^2}\le$$
$$\le\omega_n\beta_n\left(\frac{\sum_{\gamma=1}^{3}\left|x_\gamma^{(k)}(t)-\bar{x}_\gamma^{(k)}(t)\right|+\sum_{\gamma=1}^{3}\left|x_\gamma^{(n)}(t-\tau_{kn})-\bar{x}_\gamma^{(n)}(t-\tau_{kn})\right|}{c^3\tau_{kn}{}^3}+\frac{\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}(t)-\bar{u}_\gamma^{(k)}(t)\right|}{c^3\tau_{kn}{}^2}\right)+\frac{(1+\beta_k)\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{(n)}(t-\tau_{kn})-\dot{\bar{u}}_\gamma^{(n)}(t-\tau_{kn})\right|}{c^3\tau_{kn}{}^2};$$

$$\left|B_{kn}-\bar{B}_{kn}\right|\le\frac{\sum_{\gamma=1}^{3}\left|\xi_\gamma^{(kn)}\dot{u}_\gamma^{(n)}-\bar{\xi}_\gamma^{(kn)}\dot{u}_\gamma^{(n)}\right|-\sum_{\gamma=1}^{3}\left|\bar{\xi}_\gamma^{(kn)}\dot{u}_\gamma^{(n)}-\bar{\xi}_\gamma^{(kn)}\dot{\bar{u}}_\gamma^{(n)}\right|}{c^4\tau_{kn}{}^2}\le$$
$$\le\frac{\omega_nU_0e^{\mu_kT^{(k)}}\sum_{\gamma=1}^{3}\left|x_\gamma^{(k)}(t)-\bar{x}_\gamma^{(k)}(t)\right|}{c^4\tau_{kn}{}^2}+\frac{\omega_nU_0e^{\mu_nT^{(n)}}\sum_{\gamma=1}^{3}\left|x_\gamma^{(n)}(t-\tau_{kn})-\bar{x}_\gamma^{(n)}(t-\tau_{kn})\right|}{c^4\tau_{kn}{}^2}+\frac{c\tau_{kn}\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{(n)}(t-\tau_{kn})-\dot{\bar{u}}_\gamma^{(n)}(t-\tau_{kn})\right|}{c^4\tau_{kn}{}^2}\le$$

$$\le\frac{\omega_n\beta_n\sum_{\gamma=1}^{3}\left|x_\gamma^{(k)}(t)-\bar{x}_\gamma^{(k)}(t)\right|+\omega_n\beta_n\sum_{\gamma=1}^{3}\left|x_\gamma^{(n)}(t-\tau_{kn})-\bar{x}_\gamma^{(n)}(t-\tau_{kn})\right|}{c^3\tau_{kn}{}^2}+\frac{\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{(n)}(t-\tau_{kn})-\dot{\bar{u}}_\gamma^{(n)}(t-\tau_{kn})\right|}{c^3\tau_{kn}};$$

$$\left|G_\alpha^{(kn)}-\bar{G}_\alpha^{(kn)}\right|=\frac{\left|e_ke_n\right|}{m_kc}\left|A_{kn}\xi_\alpha^{(kn)}-\bar{A}_{kn}\bar{\xi}_\alpha^{(kn)}-B_{kn}u_\alpha^{(n)}+\bar{B}_{kn}\bar{u}_\alpha^{(n)}+C_{kn}\dot{u}_\alpha^{(n)}-\bar{C}_{kn}\dot{\bar{u}}_\alpha^{(n)}\right|\le$$
$$\le\frac{\left|e_ke_n\right|}{m_kc}\left(\left|\xi_\alpha^{(kn)}\right|\left|A_{kn}-\bar{A}_{kn}\right|+\left|\bar{A}_{kn}\right|\left|\xi_\alpha^{(kn)}-\bar{\xi}_\alpha^{(kn)}\right|+\left|B_{kn}u_\alpha^{(n)}-\bar{B}_{kn}u_\alpha^{(n)}\right|+\left|\bar{B}_{kn}u_\alpha^{(n)}-\bar{B}_{kn}\bar{u}_\alpha^{(n)}\right|+\left|C_{kn}\dot{u}_\alpha^{(n)}-\bar{C}_{kn}\dot{u}_\alpha^{(n)}\right|+\left|\bar{C}_{kn}\dot{u}_\alpha^{(n)}-\bar{C}_{kn}\dot{\bar{u}}_\alpha^{(n)}\right|\right)\le$$
$$\le\frac{\left|e_ke_n\right|}{m_kc}\left(c\tau_{kn}\left|A_{kn}-\bar{A}_{kn}\right|+\left|\bar{A}_{kn}\right|\left|\xi_\alpha^{(kn)}-\bar{\xi}_\alpha^{(kn)}\right|+c_n\left|B_{kn}-\bar{B}_{kn}\right|+\left|\bar{B}_{kn}\right|\left|u_\alpha^{(n)}-\bar{u}_\alpha^{(n)}\right|+\omega_nc_n\left|C_{kn}-\bar{C}_{kn}\right|+\left|\bar{C}_{kn}\right|\left|\dot{u}_\alpha^{(n)}-\dot{\bar{u}}_\alpha^{(n)}\right|\right)\le$$

$$\le\frac{\left|e_ke_n\right|}{m_kc}\left[c\tau_{kn}\left(\frac{\omega_n\beta_n}{c^3\tau_{kn}{}^3}\left(\sum_{\gamma=1}^{3}\left|x_\gamma^{(k)}-\bar{x}_\gamma^{(k)}\right|+\sum_{\gamma=1}^{3}\left|x_\gamma^{(n)}-\bar{x}_\gamma^{(n)}\right|\right)+\frac{1}{c^3\tau_{kn}{}^2}\omega_n\beta_n\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}-\bar{u}_\gamma^{(k)}\right|+\frac{1}{c^3\tau_{kn}{}^2}(1+\beta_k)\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{(n)}-\dot{\bar{u}}_\gamma^{(n)}\right|\right)+\right.$$
$$+\left(\frac{1}{c^2\tau_{kn}{}^3}+\frac{\beta_n\omega_n}{c^2\tau_{kn}{}^2}\right)\left(\sum_{\gamma=1}^{3}\left|x_\gamma^{(k)}(t)-\bar{x}_\gamma^{(k)}(t)\right|+\sum_{\gamma=1}^{3}\left|x_\gamma^{(n)}(t-\tau_{kn})-\bar{x}_\gamma^{(n)}(t-\tau_{kn})\right|\right)+$$

$$+c_n\left(\frac{\omega_n\beta_n}{c^3\tau_{kn}{}^2}\left(\sum_{\gamma=1}^{3}\left|x_\gamma^{(k)}(t)-\bar{x}_\gamma^{(k)}(t)\right|+\sum_{\gamma=1}^{3}\left|x_\gamma^{(n)}(t-\tau_{kn})-\bar{x}_\gamma^{(n)}(t-\tau_{kn})\right|\right)+\frac{1}{c^3\tau_{kn}}\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{(n)}(t-\tau_{kn})-\dot{\bar{u}}_\gamma^{(n)}(t-\tau_{kn})\right|\right)+$$
$$\left.+\left(\frac{1}{c^2\tau_{kn}{}^2}+\frac{\omega_n\beta_n}{c^2\tau_{kn}}\right)\sum_{\gamma=1}^{3}\left|u_\gamma^{(n)}(t-\tau_{kn})-\bar{u}_\gamma^{(n)}(t-\tau_{kn})\right|+\frac{1}{c^2\tau_{kn}}\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{(n)}(t-\tau_{kn})-\dot{\bar{u}}_\gamma^{(n)}(t-\tau_{kn})\right|\right]\le$$

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$$\le \frac{|e_k e_n|}{m_k c}\left[\frac{\omega_n\beta_n}{c^2\tau_{kn}^{\ 2}}\sum_{\gamma=1}^{3}\left|x_\gamma^{\ (k)}-\overline{x}_\gamma^{\ (k)}\right|+\frac{\omega_n\beta_n}{c^2\tau_{kn}^{\ 2}}\sum_{\gamma=1}^{3}\left|x_\gamma^{\ (n)}-\overline{x}_\gamma^{\ (n)}\right|+\frac{\omega_n\beta_n}{c^2\tau_{kn}}\sum_{\gamma=1}^{3}\left|u_\gamma^{\ (k)}-\overline{u}_\gamma^{\ (k)}\right|+\frac{(1+\beta_k)}{c^2\tau_{kn}}\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{\ (n)}-\dot{\overline{u}}_\gamma^{\ (n)}\right|+\right.$$

$$+\left(\frac{1}{c^2\tau_{kn}^{\ 3}}+\frac{\beta_n\omega_n}{c^2\tau_{kn}^{\ 2}}\right)\sum_{\gamma=1}^{3}\left|x_\gamma^{\ (k)}-\overline{x}_\gamma^{\ (k)}\right|+\left(\frac{1}{c^2\tau_{kn}^{\ 3}}+\frac{\beta_n\omega_n}{c^2\tau_{kn}^{\ 2}}\right)\sum_{\gamma=1}^{3}\left|x_\gamma^{\ (n)}-\overline{x}_\gamma^{\ (n)}\right|+$$

$$\left.+\frac{\omega_n\beta_n^{\ 2}}{c^2\tau_{kn}^{\ 2}}\sum_{\gamma=1}^{3}\left|x_\gamma^{\ (k)}-\overline{x}_\gamma^{\ (k)}\right|+\frac{\omega_n\beta_n^{\ 2}}{c^2\tau_{kn}^{\ 2}}\sum_{\gamma=1}^{3}\left|x_\gamma^{\ (n)}-\overline{x}_\gamma^{\ (n)})\right|+\frac{\beta_n}{c^2\tau_{kn}}\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{\ (n)}-\dot{\overline{u}}_\gamma^{\ (n)}\right|+\left(\frac{1}{c^2\tau_{kn}^{\ 2}}+\frac{\omega_n\beta_n}{c^2\tau_{kn}}\right)\sum_{\gamma=1}^{3}\left|u_\gamma^{\ (n)}-\overline{u}_\gamma^{\ (n)}\right|+\frac{1}{c^2\tau_{kn}}\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{\ (n)}-\dot{\overline{u}}_\gamma^{\ (n)}\right|\right]\le$$

$$\le \frac{|e_k e_n|}{m_k c}\left[\left(\frac{\omega_n\beta_n}{c^2\tau_{kn}^{\ 2}}+\frac{1}{c^2\tau_{kn}^{\ 3}}+\frac{\beta_n\omega_n}{c^2\tau_{kn}^{\ 2}}+\frac{\omega_n\beta_n^{\ 2}}{c^2\tau_{kn}^{\ 2}}\right)\sum_{\gamma=1}^{3}\left|x_\gamma^{\ (k)}-\overline{x}_\gamma^{\ (k)}\right|+\left(\frac{\omega_n\beta_n}{c^2\tau_{kn}^{\ 2}}+\frac{1}{c^2\tau_{kn}^{\ 3}}+\frac{\beta_n\omega_n}{c^2\tau_{kn}^{\ 2}}+\frac{\omega_n\beta_n^{\ 2}}{c^2\tau_{kn}^{\ 2}}\right)\sum_{\gamma=1}^{3}\left|x_\gamma^{\ (n)}-\overline{x}_\gamma^{\ (n)}\right|+\right.$$

$$\left.+\left(\frac{1}{c^2\tau_{kn}^{\ 2}}+\frac{\omega_n\beta_n}{c^2\tau_{kn}}+\frac{\omega_n\beta_n}{c^2\tau_{kn}}\right)\sum_{\gamma=1}^{3}\left|u_\gamma^{\ (k)}-\overline{u}_\gamma^{\ (k)}\right|+\left(\frac{1+\beta_k}{c^2\tau_{kn}}+\frac{\beta_n}{c^2\tau_{kn}}+\frac{1}{c^2\tau_{kn}}\right)\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{\ (n)}-\dot{\overline{u}}_\gamma^{\ (n)}\right|\right]\simeq$$

$$\simeq \frac{|e_k e_n|}{m_k}\left[\left(\frac{1}{c^3\tau_{kn}^{\ 3}}+\frac{2\beta_n\omega_n}{c^3\tau_{kn}^{\ 2}}\right)\sum_{\gamma=1}^{3}\left|x_\gamma^{\ (k)}-\overline{x}_\gamma^{\ (k)}\right|+\left(\frac{1}{c^3\tau_{kn}^{\ 3}}+\frac{2\beta_n\omega_n}{c^3\tau_{kn}^{\ 2}}\right)\sum_{\gamma=1}^{3}\left|x_\gamma^{\ (n)}-\overline{x}_\gamma^{\ (n)}\right|+\left(\frac{1}{c^3\tau_{kn}^{\ 2}}+\frac{2\omega_n\beta_n}{c^3\tau_{kn}}\right)\sum_{\gamma=1}^{3}\left|u_\gamma^{\ (k)}-\overline{u}_\gamma^{\ (k)}\right|+\right.$$

$$\left.+\frac{2+\beta_k+\beta_n}{c^3\tau_{kn}}\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{\ (n)}-\dot{\overline{u}}_\gamma^{\ (n)}\right|\right].$$

We apply the inequality $\sqrt{\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}-\overline{u}_\gamma^{(k)}\right|^2}\le\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}-\overline{u}_\gamma^{(k)}\right|$ and obtain

$$\left|G_\alpha^{(k)rad}-\overline{G}_\alpha^{(k)rad}\right|\le\frac{e_k^2}{m_k c^5}\left|u_\alpha^{(k)}\left\langle\vec{u}^{(k)},\ddot{\vec{u}}^{(k)}\right\rangle-\overline{u}_\alpha^{(k)}\left\langle\vec{\overline{u}}^{(k)},\ddot{\vec{\overline{u}}}^{(k)}\right\rangle+c^2\ddot{u}_\alpha^{\ (k)}-c^2\ddot{\overline{u}}_\alpha^{\ (k)}\right|\le$$

$$\le\frac{e_k^2}{m_k c^5}\left|u_\alpha^{(k)}\left\langle\vec{u}^{(k)},\ddot{\vec{u}}^{(k)}\right\rangle-\overline{u}_\alpha^{(k)}\left\langle\vec{u}^{(k)},\ddot{\vec{u}}^{(k)}\right\rangle+\overline{u}_\alpha^{(k)}\left\langle\vec{u}^{(k)},\ddot{\vec{u}}^{(k)}\right\rangle-\overline{u}_\alpha^{(k)}\left\langle\vec{\overline{u}}^{(k)},\ddot{\vec{u}}^{(k)}\right\rangle+\overline{u}_\alpha^{(k)}\left\langle\vec{\overline{u}}^{(k)},\ddot{\vec{u}}^{(k)}\right\rangle-\overline{u}_\alpha^{(k)}\left\langle\vec{\overline{u}}^{(k)},\ddot{\vec{\overline{u}}}^{(k)}\right\rangle+c^2(\ddot{u}_\alpha^{\ (k)}-\ddot{\overline{u}}_\alpha^{\ (k)})\right|\le$$

$$\le\frac{e_k^2}{m_k c^5}\left(\left|u_\alpha^{(k)}-\overline{u}_\alpha^{(k)}\right|c_k\omega_k^{\ 2}U_0 e^{\mu_k(t-T_p^{(k)})}+c_k\left|\left\langle\vec{u}^{(k)}-\vec{\overline{u}}^{(k)},\ddot{\vec{u}}^{(k)}\right\rangle\right|+\overline{u}_\alpha^{(k)}\left|\left\langle\vec{\overline{u}}^{(k)},\ddot{\vec{u}}^{(k)}-\ddot{\vec{\overline{u}}}^{(k)}\right\rangle\right|+c^2\left|\ddot{u}_\alpha^{\ (k)}-\ddot{\overline{u}}_\alpha^{\ (k)}\right|\right)\simeq$$

$$\simeq\frac{e_k^2}{m_k c^3}\left(\left|u_\alpha^{(k)}-\overline{u}_\alpha^{(k)}\right|\beta_k^{\ 2}\omega_k^{\ 2}+\left\|\vec{u}^{(k)}-\vec{\overline{u}}^{(k)}\right\|\omega_k^{\ 2}\beta_k^{\ 2}\sqrt{3}+\left|\ddot{u}_\alpha^{\ (k)}-\ddot{\overline{u}}_\alpha^{\ (k)}\right|\right)\le$$

$$\le\frac{e_k^2}{m_k c^3}\left(\beta_k^{\ 2}\omega_k^{\ 2}\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}-\overline{u}_\gamma^{(k)}\right|+\sqrt{3}\omega_k^{\ 2}\beta_k^{\ 2}\sqrt{\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}-\overline{u}_\gamma^{(k)}\right|^2}+\left|\ddot{u}_\alpha^{\ (k)}-\ddot{\overline{u}}_\alpha^{\ (k)}\right|\right)\le\frac{e_k^2}{m_k c^3}\left(\left(1+\sqrt{3}\right)\omega_k^{\ 2}\beta_k^{\ 2}\sum_{\gamma=1}^{3}\left|u_\gamma^{(k)}-\overline{u}_\gamma^{(k)}\right|+\sum_{\gamma=1}^{3}\left|\ddot{u}_\gamma^{(k)}-\ddot{\overline{u}}_\gamma^{(k)}\right|\right).$$

Therefore

$$\left|B_\alpha^{(k)}(u)(t)-B_\alpha^{(k)}(\overline{u})(t)\right|\le\int_{T_p}^{t}\left|U_\alpha^{(k)}(u)-U_\alpha^{(k)}(\overline{u})\right|ds+\left|\frac{t-T_p}{T}-\frac{1}{2}\right|\int_{T_p}^{T_{p+1}}\left|U_\alpha^{(k)}(u)-U_\alpha^{(k)}(\overline{u})\right|ds+\frac{1}{T}\int_{T_p}^{T_{p+1}}\int_{T_p}^{\theta}\left|U_\alpha^{(k)}(u)-U_\alpha^{(k)}(\overline{u})\right|dsd\theta\ \le$$

$$\le\int_{T_p}^{t}\left|U_\alpha^{(k)}(u)-U_\alpha^{(k)}(\overline{u})\right|ds+\int_{T_p}^{T_{p+1}}\left|U_\alpha^{(k)}(u)-U_\alpha^{(k)}(\overline{u})\right|ds\le 2\int_{T_p}^{T_{p+1}}\left|U_\alpha^{(k)}(u)-U_\alpha^{(k)}(\overline{u})\right|ds\simeq$$

$$\simeq\sum_{n=1,n\ne k}^{4}\frac{|e_k e_n|}{m_k}\sum_{\gamma=1}^{3}\int_{T_p}^{T_{p+1}}\left[\left(\frac{1}{c^3\tau_{kn}^{\ 3}}+\frac{2\beta_n\omega_n}{c^3\tau_{kn}^{\ 2}}\right)\sum_{\gamma=1}^{3}\left|x_\gamma^{\ (k)}-\overline{x}_\gamma^{\ (k)}\right|+\left(\frac{1}{c^3\tau_{kn}^{\ 3}}+\frac{2\beta_n\omega_n}{c^3\tau_{kn}^{\ 2}}\right)\sum_{\gamma=1}^{3}\left|x_\gamma^{\ (n)}-\overline{x}_\gamma^{\ (n)}\right|+\right.$$

$$\left.+\left(\frac{1}{c^3\tau_{kn}^{\ 2}}+\frac{2\omega_n\beta_n}{c^3\tau_{kn}}+\frac{2\left(1+\sqrt{3}\right)\omega_k^{\ 2}\beta_k^{\ 2}}{c^3}\right)\sum_{\gamma=1}^{3}\left|u_\gamma^{\ (k)}-\overline{u}_\gamma^{\ (k)}\right|+\frac{2+\beta_k+\beta_n}{c^3\tau_{kn}}\sum_{\gamma=1}^{3}\left|\dot{u}_\gamma^{\ (n)}-\dot{\overline{u}}_\gamma^{\ (n)}\right|\right]ds+\frac{2}{c^3}\sum_{\gamma=1}^{3}\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\left|\ddot{u}_\gamma^{(k)}(t)-\ddot{\overline{u}}_\gamma^{(k)}(t)\right|ds\simeq$$

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$$
\begin{aligned}
&\simeq \sum_{n=1,n\neq k}^{4}\frac{|e_k e_n|}{m_k}\sum_{\gamma=1}^{3}\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}\Bigg[\Bigg(\frac{1}{c^3\tau_{kn}^{\;3}}+\frac{2\beta_n\omega_n}{c^3\tau_{kn}^{\;2}}\Bigg)\Bigg(2\frac{\omega_k^{\;h}e^{\mu_k T^{(k)}}}{\mu_k^{\;h+1}}+\frac{\omega_k^{\;h-1}}{\mu_k^{\;h+1}}\Bigg)\sum_{\gamma=1}^{3}\rho_{(p,h)}(u_\gamma^{(k)},\bar{u}_\gamma^{(k)})e^{\mu_k(t-T_p^{(k)})}+\\
&\Bigg(\frac{1}{c^3\tau_{kn}^{\;3}}+\frac{2\beta_n\omega_n}{c^3\tau_{kn}^{\;2}}\Bigg)\sum_{\gamma=1}^{3}\frac{\omega_n^{\;h}}{\mu_n^{\;h}}\frac{e^{-2c\mu_n/\hat{r}_{kn}}+1}{\mu_n}e^{\mu_n T^{(n)}}\rho_{(p-1,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})e^{\mu_n(t-T_p^{(n)})}+\Bigg(\frac{1}{c^3\tau_{kn}^{\;3}}+\frac{2\beta_n\omega_n}{c^3\tau_{kn}^{\;2}}\Bigg)\sum_{\gamma=1}^{3}\Bigg(\frac{2}{\mu_n}\frac{\omega_n^{\;h}}{\mu_n^{\;h}}\rho_{(p,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})\Bigg)e^{\mu_n(t-T_p^{(n)})}+\\
&+\Bigg(\frac{1}{c^3\tau_{kn}^{\;2}}+\frac{2\omega_n\beta_n}{c^3\tau_{kn}}+\frac{2\left(1+\sqrt{3}\right)\omega_k^{\;2}\beta_k^{\;2}}{c^3}\Bigg)\sum_{\gamma=1}^{3}2\frac{\omega_k^{\;h}}{\mu_k^{\;h}}\rho_{(p,h)}(u_\gamma^{(k)},\bar{u}_\gamma^{(k)})e^{\mu_k(t-T_p^{(k)})}+\\
&+\frac{2+\beta_k+\beta_n}{c^3\tau_{kn}}\sum_{\gamma=1}^{3}2\frac{\omega_k^{\;h+1}}{\mu_k^{\;h}}\rho_{(p-1,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})e^{-2c\mu_n/\hat{r}_{kn}}e^{\mu_n(t-T_{p-1}^{(n)})}\Bigg]ds+\frac{2}{c^3}\sum_{\gamma=1}^{3}\frac{\omega_k^{\;h+2}}{\mu_k^{\;h}}\rho_{(p,h)}(u_\gamma^{(k)},\bar{u}_\gamma^{(k)})e^{\mu_k(T_{p+1}^{(k)}-T_p^{(k)})}\simeq
\end{aligned}
$$

$$
\begin{aligned}
&\simeq \sum_{n=1,n\neq k}^{4}\frac{|e_k e_n|}{m_k}\sum_{\gamma=1}^{3}\int_{T_p^{(k)}}^{T_{p+1}^{(k)}}e^{\mu_k(t-T_p^{(k)})}\Bigg[\Bigg(\frac{8}{\hat{r}_{kn}^{\;3}}+\frac{8\beta_n\omega_n}{c\hat{r}_{kn}^{\;2}}\Bigg)\Bigg(2\frac{\omega_k^{\;h}e^{\mu_k T^{(k)}}}{\mu_k^{\;h+1}}+\frac{\omega_k^{\;h-1}}{\mu_k^{\;h+1}}\Bigg)\sum_{\gamma=1}^{3}\rho_{(p,h)}(u_\gamma^{(k)},\bar{u}_\gamma^{(k)})+\\
&+\Bigg(\frac{8}{\hat{r}_{kn}^{\;3}}+\frac{8\beta_n\omega_n}{c\hat{r}_{kn}^{\;2}}\Bigg)\sum_{\gamma=1}^{3}\frac{\omega_n^{\;h}}{\mu_n^{\;h}}\frac{e^{-2c\mu_n/\hat{r}_{kn}}+1}{\mu_n}e^{2\mu_n T^{(n)}}\rho_{(p-1,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})+\frac{2}{\mu_n}\frac{\omega_n^{\;h}}{\mu_n^{\;h}}\Bigg(\frac{8}{\hat{r}_{kn}^{\;3}}+\frac{8\beta_n\omega_n}{\hat{r}_{kn}^{\;2}}\Bigg)e^{\mu_n T^{(n)}}\sum_{\gamma=1}^{3}\rho_{(p,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})+\\
&+2\Bigg(\frac{4}{c\hat{r}_{kn}^{\;2}}+\frac{4\omega_n\beta_n}{c^2\hat{r}_{kn}}+\frac{2\left(1+\sqrt{3}\right)\omega_k^{\;2}\beta_k^{\;2}}{c^3}\Bigg)\frac{\omega_k^{\;h}}{\mu_k^{\;h}}\sum_{\gamma=1}^{3}\rho_{(p,h)}(u_\gamma^{(k)},\bar{u}_\gamma^{(k)})+\frac{4(2+\beta_k+\beta_n)}{c^2\hat{r}_{kn}}e^{-2c\mu_n/\hat{r}_{kn}}e^{\mu_n T^{(n)}}\frac{\omega_k^{\;h+1}}{\mu_k^{\;h}}\sum_{\gamma=1}^{3}\rho_{(p-1,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})\Bigg]ds+\\
&+\frac{2}{c^3}\frac{\omega_k^{\;h+2}e^{\mu_k T^{(k)}}}{\mu_k^{\;h}}\sum_{\gamma=1}^{3}\rho_{(p,h)}(u_\gamma^{(k)},\bar{u}_\gamma^{(k)})\simeq\\
&\simeq \sum_{n=1,n\neq k}^{4}\frac{|e_k e_n|}{m_k}\frac{e^{\mu_k T^{(k)})}-1}{\mu_k}\Bigg[16\Bigg(\frac{1}{\hat{r}_{kn}^{\;3}}+\frac{\beta_n\omega_n}{c\hat{r}_{kn}^{\;2}}\Bigg)\Bigg(\frac{\omega_k^{\;h}e^{\mu_k T^{(k)}}}{\mu_k^{\;h+1}}+\frac{\omega_k^{\;h-1}}{\mu_k^{\;h+1}}\Bigg)+\frac{1}{\mu_n}\frac{\omega_n^{\;h}}{\mu_n^{\;h}}\Bigg(\frac{8}{\hat{r}_{kn}^{\;3}}+\frac{8\beta_n\omega_n}{\hat{r}_{kn}^{\;2}}\Bigg)+\\
&+4\Bigg(\frac{2}{c\hat{r}_{kn}^{\;2}}+\frac{2\omega_n\beta_n}{c^2\hat{r}_{kn}}+\frac{\left(1+\sqrt{3}\right)\omega_k^{\;2}\beta_k^{\;2}}{c^3}\Bigg)\frac{\omega_k^{\;h}}{\mu_k^{\;h}}\Bigg]\sum_{\gamma=1}^{3}\rho_{(p,h)}(u_\gamma^{(k)},\bar{u}_\gamma^{(k)})+\\
&+\sum_{n=1,n\neq k}^{4}\frac{|e_k e_n|}{m_k}\Bigg[\Bigg(\frac{8}{\hat{r}_{kn}^{\;3}}+\frac{8\beta_n\omega_n}{c\hat{r}_{kn}^{\;2}}\Bigg)\frac{\omega_n^{\;h}}{\mu_n^{\;h}}\frac{e^{-2c\mu_n/\hat{r}_{kn}}+1}{\mu_n}e^{2\mu_n T^{(n)}}+\frac{4(2+\beta_k+\beta_n)}{c^2\hat{r}_{kn}}e^{-2c\mu_n/\hat{r}_{kn}}e^{\mu_n T^{(n)}}\frac{\omega_k^{\;h+1}}{\mu_k^{\;h}}\Bigg]\sum_{\gamma=1}^{3}\rho_{(p-1,h)}(u_\gamma^{(n)},\bar{u}_\gamma^{(n)})\leq\\
&\leq K_{(p,h)}\rho_{(p,h)}\left((u_1^{(1)},\ldots,u_3^{(4)}),(\bar{u}_1^{(1)},\ldots,\bar{u}_3^{(4)})\right)+K_{(p-1,h)}\rho_{(p-1,h)}\left((u_1^{(1)},\ldots,u_3^{(4)}),(\bar{u}_1^{(1)},\ldots,\bar{u}_3^{(4)})\right),
\end{aligned}
$$

where

$$
\begin{aligned}
K_{(p,h)}&=\sum_{n=1,n\neq k}^{4}\frac{|e_k e_n|}{m_k}\frac{e^{\mu_k T^{(k)})}-1}{\mu_k}\Bigg[16\Bigg(\frac{1}{\hat{r}_{kn}^{\;3}}+\frac{\beta_n\omega_n}{c\hat{r}_{kn}^{\;2}}\Bigg)\Bigg(\frac{\omega_k^{\;h}e^{\mu_k T^{(k)}}}{\mu_k^{\;h+1}}+\frac{\omega_k^{\;h-1}}{\mu_k^{\;h+1}}\Bigg)+\frac{1}{\mu_n}\frac{\omega_n^{\;h}}{\mu_n^{\;h}}\Bigg(\frac{8}{\hat{r}_{kn}^{\;3}}+\frac{8\beta_n\omega_n}{\hat{r}_{kn}^{\;2}}\Bigg)+\\
&+4\Bigg(\frac{2}{c\hat{r}_{kn}^{\;2}}+\frac{2\omega_n\beta_n}{c^2\hat{r}_{kn}}+\frac{\left(1+\sqrt{3}\right)\omega_k^{\;2}\beta_k^{\;2}}{c^3}\Bigg)\frac{\omega_k^{\;h}}{\mu_k^{\;h}}\Bigg],
\end{aligned}
$$

$$
K_{(p-1,h)}=\sum_{n=1,n\neq k}^{4}\frac{|e_k e_n|}{m_k}\Bigg[\Bigg(\frac{8}{\left(\hat{r}_{kn}^{\;m}\right)^3}+\frac{8\beta_n\omega_n}{c\left(\hat{r}_{kn}^{\;m}\right)^2}\Bigg)\frac{\omega_n^{\;h}}{\mu_n^{\;h}}\frac{e^{-2c\mu_n/\hat{r}_{kn}}+1}{\mu_n}e^{2\mu_n T^{(n)}}+\frac{4(2+\beta_k+\beta_n)}{c^2\hat{r}_{kn}^{\;m}}e^{-2c\mu_n/\hat{r}_{kn}}e^{\mu_n T^{(n)}}\frac{\omega_k^{\;h+1}}{\mu_k^{\;h}}\Bigg]
$$

and $K_{(p,h)}+K_{(p-1,h)}<1$.

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