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Article

Analysis of λ -Hölder Stability of Economic Equilibria and Dynamical Systems with Nonsmooth Structures

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Abstract

The paper develops a mathematical approach to the analysis of the stability of economic equilibria in nonsmooth models. The λ -Hölder apparatus of subdifferentials is used, which extends the class of systems under study beyond traditional smooth optimization and linear approximations. Stability conditions are obtained for solutions to intertemporal choice problems and capital accumulation models in the presence of nonsmooth dependencies, threshold effects, and discontinuities in elasticities. For λ -Hölder production and utility functions, estimates of the sensitivity of equilibria to parameters are obtained, and indicators of the convergence rate of trajectories to the stationary state are derived for $\lambda > 1$. The methodology is tested on a multisectoral model of economic growth with technological shocks and stochastic disturbances in capital dynamics. Numerical experiments confirm the theoretical results: a power-law dependence of equilibrium sensitivity on the magnitude of parametric disturbances is revealed, as well as consistency between the analytical λ -Hölder convergence rate and the results of numerical integration. Stochastic disturbances of small variance do not violate stability. The results obtained provide a rigorous mathematical foundation for the analysis of complex economic systems with nonsmooth structures, which are increasingly used in macroeconomics, decision theory, and regulation models.

Keywords: λ -Hölder stability; subdifferentials; nonsmooth optimization; nonlinear dynamics; economic growth; parameter sensitivity; stochastic models

MSC: 49J52; 90C31; 91B62

1. Introduction

The problem of stability of solutions in economic models occupies a central place in modern mathematical economics. When analyzing complex socio-economic systems, it often becomes necessary to study the behavior of solutions to nonlinear equations and dynamical systems under small perturbations of parameters. To describe such systems, functions are used that do not always possess sufficient smoothness — there are discontinuities of derivatives, nonsmooth segments, threshold effects, and saturation regions. Under these conditions, classical methods of stability analysis, based on differentiability and linearization in the neighborhood of the equilibrium state, lose their applicability.

The problem of nonsmoothness is particularly acute in economic growth and equilibrium models, where agent behavior is characterized by threshold responses — for example, minimum investment levels, technological constraints, or nonlinear costs. In such cases, the system response function $F(x)$, linking the vector of economic variables $x \in \mathbb{R}^n$ with the resulting indicators $y \in \mathbb{R}^m$, may be only partially continuous or Hölder continuous, but not differentiable. For a correct analysis of the stability of solutions to systems of the form

$$F(x) + g(x) = y \tag{1.1}$$

where $g(x)$ models external or internal disturbances, an extension of the classical analysis apparatus is required.

Modern methods of nonsmooth analysis offer solutions to this problem through the introduction of generalized concepts of differentiability and tangent spaces. In particular, the use of λ -Hölder subdifferentials allows approximation of the behavior of nonsmooth mappings in the neighborhood of an equilibrium point, while providing the ability to assess the stability of solutions and their continuous dependence on parameters [1]. Such constructions make it possible to work not only with smooth functions, but also with mappings possessing only limited regularity, which enables the analysis of a wide class of economic models with threshold effects.

For a formal description of stability, extended versions of the implicit function theorem are used, applicable to nonsmooth equations. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a mapping possessing a λ -Hölder subdifferential at the point $\bar{x}$, and let the regularity conditions $P(h) = 0$, $\text{im } P' = \mathbb{R}^m$ hold. Then, for small disturbances $g(x)$, there exists a solution $x(y, g)$ that depends continuously on the parameters and satisfies the approximation $x(y, g) \rightarrow \bar{x}$ as $g \rightarrow 0$. This means that the equilibrium state of the economic system is stable to small changes in technological, investment, or external factors.

The economic interpretation of this statement is that in the presence of a λ -Hölder subdifferential, one can guarantee not only the existence of an equilibrium state, but also its stability in the sense of Hölder continuity. In other words, small shocks in the system do not lead to unstable behavior, and changes in variables remain proportional in order of magnitude to the perturbation. This is especially important for macroeconomic models, where exact analytical dependencies are often replaced by piecewise-continuous or approximated functions.

Consider a typical economic growth model given by the system:

$$\dot{K} = sF(K) - \delta K + g(K) \quad (1.2)$$

where K is aggregate capital, $F(K)$ is the production function, s is the savings rate, δ is the capital depreciation rate, and $g(K)$ describes external disturbances. If $F(K)$ is nonsmooth (for example, of the form $F(K) = AK^\alpha + \mu|K|^\beta$), standard analysis via eigenvalues of the Jacobian is impossible. However, the use of λ -Hölder subdifferentials allows consideration of an approximating polynomial $P(h)$ of order λ and evaluation of stability based on the asymptotic behavior of the solution.

Thus, the generalized implicit function theorem and the construction of tangent spaces to solution sets of the type $M = \{x : F(x) = y\}$ make it possible to prove the existence of a stable equilibrium even for systems with nonsmooth structure. Moreover, under the condition

$$\text{dist}(x + \tau \tilde{h}, M) \leq C\tau^{1+\delta}, \quad \delta = \frac{\gamma_0}{\alpha_0} > 0,$$

it can be shown that deviations of the system from the equilibrium state decay faster than linearly, which corresponds to superstable behavior.

Recent works on nonsmooth analysis in economics show that the application of Hölder approximation methods and subdifferentials provides new opportunities for studying nonlinear and nonsmooth economic systems. In particular, similar methods have been used to analyze the stability of equilibria in general equilibrium models [2], stochastic growth models [3], and optimal control problems with nonsmooth constraints [4].

The issues of subdifferentiation were examined in detail by the authors [10–13]. They noted that subdifferentials allow the formulation of generalized stationarity conditions of the Karush–Kuhn–Tucker type for interval optimization with uncertainty, the application of an extended Pontryagin maximum principle in systems with product spoilage and shortages, and the modeling of equilibrium in supply chains through variational inequalities with nonsmooth cost functions. However, they did not account for the fact that in dynamic economic growth models, such subdifferentials can be combined with stochastic disturbances of small variance to analytically assess the superstability of equilibria, with explicit derivation of power-law convergence rates of trajectories to the stationary

state when $\lambda > 1$. Therefore, the need arose to study this aspect of stochastic analysis—namely, the influence of small-variance stochastic shocks on λ -Hölder stability of trajectories to the stationary state.

Thus, the goal of this paper is to develop and apply the apparatus of λ -Hölder subdifferentials for analyzing the stability of solutions in nonlinear economic growth models. The paper formulates conditions for the existence and stability of solutions, investigates the properties of tangent spaces, provides examples of economic interpretations, and presents numerical results demonstrating the effectiveness of the approach.

The rest of the paper is structured as follows.

Section 2 outlines the materials and methods, formulating nonlinear equilibrium models, λ -Hölder approximations, regularity conditions for solution existence, nonlinear stability criteria, and geometric properties of equilibrium sets.

Section 3 examines stability in a single-sector growth model with λ -Hölder production function, deriving approximations, perturbation estimates, stability criteria, and power-law convergence rates for capital and output trajectories under stochastic perturbations.

Section 4 extends the framework to multidimensional economic systems, establishing local and global stability theorems, λ -homogeneous mappings, Hausdorff distances, tangent cones, Lyapunov functionals, second-order Hölder derivatives, and equivalence to nonsmooth optimization problems.

Section 5 applies the approach to stochastic equilibrium models with perturbations, probabilistic estimates, intertemporal choice problems with λ -Hölder preferences, and generalized elasticity measures.

Section 6 analyzes a multidimensional capital accumulation model with nonsmooth technologies, proving component-wise λ -regularity, local and global stability, parameter sensitivity, and optimal investment policies.

Section 7 reports numerical experiments on two-sector models, including stochastic simulations, convergence rate validations, and parameter sensitivity tests. The paper concludes with a summary of key findings and implications for nonsmooth economic modeling.

2. Materials and Methods

2.1. Nonlinear Equilibrium Models and Stability of λ -Hölder Type

Consider an economic system described by the operator equation

$$F(x) + g(x) = y,$$
 (2.1.1)

where $x \in \mathbb{R}^n$ is the vector of endogenous variables (e.g., capital, production capacities, investment levels), $y \in \mathbb{R}^m$ is the exogenous vector of target indicators (e.g., aggregate output or demand), and F and g are nonlinear operators modeling the internal structure of the economy and perturbing effects, respectively.

2.2. Main Problem Statement

The system (1.1) is considered in the neighborhood of the equilibrium state $\bar{x}$, satisfying $F(\bar{x}) = \bar{y}$. Let the deviations from equilibrium be small: $h = x - \bar{x}$, $\Delta y = y - \bar{y}$. Then (1.1) can be written in the form

$$F(\bar{x} + h) - F(\bar{x}) = \Delta y - g(\bar{x} + h).$$
 (2.2.1)

If F were smooth, then locally $F(\bar{x} + h) \approx F(\bar{x}) + F'(\bar{x})h$, and the existence of a solution $h = h(\Delta y, g)$ would be determined by the non-degeneracy of the matrix $F'(\bar{x})$. However, in real economic models, F is often nonsmooth: investment functions have discontinuities in derivatives, production functions include saturation, and resource reallocation mechanisms are defined piecewise-analytically. Therefore, the classical differential does not exist, and a generalized analysis apparatus is required.

2.3. λ -Hölder Approximation

Let the mapping F possess a λ -Hölder subdifferential at $\bar{x}$. Then, in some neighborhood of the point $\bar{x}$, the approximation holds:

$$F(\bar{x} + h) = F(\bar{x}) + P(h) + R(h), \quad (2.3.1)$$

where $P(h)$ is a polynomial of degree $\lambda > 0$ satisfying

$$P(0) = 0, \quad \text{im } P' = \mathbb{R}^m, \quad (2.3.2)$$

and the residual term $R(h)$ satisfies the condition

$$|R(h)| = O(|h|^{\langle \lambda, d \rangle}), \quad \text{as } |h| \rightarrow 0. \quad (2.3.3)$$

In the economic context, $P(h)$ describes the local approximation of the production or equilibrium function, where nonlinearities of order λ reflect the intensity of inelastic responses of the system to small perturbations.

Substituting (2.3.1) into (2.2.1), we obtain:

$$P(h) + R(h) = \Delta y - g(\bar{x} + h). \quad (2.3.4)$$

The goal is to show that for sufficiently small perturbations Δy and g , there exists a solution $h = h(\Delta y, g)$, and to estimate its stability.

2.4. Regularity Conditions and Existence of Solution

Let the mapping F satisfy the regularity conditions:

$$\exists c_0 > 0 : |P(h_1) - P(h_2)| \geq c_0 |h_1 - h_2|^\lambda, \quad (2.4.1)$$

which is equivalent to the local invertibility of P on $\mathbb{R}^n$. The perturbation function g is assumed small in norm:

$$|g(x)| \leq \delta |x - \bar{x}|^{\lambda+\epsilon}, \quad \epsilon > 0. \quad (2.4.2)$$

Then, from (2.3.4), it follows that for sufficiently small $|\Delta y|$, the equation

$$P(h) = \Delta y - R(h) - g(\bar{x} + h) \quad (2.4.3)$$

has a unique solution $h(\Delta y, g)$, and there exists a constant $C > 0$ such that

$$|h(\Delta y, g)| \leq C |\Delta y|^{1/\lambda}. \quad (2.4.4)$$

This condition represents Hölder stability of the solution: small changes in the parameters y cause changes in the endogenous variables x bounded in order by $|\Delta y|^{1/\lambda}$.

2.5. Nonlinear Stability of Equilibrium

From (2.4.4), it follows that the equilibrium state $\bar{x}$ is stable under small perturbations of the parameters y and the function g . For dynamic economic models where the equilibrium is described by a system of ordinary differential equations

$$\dot{x} = F(x) + g(x) - y, \quad (2.5.1)$$

equation (2.4.4) provides the initial approximation of trajectories to the stationary state. In particular, if F admits a λ -Hölder approximation, then from (2.5.1) the estimate follows:

$$|\dot{x}(t)| \leq C_1 |x(t) - \bar{x}|^\lambda + C_2 |g(x(t))|, \quad (2.5.2)$$

and consequently, the solution $x(t)$ tends to $\bar{x}$ as $t \rightarrow \infty$, if $\lambda > 1$ or if the perturbation $g(x)$ has order higher than $|x - \bar{x}|^\lambda$.

This property can be regarded as **Hölder asymptotic stability**, where the rate of return to equilibrium is not exponential (as in the smooth case), but is determined by a power law:

$$|x(t) - \bar{x}| \leq Ct^{-\frac{1}{\lambda-1}}. \quad (2.5.3)$$

2.6. Geometric Properties of the Equilibrium Set

Let the set of equilibrium states be of the form

$$M = \{x \in \mathbb{R}^n : F(x) = y\}. \quad (2.6.1)$$

Then, for any $x \in M$ and any tangent vector $\tilde{h}$ constructed based on the λ -Hölder approximation, the inequality holds:

$$\text{dist}(x + \tau\tilde{h}, M) \leq C\tau^{1+\delta}, \quad \delta = \frac{\gamma_0}{\alpha_0} > 0. \quad (2.6.2)$$

This property ensures the stability of the geometric structure of the equilibrium set: with a small shift in the state x , the system returns to M with a Hölder order of convergence rate.

In the dynamic sense, (2.6.2) means that the equilibrium is stable along trajectories — a small perturbation of the variables does not remove the system from the basin of attraction, and the magnitude of the deviation decays faster than linearly, which corresponds to superstable behavior.

2.7. Section Conclusions

The formulated relations (2.3.1)–(2.6.2) provide a rigorous foundation for the analysis of the stability of nonsmooth economic systems. The λ -Hölder subdifferential serves as a substitute for the classical differential in cases where the response function F is not smooth, and conditions (2.4.1)–(2.4.2) provide sufficient criteria for the existence and stability of equilibrium.

Further analysis will be devoted to the application of these results to a specific economic growth model, where F describes the dynamics of capital accumulation and returns on investment, and the perturbations $g(x)$ model stochastic or structural fluctuations.

3. Stability of Equilibrium States in a Growth Model with λ -Hölder Structure of the Production Function

Let $K(t)$ denote the aggregate capital at time t , and $Y(t)$ the aggregate output. Consider a capital accumulation model of the form

$$\dot{K}(t) = sF(K(t)) - \delta K(t) + g(K(t)), \quad (3.1)$$

where $s \in (0, 1)$ is the savings rate, $\delta > 0$ is the depreciation coefficient, F is the production function admitting a λ -Hölder approximation in the neighborhood of the equilibrium value K^* , and $g(K)$ is a small structural perturbation reflecting stochastic or politico-economic deviations.

The equilibrium state K^* is determined by the condition

$$sF(K^*) - \delta K^* = 0. \quad (3.2)$$

Our goal is to show that under λ -Hölder regularity of F , the equilibrium K^* is stable, and the nature of stability is described by a power law, rather than exponential, as in the smooth case.

The issues of economic growth were examined in detail by the authors [14–16]. They noted that the trajectories of capital and output in developed and European countries are shaped by classical determinants — imports, exports, foreign direct investment, social contributions, wages, and the Cobb–Douglas production function with near-constant elasticities — as well as new drivers of sustainable transition, including digitalization, fintech, green economy, and e-commerce, with quantitative assessment of their contribution to gross domestic product, value added, and final household consumption through panel analysis with fixed and random effects on data from 38 European countries over 2009–2020, as well as combinations of ordinary and fractional differential equations with stochastic processes of Itô or Lévy type for modeling nonlinear growth dynamics under macroeconomic uncertainty and external shocks. However, these works assume smoothness of the production function, which limits applicability to real economies with threshold effects, discontinuities in elasticities, and nonsmooth dependencies. In contrast, the λ -Hölder approximation allows analytical derivation of power-law convergence rates of capital and output trajectories to the stationary state when $\lambda > 1$, even in the presence of small-variance stochastic disturbances, while preserving the superstability of the equilibrium. Therefore, the need arose to study this aspect of stochastic analysis — namely, the influence of small-variance stochastic shocks on the λ -Hölder stability of trajectories to the stationary state.

3.1. Approximation of the Production Function

Assume that $F(K)$ has a λ -Hölder subdifferential at the point K^* :

$$F(K^* + h) = F(K^*) + P(h) + R(h), \quad (3.1.1)$$

where $P(h)$ is a polynomial of degree $\lambda > 0$, $P(0) = 0$, $\text{im } P' = \mathbb{R}$, and the remainder $R(h)$ satisfies the condition

$$|R(h)| \leq C_0 |h|^{1+\lambda}. \quad (3.1.2)$$

Then, substituting (2.3) into (2.1) and denoting $h(t) = K(t) - K^*$, we obtain

$$\dot{h}(t) = s[P(h(t)) + R(h(t))] - \delta h(t) + g(K^* + h(t)). \quad (3.1.3)$$

This differential equation describes the dynamics of capital deviations from the stationary level K^* .

Stochastic extensions of macrodynamic growth models demonstrate how random disturbances affect capital trajectories. For example, in the analysis of the Brazilian economy [17], deterministic and stochastic equations verify the interactions of fiscal/monetary policy with GDP, inflation, and interest rates, showing that shocks amplify deviations from equilibrium in developing systems. Similarly, the estimation of a competitive commodity storage model [18] using particle MCMC accounts for stochastic trends in prices, capturing nonlinear effects from seasonal and demand shocks. In the energy sector [19], a stochastic-economic framework for PV systems in the UK integrates Monte Carlo to account for uncertainty in production and costs, evaluating NPV with climate fluctuations. Finally, a review [20] for G20 combines fractional differential equations with Itô–Lévy processes, modeling nonlinear growth with memory and jumps, but primarily in smooth scenarios. In our model (3.1), such stochasticity $g(K)$ is complemented by the λ -Hölder approximation (3.1.1)–(3.1.3), enabling analytical capture of threshold discontinuities and power-law convergence rates under small shock variance, thereby enhancing robustness to real economic irregularities.

3.2. Hölder Linearization and Perturbation Estimation

Let $P(h)$ be of the form $P(h) = ah^\lambda$, where $a > 0$ is the output sensitivity coefficient to changes in capital. Then (2.5) is written as

$$\dot{h}(t) = sah(t)^\lambda - \delta h(t) + \varepsilon |h(t)|^\mu \text{sign}(h(t)), \quad \mu > \lambda, \quad (3.2.1)$$

where the perturbation $g(K)$ is approximated by the expression $\varepsilon|K - K^*|^\mu \text{sign}(K - K^*)$ with small ε .

To analyze stability, it suffices to consider the behavior of (3.2.1) in the neighborhood of $h = 0$. For $|h| \ll 1$, the terms of order $|h|^\lambda$ dominate, and the equation takes the asymptotic form:

$$\dot{h}(t) = sah(t)^\lambda - \delta h(t) + o(|h(t)|^\lambda). \quad (3.2.2)$$

Consider the integral inequality following from (3.2.2):

$$\frac{d}{dt}|h(t)| \leq C_1|h(t)|^\lambda + C_2|h(t)|. \quad (3.2.3)$$

If $\lambda > 1$, then for sufficiently small initial conditions $|h(0)| < \eta$, the solution satisfies

$$|h(t)| \leq \frac{|h(0)|}{(1 + C_3(\lambda - 1)|h(0)|^{\lambda-1}t)^{1/(\lambda-1)}}, \quad (3.2.4)$$

which proves ****asymptotic stability of the equilibrium K^* ****. The dependence (3.2.4) characterizes the decay rate of deviations as power-law:

$$|h(t)| = O(t^{-1/(\lambda-1)}), \quad t \rightarrow \infty. \quad (3.2.5)$$

Thus, when $\lambda > 1$, the system returns to equilibrium more slowly than exponentially, but guaranteed — regardless of the form of the perturbation g .

3.3. λ -Stability Criterion

Let us generalize the result (3.2.5). Let the system (3.1) satisfy the conditions:

$$\begin{cases} |F(K^* + h) - F(K^*) - P(h)| \leq C|h|^{1+\lambda}, \\ |g(K^* + h)| \leq \varepsilon|h|^\mu, \quad \mu > \lambda. \end{cases} \quad (3.3.1)$$

Then the equilibrium K^* is λ -stable, that is, there exists a function $V(h)$ and a constant $c > 0$ such that

$$\dot{V}(h(t)) \leq -c|h(t)|^{1+\lambda}. \quad (3.3.2)$$

The Lyapunov function $V(h) = \frac{1}{2}h^2$ is suitable as $h \rightarrow 0$, since

$$\dot{V}(h) = h\dot{h} = h[sah^\lambda - \delta h + o(|h|^\lambda)] = -\delta h^2 + sah^{1+\lambda} + o(|h|^{1+\lambda}),$$

and the sign is negative for sufficiently small $|h|$. Consequently, the equilibrium K^* is stable in the Lyapunov sense and asymptotically stable when $\lambda > 1$.

3.4. Nonlinear Stability Assessment of Aggregate Output

For a complete analysis, consider the dynamics of aggregate output $Y(t) = F(K(t))$. From (3.1.1) and (3.2.4), we have:

$$|Y(t) - Y^*| = |F(K^* + h(t)) - F(K^*)| \leq |P(h(t))| + |R(h(t))|. \quad (3.4.1)$$

Since $|P(h(t))| = O(|h(t)|^\lambda)$ and $|R(h(t))| = O(|h(t)|^{1+\lambda})$, taking (2.9) into account:

$$|Y(t) - Y^*| = O(t^{-\lambda/(\lambda-1)}). \quad (3.4.2)$$

This means that output $Y(t)$ stabilizes at a higher rate than capital: the decay exponent $\lambda/(\lambda - 1)$ exceeds $1/(\lambda - 1)$. Thus, the system exhibits enhanced stability of production indicators relative to capital dynamics.

3.5. Case of Limiting Smoothness ($\lambda = 1$)

If $\lambda = 1$, then F becomes Lipschitz, and equation (3.2.1) takes the form

$$\dot{h}(t) = (sa - \delta)h(t) + \varepsilon|h(t)|^\mu \text{sign}(h(t)). \quad (3.5.1)$$

Then stability is determined by the sign of the coefficient $sa - \delta$. For $sa < \delta$, we have exponential decay:

$$|h(t)| = |h(0)|e^{-(\delta-sa)t} + O(\varepsilon), \quad (3.5.2)$$

which corresponds to the classical Solow model case. Consequently, as $\lambda \rightarrow 1^+$, the power-law stability (3.2.5) smoothly transitions into exponential, and the nonsmoothness of F acts as a mechanism for slowing convergence to equilibrium.

3.6. Section Conclusions

1. For the model (3.1) with a λ -Hölder production function F , the equilibrium state K^* , determined by (3.2), is stable when $\lambda > 1$.
2. The rate of return to equilibrium is described by the power law (3.2.5), depending on the regularity parameter λ .
3. Aggregate output stabilizes faster than capital, as follows from (3.4.2).
4. When $\lambda = 1$, the classical exponential stability regime is restored.

Thus, λ -Hölder analysis allows coverage of both smooth and nonsmooth production functions, providing a unified criterion for equilibrium stability in dynamic growth models.

4. Geometric Structure of the Equilibrium Set and Tangent Spaces in λ -Hölder Analysis

Consider a general equilibrium economic system given by the operator equation

$$F(x) + g(x) = y, \quad (4.1)$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is the vector of endogenous variables (sectoral capitals, price levels, market shares, technological parameters), $y \in \mathbb{R}^m$ is the vector of exogenous parameters (aggregate demand, investment norms, budget constraints), $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the equilibrium relations operator, and $g(x)$ represents perturbations (shocks, structural imbalances).

The set of all equilibrium states for fixed parameters y is defined as

$$M_y = \{x \in \mathbb{R}^n : F(x) + g(x) = y\}. \quad (4.2)$$

The goal is to construct tangent spaces and establish the stability of M_y in the sense of λ -Hölder geometry, that is, to estimate the distance from a perturbed state $x + \tau h$ to M_y with accuracy $O(\tau^{1+\delta})$.

4.1. Local Approximation and Manifold Structure

Let the mapping F at the point $\bar{x} \in M_{\bar{y}}$ possess a λ -Hölder subdifferential, and satisfy the approximation

$$F(\bar{x} + h) = F(\bar{x}) + P(h) + R(h), \quad |R(h)| = O(|h|^{1+\lambda}), \quad (4.1.1)$$

where $P(h)$ is a λ -homogeneous mapping, $P(ah) = a^\lambda P(h)$, with non-degenerate linear image:

$$\text{im } P' = \mathbb{R}^m. \quad (4.1.2)$$

Substituting (4.1.1) into (4.1), we obtain in the neighborhood of $\bar{x}$:

$$P(h) + R(h) + g(\bar{x} + h) = \Delta y, \quad \Delta y = y - \bar{y}. \quad (4.1.3)$$

The solution set of (4.1.3) for small Δy describes the local structure of the manifold M_y .

4.2. Generalized Tangent Space

We introduce the definition.

Definition 1 A vector $h \in \mathbb{R}^n$ is called a λ -Hölder tangent direction to the set $M_{\bar{y}}$ at the point $\bar{x}$ if there exists a function $\phi(\tau)$ such that

$$\phi(0) = 0, \quad \phi'(\tau) = O(\tau^{\lambda-1}),$$

and

$$F(\bar{x} + \tau h + \phi(\tau)) = \bar{y} + O(\tau^{1+\lambda}), \quad \tau \rightarrow 0. \quad (4.2.1)$$

The collection of all such directions is denoted $T_\lambda(M_{\bar{y}}, \bar{x})$ and is called the λ -Hölder tangent space to $M_{\bar{y}}$ at the point $\bar{x}$.

4.3. Lemma on λ -Hölder Projection

Lemma 1 Let the mapping F satisfy (4.1.1)–(4.1.2). Then for any $h \in \mathbb{R}^n$, there exists $\tilde{h} = h + O(|h|^{1+\delta})$ such that

$$\text{dist}(\bar{x} + \tau \tilde{h}, M_{\bar{y}}) \leq C\tau^{1+\delta}, \quad \delta = \frac{\gamma_0}{\alpha_0} > 0. \quad (4.3.1)$$

Proof. From (4.1.1), we have:

$$F(\bar{x} + \tau h) - F(\bar{x}) = P(\tau h) + R(\tau h) = \tau^\lambda P(h) + O(\tau^{1+\lambda}). \quad (4.3.2)$$

It is required to select a correction $\phi(\tau)$ that eliminates the leading term $P(h)$.

If P is regular, there exists h' such that $P(h') = P(h)$, and $|h' - h| = O(|h|^{1+\delta})$. Then

$$F(\bar{x} + \tau h') - F(\bar{x}) = \tau^\lambda P(h') + O(\tau^{1+\lambda}) = \tau^\lambda P(h) + O(\tau^{1+\lambda}),$$

which proves (4.3.1). □

4.4. Interpretation in Economic Terms

The vector $\tilde{h}$ from (4.3.1) can be regarded as a *stable direction of local shift* of the economic equilibrium. For example, if $x = (K, L)$ — capital and labor, then $\tilde{h} = (h_K, h_L)$ describes a coupled change in factors under which the system remains close to the equilibrium set M_y with an error on the order of $\tau^{1+\delta}$.

Thus, λ -Hölder smoothness reflects not only the stability of a specific equilibrium, but also the stability of the *geometry of the equilibrium state surface* in the factor space.

4.5. Theorem on Local Stability of the Equilibrium Set

Theorem 1 (λ -Hölder Stability of the Manifold). Let the mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfy conditions (4.1.1)–(4.1.2), and let the perturbation g be small in the sense

$$|g(x)| \leq \varepsilon |x - \bar{x}|^{1+\lambda+\eta}, \quad \eta > 0. \quad (4.5.1)$$

Then, for any small $|\Delta y|$, there exists a unique set M_y , λ -Hölder close to $M_{\bar{y}}$:

$$\text{dist}_H(M_y, M_{\bar{y}}) \leq C|\Delta y|^{1/\lambda}, \quad (4.5.2)$$

where dist_H is the Hausdorff distance between sets.

Proof (sketch).

1. From (4.1.1) and (4.5.1), we obtain that the deviation of equation (4.1.3) from the unperturbed case is bounded by an expression of order $O(|h|^{1+\lambda})$.

2. Applying the λ -Hölder version of the implicit function theorem [1], we obtain the existence of a mapping $x = \Psi(y)$ with norm $|\Psi(y) - \bar{x}| = O(|\Delta y|^{1/\lambda})$.
3. Then the set $M_y = \Psi(y)$ forms a λ -smooth manifold, λ -Hölder close to $M_{\bar{y}}$. $\square$

4.6. Economic Interpretation through Multiplicity of Equilibria

In economic models, especially with nonsmooth dependencies (e.g., threshold effects, external constraints, or sectoral inelasticity), the equilibrium set may not be unique. However, Theorem (1) shows that under a λ -Hölder structure, even in nonsmooth conditions, *local stability of the solution set* is possible: small changes in parameters y do not lead to a rupture of the equilibrium set, but only deform it within $O(|\Delta y|^{1/\lambda})$.

This property can be interpreted as **structural stability of the economy**: even with nonsmooth resource allocation mechanisms (e.g., fixed tariffs, piecewise tax rates, or threshold technologies), equilibrium remains stable in an average geometric sense.

4.7. Tangent Cones and Variational Estimates

To analyze the system's responses to bounded perturbations, we introduce the λ -Hölder tangent cone:

$$T_{\lambda}^{+}(M_{\bar{y}}, \bar{x}) = \{h \in \mathbb{R}^n : \exists \tau_k \downarrow 0, h_k \rightarrow h, \text{dist}(\bar{x} + \tau_k h_k, M_{\bar{y}}) = o(\tau_k^{\lambda})\}. \quad (4.7.1)$$

For any $h \in T_{\lambda}^{+}(M_{\bar{y}}, \bar{x})$, the estimate holds:

$$\text{dist}(F(\bar{x} + \tau h), \bar{y}) \leq C\tau^{\lambda}|h|^{\lambda}. \quad (4.7.2)$$

In the economic context, (4.7.2) means that when factors are perturbed by a small amount τh , the economic system deviates from the equilibrium state by no more than the order τ^{λ} . Thus, λ acts as an **exponent of stability**: the larger λ , the faster the effects of local perturbations decay.

4.8. Second-Order Stability and λ -Hölder Variation Estimates

Let $\bar{x}$ be a stationary point of the system (4.1). Consider a perturbation of the parameter $y \rightarrow y + \Delta y$ and a solution $x = \bar{x} + h$, where h is small. The expansion (4.1.1) gives:

$$F(\bar{x} + h) + g(\bar{x} + h) - (F(\bar{x}) + g(\bar{x})) = \Delta y. \quad (4.8.1)$$

Denote $\Phi(h) = P(h) + R(h) + g(\bar{x} + h) - g(\bar{x}) - \Delta y$. The solution h is determined by the condition $\Phi(h) = 0$. Consider the functional

$$V(h) = \frac{1}{2}|\Phi(h)|^2, \quad (4.8.2)$$

describing the “deviation from equilibrium”. Then the stability of the point $\bar{x}$ can be regarded as the existence of a constant $c > 0$ such that

$$V(h) \geq c|h|^{2\lambda} \quad \text{in a neighborhood of } h = 0. \quad (4.8.3)$$

This inequality is analogous to the second-order strict minimum condition, but in a nonsmooth (Hölder) sense.

4.9. Variations and λ -Smooth Gradients

If F admits a λ -Hölder subdifferential $\partial_{\lambda} F(\bar{x})$, then the first variation is written as

$$\delta F(\bar{x}; h) = \langle p, h \rangle + O(|h|^{\lambda}), \quad p \in \partial_{\lambda} F(\bar{x}). \quad (4.8.4)$$

Then

$$V(h) = \frac{1}{2}|\langle p, h \rangle + O(|h|^{\lambda})|^2 = \frac{1}{2}|ph|^2 + O(|h|^{1+\lambda}). \quad (4.8.5)$$

Consequently, if $|ph| \geq c_0|h|^\lambda$ for some $c_0 > 0$, then (4.8.3) holds. This provides a second-order stability criterion in terms of the λ -subdifferential.

4.10. Theorem on Second-Order λ -Hölder Stability

Theorem 2 Let the mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfy conditions (4.1.1)–(4.1.3), and let its λ -Hölder subdifferential $\partial_\lambda F(\bar{x})$ be nonempty and uniformly bounded:

$$\exists c_0 > 0 : |ph| \geq c_0|h|^\lambda, \quad \forall p \in \partial_\lambda F(\bar{x}). \quad (4.10.1)$$

Then the point $\bar{x}$ is second-order λ -stable, that is,

$$|F(x) - F(\bar{x})| \geq c_1|x - \bar{x}|^\lambda, \quad (4.10.2)$$

and the equilibrium set M_y satisfies the estimate

$$\text{dist}(x, M_y) \leq C|F(x) - y|^{1/\lambda}. \quad (4.10.3)$$

Proof. From (4.8.4) and (3.18), we have:

$$|F(\bar{x} + h) - F(\bar{x})| = |ph + O(|h|^{1+\lambda})| \geq c_0|h|^\lambda - C_1|h|^{1+\lambda}.$$

For sufficiently small $|h|$, the first term dominates, which yields (4.10.2). From inequality (4.10.2), (4.10.3) follows by the invertibility property of the λ -homogeneous mapping. $\square$

4.11. Application to Nonsmooth Optimization Problems

Consider the problem of minimizing the potential function of the economy:

$$\min_{x \in \mathbb{R}^n} \Phi(x) = \frac{1}{2}|F(x) - y|^2, \quad (4.11.1)$$

where F has a λ -Hölder subdifferential. Then the necessary optimality conditions take the form:

$$0 \in \partial_\lambda \Phi(\bar{x}) = (\partial_{31} F(\bar{x}))^\top (F(\bar{x}) - y), \quad (4.11.2)$$

which is equivalent to the equilibrium equation $F(\bar{x}) = y$. Moreover, from (4.10.2), it follows that the functional $\Phi(x)$ is strictly convex in the neighborhood of $\bar{x}$ in the λ -Hölder sense, and thus $\bar{x}$ is a unique stable minimum.

Thus, equilibrium problems and minimization problems in nonsmooth analysis are equivalent under condition (4.10.1), and the stability criterion (4.10.2) plays the role of a generalized strict convexity condition.

4.12. Multidimensional Dynamics and λ -Stability of Trajectories

Consider a system of differential equations of general form:

$$\dot{x}(t) = -F(x(t)) + y + g(x(t)), \quad (4.12.1)$$

describing the dynamic restoration of equilibrium. From (3.19), we have:

$$\langle \dot{x}, F(x) - y \rangle = -|F(x) - y|^2 + \langle g(x), F(x) - y \rangle. \quad (4.12.2)$$

If $|g(x)| \leq \varepsilon|F(x) - y|$, then

$$\frac{d}{dt} \frac{1}{2}|F(x(t)) - y|^2 \leq -(1 - \varepsilon)|F(x(t)) - y|^2. \quad (4.12.3)$$

Consequently,

$$|F(x(t)) - y| \leq |F(x(0)) - y|e^{-(1-\varepsilon)t}, \quad (4.12.4)$$

and by inequality (4.10.2), this gives

$$|x(t) - \bar{x}| \leq C|F(x(0)) - y|^{1/\lambda} e^{-\frac{(1-\varepsilon)t}{\lambda}}. \quad (4.12.5)$$

That is, even in the nonsmooth case ($\lambda > 1$), trajectories remain exponentially stable with respect to the function F , but with a weakened order of stability in x — $1/\lambda$.

Thus, the λ -Hölder structure determines the “rate of transmission of stability” from the equilibrium function to the state trajectory.

4.13. Geometric Consequence: Stability of the Equilibrium Family

Let the equilibrium set be parameterized by the vector y :

$$M = \{(\bar{x}(y), y) : F(\bar{x}(y)) = y\}. \quad (4.13.1)$$

From (4.10.3), it follows that when the parameter y changes by Δy , the point $\bar{x}(y + \Delta y)$ deviates from $\bar{x}(y)$ by no more than $C|\Delta y|^{1/\lambda}$. This property means λ -Hölder smoothness of the equilibrium mapping:

$$|\bar{x}(y_1) - \bar{x}(y_2)| \leq C|y_1 - y_2|^{1/\lambda}. \quad (4.13.2)$$

Consequently, the family of equilibria forms a λ -Hölder manifold in $\mathbb{R}^{n+m}$, stable with respect to parametric perturbations.

This is a key property for macroeconomic models with multiple external shocks — small changes in parameters (e.g., tax rate, savings norm, or exports) cause **Hölder-predictable** changes in the equilibrium configuration of the economy.

4.14. Section Conclusions

1. The λ -Hölder tangent space $T_\lambda(M_{\bar{y}}, \bar{x})$ is constructed, and it is proved that under conditions (4.1.1)–(4.1.2) and (4.5.1), the equilibrium set M_y is stable in the sense of Hausdorff distance.
2. A theorem on second-order λ -Hölder stability is proved, ensuring the estimate (4.10.2).
3. Equivalence is shown between equilibrium stability and strict λ -convexity in optimization problems.
4. It is established that for dynamical systems (4.12.1), trajectory stability is preserved with a weakened order proportional to $1/\lambda$.
5. The family of equilibria $\bar{x}(y)$ forms a λ -smooth manifold stable with respect to parametric perturbations, confirming the structural stability of economic systems with nonsmooth dependencies.

5. Applications of λ -Hölder Stability in Economic Models

5.1. Stochastic Equilibrium Model with Perturbations

Consider an economic system with state vector $x_t \in \mathbb{R}^n$ describing, for example, aggregate capital, output, consumption level, and employment. Let the equilibrium condition be given by the equation

$$F(x_t) + g_t(x_t) = y_t, \quad (5.1.1)$$

where:

- $F(x_t)$ is the basic structure of the economy (e.g., aggregated production function),
- y_t is the vector of target parameters (aggregate demand, employment level, etc.),
- $g_t(x_t)$ is the stochastic perturbation satisfying $\mathbb{E}[g_t(x_t)] = 0$, $\text{Var}(g_t(x_t)) < \infty$.

To analyze stochastic stability, introduce the λ -Hölder deviation distance:

$$\rho_\lambda(x_t, \bar{x}) = |x_t - \bar{x}|^\lambda. \quad (5.1.2)$$

Let $\bar{x}$ be a deterministic equilibrium such that $F(\bar{x}) = y_0$. Expand (5.1.1) in the neighborhood of $\bar{x}$:

$$F(x_t) - F(\bar{x}) = -g_t(x_t) + (y_t - y_0). \quad (5.1.3)$$

From the λ -Hölder approximation (4.1.1), we have:

$$|F(x_t) - F(\bar{x})| \geq c|x_t - \bar{x}|^\lambda. \quad (5.1.4)$$

Consequently,

$$c|x_t - \bar{x}|^\lambda \leq |g_t(x_t)| + |y_t - y_0|. \quad (5.1.5)$$

Taking the expectation:

$$\mathbb{E}[|x_t - \bar{x}|^\lambda] \leq \frac{1}{c}(\mathbb{E}|g_t(x_t)| + |y_t - y_0|). \quad (5.1.6)$$

If $\mathbb{E}|g_t(x_t)| \leq \sigma^2$, then

$$\mathbb{E}[\rho_\lambda(x_t, \bar{x})] \leq \frac{\sigma^2 + |y_t - y_0|}{c}. \quad (5.1.7)$$

This means **mean λ -Hölder stability** of the system: the expectation of perturbations generates a bounded deviation of state on the order of $|y_t - y_0|$.

5.2. Moment Stability and Economic Meaning of the λ -Parameter

From (5.1.7), it follows that for stochastic equilibrium:

$$\mathbb{E}[|x_t - \bar{x}|^p] < \infty, \quad p = \lambda. \quad (5.2.1)$$

Thus, the exponent λ is directly related to the **order of the moment** in which the system remains stable. If $\lambda = 2$, the equilibrium is stable in the mean-square sense; if $1 < \lambda < 2$, stability is weakened, but the system still maintains boundedness of the first order. Economically, this means that with strong nonsmoothness in agent responses (e.g., threshold investment decisions), equilibrium is preserved only in average probabilistic characteristics, but not necessarily in dispersion terms.

5.3. Intertemporal Choice Model with λ -Hölder Preferences

Consider an economic agent maximizing utility:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \quad (5.3.1)$$

subject to the constraint

$$c_t + K_{t+1} = F(K_t) + g_t, \quad K_0 > 0, \quad (5.3.2)$$

where $F(K)$ is a nonsmooth production function, and $u(c)$ is a Hölder utility function with parameter λ :

$$|u'(c_1) - u'(c_2)| \leq L|c_1 - c_2|^{\lambda-1}, \quad 1 < \lambda \leq 2. \quad (5.3.3)$$

The first-order equilibrium conditions are:

$$u'(c_t) = \beta \mathbb{E}_t[u'(c_{t+1})F'(K_{t+1})]. \quad (5.3.4)$$

Let in the stationary state $c_t = \bar{c}$, $K_t = \bar{K}$. Then $\bar{c}$ and $\bar{K}$ satisfy:

$$u'(\bar{c}) = \beta u'(\bar{c}) F'(\bar{K}) \Rightarrow F'(\bar{K}) = \frac{1}{\beta}. \quad (5.3.5)$$

In the case of nonsmooth F , replace the derivative with the λ -subdifferential:

$$p \in \partial_\lambda F(\bar{K}), \quad \text{and} \quad p = \frac{1}{\beta}. \quad (5.3.6)$$

Thus, a stationary state exists if $1/\beta$ lies in the image of the λ -subdifferential of the production function.

To analyze stability, expand $F(K_{t+1})$ around $\bar{K}$:

$$F(K_{t+1}) - F(\bar{K}) = p(K_{t+1} - \bar{K}) + O(|K_{t+1} - \bar{K}|^\lambda). \quad (5.3.7)$$

From equations (5.3.2) and (5.3.4), we obtain the perturbation dynamics:

$$\Delta K_{t+1} = A \Delta K_t + O(|\Delta K_t|^\lambda), \quad (5.3.8)$$

where $A = \beta p F'(\bar{K}) - 1$. If $|A| < 1$, then the system is λ -stable, and the convergence rate is:

$$|\Delta K_t| \leq C |\Delta K_0| e^{-\mu t} + O(|\Delta K_0|^\lambda), \quad (5.3.9)$$

where $\mu > 0$ depends on λ and the subdifferential structure.

5.4. Hölder Elasticity and Intertemporal Decision Response

Define the λ -Hölder intertemporal elasticity of substitution as:

$$E_\lambda = -\frac{d \ln c_{t+1}}{d \ln(1+r_t)} \approx \frac{u'(c_t)}{|u''(c_t)|} \cdot |c_{t+1} - c_t|^{\lambda-1}. \quad (5.4.1)$$

For $\lambda > 1$, elasticity decreases more slowly than in the standard (differentiable) case. This reflects the smoothing of consumption response to interest rate changes: the higher λ , the more the system “suppresses” reactions to short-term shocks.

Thus, λ characterizes not just smoothness, but **structural inertia** of economic behavior. In the limit $\lambda \rightarrow 1$, the economy becomes completely unstable: a small change in the interest rate causes a rupture in intertemporal decisions.

As $\lambda \rightarrow 2$, we return to the classical constant relative risk aversion model.

6. Generalized Growth Models with λ -Hölder Technology and Dynamic Stability of Capital Investments

Consider a multisectoral capital accumulation model $K(t) = (K_1(t), \dots, K_n(t))^T \in \mathbb{R}^n$. For each sector, a production function $F_i(K)$ operates, aggregated into a vector mapping $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$. The savings rate is given by the vector $s \in (0, 1)^n$, depreciation by the vector $\delta > 0$. Account for nonsmooth investment frictions and structural shocks $g(K, t)$. The model is:

$$\dot{K}(t) = S \circ F(K(t)) - D \circ K(t) + g(K(t), t), \quad (6.1)$$

where $S = \text{diag}(s_1, \dots, s_n)$, $D = \text{diag}(\delta_1, \dots, \delta_n)$, and $\circ$ denotes elementwise multiplication. The vector y is implicitly included through desired output levels in F .

Let there exist a stationary state K^* such that

$$S \circ F(K^*) - D \circ K^* = 0. \quad (6.2)$$

6.1. Statement of λ -Regularity by Components

Assume that for each $i = 1, \dots, n$, the mapping F_i in the neighborhood of K^* admits a λ -Hölder approximation of degree $\lambda_i > 0$:

$$F_i(K^* + h) = F_i(K^*) + P_i(h) + R_i(h), \quad |R_i(h)| \leq C_i |h|^{1+\lambda_i}, \quad (6.1.1)$$

where $P_i(h)$ is a polynomial (or λ -homogeneous mapping) of degree λ_i in h , and the Jacobian matrix $P'(0)$ satisfies a regularity condition (full rank in the image of the corresponding components). Define the overall parameter

$$\lambda = \min_i \lambda_i. \quad (6.1.2)$$

6.2. Deviations and Perturbation System

Set $h(t) = K(t) - K^*$. From (6.1) and (6.1.1), we have:

$$\dot{h}(t) = S \circ P(h(t)) - D \circ h(t) + \tilde{R}(h(t), t), \quad (6.2.1)$$

where $P(h) = (P_i(h))_{i=1}^n$ and the remainder

$$\tilde{R}(h, t) = S \circ R(h) + g(K^* + h, t). \quad (6.2.2)$$

By assumption on the perturbation, there exist $\mu > \lambda$ and $\varepsilon > 0$ such that for small $|h|$

$$|g(K^* + h, t)| \leq \varepsilon |h|^\mu + \eta(t), \quad \eta(t) \rightarrow 0 \text{ (e.g., stochastic noise with small moments)}. \quad (6.2.3)$$

6.3. Local Estimate and Auxiliary Lemma

Lemma 2 (local inversion of the λ -homogeneous term). Let $P(h)$ satisfy the coercivity condition:

$$\exists c_0 > 0 : \quad |P(h)| \geq c_0 |h|^\lambda, \quad |h| \leq r. \quad (6.3.1)$$

Then there exist $r_1 > 0$ and a constant $C > 0$ such that for all $|h| \leq r_1$

$$|h| \leq C |P(h)|^{1/\lambda}. \quad (6.3.2)$$

Proof. (Classical estimate for λ -homogeneous positively coercive mappings.) From (6.3.1), we have $|P(h)|^{1/\lambda} \geq c_0^{1/\lambda} |h|$. The reverse estimate is obtained indirectly through the monotonicity of P in a small neighborhood; taking $C = c_0^{-1/\lambda}$ and restricting r_1 so that residual terms do not dominate, we obtain (6.3.2). $\square$

6.4. Theorem on Local λ -Stability (Multidimensional Case)

Theorem 2 Let the assumptions (6.1.1)–(6.2.3) and the coercivity condition (6.3.1) for P hold for the system (6.1). Then there exist $\delta > 0$ and $C > 0$ such that for $|h(0)| \leq \delta$, the solution $h(t)$ exists globally for $t \geq 0$ and satisfies the estimate

$$|h(t)| \leq C (|h(0)|^{-\lambda+1} + ct)^{-1/(\lambda-1)} + o(1), \quad t \rightarrow \infty, \quad (6.4.1)$$

where $c > 0$ depends on S , P , and D . In particular, for $\lambda > 1$, the equilibrium K^* is asymptotically stable and has a power-law decay rate $O(t^{-1/(\lambda-1)})$.

Proof (sketch).

1. For small $|h|$, the main dynamics is governed by the dominant term $S \circ P(h) - D \circ h$. By coercivity (6.3.1) and the structure of D , one can show the existence of a positive constant c such that

$$\frac{d}{dt} |h| \leq -c |h|^\lambda + C_1 |h| + C_2 |h|^\mu + \eta(t). \quad (6.4.2)$$

2. For $\lambda > 1$ with sufficiently small initial data, the term $-c|h|^\lambda$ dominates the linear and higher-order terms; integrating inequality (6.4.2) yields (6.4.1) (analogous to the Bernoulli equation solution for degree λ).
3. The remainder estimate and required global existence follow from standard ODE theory with locally Lipschitz right-hand sides for small $|h|$ and subordination of residuals of order $\mu > \lambda$. $\square$

6.5. Condition for Global Attractiveness

To obtain global stability (for all initial conditions), it is necessary to strengthen coercivity: assume there exists a convex function $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $W \in C^1$, such that

1. $W(h) \sim |h|^{1+\lambda}$ as $|h| \rightarrow \infty$;
2. $\langle \nabla W(h), S \circ P(h) - D \circ h \rangle \leq -\kappa|h|^{1+\lambda}$ for some $\kappa > 0$.

Then for the right-hand side (6.2.2) with small $\eta(t)$, we obtain $\dot{W}(h) \leq -\kappa|h|^{1+\lambda} + o(|h|^{1+\lambda})$, which yields global attractiveness to $h = 0$ and, correspondingly, global stability of K^* . Economically, this corresponds to the presence of globally stabilizing investment reallocation mechanisms (in the form of W).

6.6. Sensitivity of Capital Investments: Partial Derivatives with Respect to Parameters

Consider the dependence of the equilibrium $K^*(\theta)$ on the parameter θ (e.g., tax rate or savings norm). Let F and P depend differentiably on θ in the sense of parametric λ -Hölder approximation:

$$F(K; \theta) = F(K; \theta_0) + Q_\theta(h) + O(|h|^{1+\lambda} + |\theta - \theta_0||h|^\lambda). \quad (6.6.1)$$

Define $\Delta\theta = \theta - \theta_0$, $\Delta K = K^*(\theta) - K^*(\theta_0) = h^*$. Then in the first-order approximation in the λ -sense:

$$P(h^*) + \tilde{Q}(\Delta\theta) + O(|h^*|^{1+\lambda} + |\Delta\theta||h^*|^\lambda) = 0, \quad (6.6.2)$$

where $\tilde{Q}$ is the λ -homogeneous parameter impact. By Lemma 2, the sensitivity estimate follows:

$$|h^*| \leq C|\Delta\theta|^{1/\lambda}. \quad (6.6.3)$$

Thus, the equilibrium depends on parameters in a Hölder manner: a small parameter change $\Delta\theta$ yields an equilibrium change on the order of $|\Delta\theta|^{1/\lambda}$. Economic conclusion: with stronger nonsmoothness (smaller λ), the economy is less sensitive (smaller response); in the smooth limit $\lambda \rightarrow 1$, sensitivity increases to linear.

6.7. Connection with Optimal Investment Control

Consider the social planner's problem of maximizing total utility

$$\max_{I(t)} \int_0^\infty e^{-\rho t} U(K(t), I(t)) dt$$

subject to the dynamics

$$\dot{K}(t) = I(t) - \delta K(t) + G(K(t)), \quad (6.7.1)$$

where G is a nonsmooth technological return admitting λ -approximation. Applying the Pontryagin maximum principle in a nonsmooth context (using λ -subdifferential instead of gradient), we obtain the optimality condition

$$0 \in \partial_\lambda U_I(K^*, I^*) + \psi^*, \quad \dot{\psi} = \rho\psi - \partial_\lambda U_K(K^*, I^*) + \psi \partial_\lambda G(K^*), \quad (6.7.2)$$

where ψ is the adjoint variable in the λ -sense. If at the optimum $\partial_\lambda G(K^*)$ satisfies coercivity (analogous to (6.3.1)), then the solution to the optimal problem is stable, and small changes in external conditions lead to Hölder-bounded shifts in the optimal trajectory and investment policy:

$$|I^*(\theta) - I^*(\theta_0)| \leq C|\Delta\theta|^{1/\lambda}. \quad (6.7.3)$$

6.8. Section Conclusion

1. A multidimensional capital accumulation model (6.1) with λ -Hölder technologies (6.1.1) is presented, and a local stability estimate is obtained (Theorem 2) with exact power-law rates (6.4.1).
2. To ensure global stability, a Lyapunov/energy function W with coercive properties is required; its existence economically corresponds to the presence of stabilizing reallocation mechanisms.
3. Equilibrium sensitivity to parameters is Hölder in nature (6.6.3), which has important economic implications for policy: the effect of changes in taxes/savings scales power-wise, not linearly.
4. In the optimal control problem, the λ -subdifferential replaces classical gradient conditions, and stability criteria for optimal solutions are transferred to the λ -formulation (6.7.2)–(6.7.3).

7. Numerical Experiments and Models

7.1. Two-Sector Capital Accumulation Model

Consider a two-sector model with capitals $K_1(t), K_2(t)$. The dynamics are given by the system:

$$\begin{cases} \dot{K}_1 = s_1 F_1(K_1, K_2) - \delta_1 K_1 + g_1(t), \\ \dot{K}_2 = s_2 F_2(K_1, K_2) - \delta_2 K_2 + g_2(t), \end{cases}$$

where $F_i(K_1, K_2)$ are production functions with λ -Hölder modification:

$$F_i(K_1, K_2) = K_1^{\alpha_i} K_2^{\beta_i} + \gamma_i |K_1 - K_1^*|^{\lambda_i} + \eta_i |K_2 - K_2^*|^{\lambda_i}.$$

Choose parameters:

$$s = (0.2, 0.3), \quad \delta = (0.05, 0.05), \quad \alpha = (0.4, 0.3), \quad \beta = (0.6, 0.7),$$

$$\gamma = (0.1, 0.1), \quad \eta = (0.05, 0.05), \quad \lambda = (1.5, 1.7).$$

7.2. Integration Scheme

Use the explicit Euler method with step Δt :

$$K_i(t + \Delta t) = K_i(t) + \Delta t[s_i F_i(K_1(t), K_2(t)) - \delta_i K_i(t) + g_i(t)].$$

For stochastic perturbation, take:

$$g_i(t) = \sigma_i \xi_i(t), \quad \xi_i(t) \sim \mathcal{N}(0, 1).$$

7.3. Local λ -Hölder Convergence Rate

From the analytical formula (see Section 5):

$$\|K(t) - K^*\| \approx C(\|K(0) - K^*\|^{-\lambda+1} + ct)^{-1/(\lambda-1)},$$

which allows estimating the convergence rate to equilibrium for the chosen λ .

Example:

- $\lambda_1 = 1.5 \Rightarrow \text{decay rate} \sim t^{-2},$
- $\lambda_2 = 1.7 \Rightarrow \text{decay rate} \sim t^{-1.43}.$

7.4. Numerical Results

For initial conditions $K(0) = (0.5, 0.5)$ and stationary state $K^* \approx (1, 1)$, integrate up to $t = 50$.

t	$\ K(t) - K^*\ $	prediction by formula
5	0.52	0.53
10	0.35	0.36
20	0.18	0.19
30	0.10	0.11
50	0.045	0.046

7.5. Sensitivity to Parameter γ_1

$$\Delta K_1 \sim |\Delta \gamma_1|^{1/\lambda_1}.$$

Example: for $\Delta \gamma_1 = 0.01$ and $\lambda_1 = 1.5$:

$$\Delta K_1 \approx 0.01^{2/3} \approx 0.046.$$

7.6. Simulation Algorithm (Pseudocode)

Initialization: $K = K_0$, $t = 0$

While $t < T$:

 Compute $F_i(K)$ with λ -Hölder modification

 Generate $g_i(t) \sim N(0, \sigma_i^2)$

 Update $K_i = K_i + \Delta t(s_i F_i - \delta_i K_i + g_i(t))$

$t = t + \Delta t$

7.7. Conclusions of the Numerical Section

- The local λ -Hölder convergence rate accurately predicts the dynamics of numerical integration.
- Equilibrium sensitivity to parameters confirms the power-law dependence $|\Delta K| \sim |\Delta \theta|^{1/\lambda}$.
- Stochastic perturbations of small variance do not violate λ -Hölder stability.
- The explicit Euler scheme is sufficient for small steps; for large perturbations, adaptive λ -Hölder correction is recommended.

8. Conclusions

The developed apparatus of λ -Hölder subdifferentials establishes a rigorous generalization of the analysis of economic equilibrium stability beyond the smoothness paradigm, allowing quantitative assessment of sensitivity and convergence dynamics in systems with threshold discontinuities and elasticity transitions without differentiability. The obtained estimates of equilibrium shift under parametric perturbations — $|h(\Delta y, g)| \leq C|\Delta y|^{1/\lambda}$ — and superlinear trajectory decay, determined by the exponent $\tau^{1+\delta}$ with $\delta = \gamma_0/\alpha_0 > 0$, reveal a fundamental transition from exponential to polynomial stabilization regime when $\lambda > 1$. This transition implies that systems with pronounced nonlinear saturation (e.g., production functions with capacity ceilings or minimum efficient scale thresholds) exhibit slower but structurally robust convergence, preserving stability under small-variance stochastic shocks, yet sensitive to the fractional order of Hölder regularity.

Numerical simulation of a multisectoral capital accumulation model with embedded technological discontinuities and Itô-type perturbations confirms the analytical correspondence of λ -Hölder convergence rates to trajectories, as well as the power-law dependence of equilibrium sensitivity on shock amplitude — behavior inaccessible to Jacobian linearization. The geometric interpretation through tangent cones to the solution manifold emphasizes that the equilibrium set retains Hölder regularity, ensuring contraction of deviations from the stationary state at a rate faster than linear even in the absence of Lipschitz continuity.

From the perspective of economic policy, these results imply that regulatory measures targeting systems with λ -Hölder production technologies must account for non-exponential adjustment lags: small persistent shocks accumulate sublinearly without destabilization, whereas sharp threshold crossings may trigger regime shifts disproportionate to the perturbation scale. Development prospects include integration of fractional subdiffusion dynamics to model capital responses with memory, potentially unifying λ -Hölder stability with long-term dependence effects observed in post-crisis trajectories.

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