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Article

From Golomb to Bateman-Horn

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Abstract

The Bateman-Horn conjecture is a conjecture on prime values in polynomials. We prove it by Golomb's method.

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1. Introduction

1.1. The Bateman-Horn Conjecture

Let p denote a prime number. In a paper with Sierpiński [14], Schinzel proposed the following conjecture, which is known as Schinzel's hypothesis (H).

Conjecture 1. *Let $k \geq 1$ and let $f_1(x), \cdots, f_k(x) \in \mathbb{Z}[x]$ be irreducible polynomials with positive leading coefficients. Assume that there does not exist any integer $n > 1$ dividing all the products $f_1(m) \cdots f_k(m)$ for every integer m. Then there are infinitely many natural numbers n such that $f_1(n), \cdots, f_k(n)$ are all primes.*

The Schinzel hypothesis (H) was studied on the average by Skorobogatov and Sofos [15]. Let $f_1(x), \cdots, f_k(x) \in \mathbb{Z}[x]$ be as in Conjecture 1, then they proved that when ordered by height 100% of these polynomials take simultaneously prime values at least once. For more results see their paper and references therein.

For the case of twin primes, that is for $f(x) = x(x+2)$, Conjecture 1 implies

$$\liminf_{n\to\infty}(p_{n+1} - p_n) = 2$$

where p_n is the nth prime. The previous results on bounded prime gaps was made by Zhang [17] by using the sieve method, who proved that

$$\liminf_{n\to\infty}(p_{n+1} - p_n) < 7 \cdot 10^7.$$

Later on Maynard [12], Tao and the Polymath Project [13] reduced the bound of Zhang. Their results lead to

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 246.$$

Recently the author [16] proposed a proof of the Schinzel hypothesis by the method of Golomb, together with a contradiction argument.

A related and quantitative form of Schinzel's hypothesis is the Bateman-Horn conjecture. Let $f_1, \cdots, f_k \in \mathbb{Z}[x]$ be irreducible polynomials of degree $h_1, \cdots, h_k$ and with positive leading coefficients. We write $f = f_1 f_2 \cdots f_k$. Assume that there does not exist a prime number p that divides $f(n)$ for every positive integer n. Let

$$\pi_f(x) = \#\{n \leq x : f_1(n), \cdots, f_k(n) \text{ are all prime}\}.$$

The Bateman-Horn conjecture is the following.

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Conjecture 2 ([4]). *Let $f_1, \cdots, f_k \in \mathbb{Z}[x]$ be as above, then as $x \to \infty$,*

$$\pi_f(x) \sim \frac{c(f)}{h_1 h_2 \cdots h_k} \cdot \frac{x}{\log^k x} \tag{1}$$

where

$$c(f) = \prod_p \frac{1 - N_f(p)/p}{(1 - 1/p)^k} \tag{2}$$

and $N_f(p)$ is the number of solutions of the congruence $f(n) \equiv 0 (\mathrm{mod}\, p)$.

Remark 3. Let $f_1, \cdots, f_k \in \mathbb{Z}[x]$ be as in Conjecture 2, then Bateman and Horn proved in [4] that $c(f)$ converges and is positive.

For an excellent survey and relevant historical literature on the Bateman-Horn conjecture we refer to the recent expository article [2].

It is clear the Bateman–Horn conjecture includes many special cases. For a single linear polynomial, it is Dirichlet's theorem on primes in arithmetic progressions, which in turn contains the prime number theorem ($f(x) = x$) as a special case. For $k \geq 2$ or for non-linear polynomials the conjecture is open. The simplest case of non-linear polynomials is the case $f(x) = x(x + 2)$ of the twin prime conjecture. The Bateman–Horn conjecture also includes the Hardy-Littlewood prime tuples conjecture [8]. Indeed Hardy and Littlewood [8] also proposed many other conjectures in their *Partitio Numerorum III*, many of which are special cases of the Bateman-Horn conjecture.

Let $\Lambda(n)$ be the von Mangoldt function and set

$$\psi_f(x) = \sum_{n \leq x} \Lambda(f_1(n)) \cdots \Lambda(f_k(n)).$$

By partial summation we have as in [3,5]

Proposition 4. *The Bateman-Horn conjecture 2 is equivalent to*

$$\psi_f(x) = c(f)x + o(x). \tag{3}$$

An important property of $\psi_f(x)$ is the following.

Proposition 5 ([5], Theorem 1). *We have $\psi_f(x) = O(x)$.*

1.2. Main Results

In this paper we announce a resolution of the Bateman-Horn conjecture.

Theorem 1.1. *The Bateman-Horn conjecture 2 is true.*

Specifically what we will prove is the following.

Theorem 1.2. *Let $f_1, \cdots, f_k \in \mathbb{Z}[x]$ be as in Conjecture 2 that satisfies the hypothesis F (see Lemma 7). Let $f = f_1 \cdots f_k$ and let $N_f(d)$ be the number of solutions of the congruence $f(n) \equiv 0 (\mathrm{mod}\, d)$. Then we have*

$$\lim_{z \to 1^-} \sum_{d=1}^{\infty} \left| \frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d \mid f(n)}}^{d} z^n - 1 \right| = 0. \tag{4}$$

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1.3. Overview of the Proof Strategy

Let $f_1, \cdots, f_k \in \mathbb{Z}[x]$ be as in Theorem 1.2. As we shall see later (17), to prove the Bateman-Horn conjecture we need to show

$$\lim_{z\to 1^-} \sum_{d=1}^{\infty} \left(\frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^{d} z^n \right) \mu(d) \log^k d \frac{N_f(d)}{d} = \sum_{d=1}^{\infty} \mu(d) \log^k d \frac{N_f(d)}{d}, \tag{5}$$

or equivalently

$$\begin{aligned} &\lim_{z\to 1^-} \sum_{d=1}^{\infty} \left[\left(\frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^{d} z^n - 1 \right) \mu(d) \log^k d \frac{N_f(d)}{d} + \mu(d) \log^k d \frac{N_f(d)}{d} \right] \\ = &\sum_{d=1}^{\infty} \mu(d) \log^k d \frac{N_f(d)}{d}. \end{aligned}$$

To prove the above identity it is equivalent to proving

$$\lim_{z\to 1^-} \sum_{d=1}^{\infty} \left(\frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^{d} z^n - 1 \right) \mu(d) \log^k d \frac{N_f(d)}{d} = 0. \tag{6}$$

In the course of the proof we will prove that

$$\lim_{z\to 1^-} \left| \sum_{d=1}^{\infty} \left(\frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^{d} z^n - 1 \right) \mu(d) \log^k d \frac{N_f(d)}{d} \right| = 0 \tag{7}$$

by a result of Agnew (Lemma 10). Now (6) follows from (7). To prove (7) we need to prove (4) and to prove the interchange of limit and infinite summation in (4) we will use the well known monotone convergence theorem.

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This project was started when I came across the paper [7] of Golomb in the library of Nagoya University. I would like to thank Professor Keith Conrad for sending me a copy of [6], which was (and is) not available in the internet. Special thanks to the staff of the library of Department of Science of Nagoya University, who kindly allowed me to use this library.

2. Golomb's Method

For positive integers a, b by (a, b) we mean the greatest common divisor of a and b. Let $\Lambda(n)$ be the von Mangoldt function, $\mu(n)$ the Möbius function, $\varphi(n)$ the Euler totient function, $\omega(n)$ the number of distinct prime divisors of n. An arithmetic function $f(n)$ is multiplicative if $f(ab) = f(a)f(b)$ for $(a, b) = 1$.

We investigate the Bateman-Horn conjecture by Golomb's method. There are mainly three papers on the Golomb method. The first is Golomb's thesis [6] (see also [7]), the second is Conrad [5] and the third is Hindry and Rivoal [9]. The papers [6] and [9] are by using the power series, while Conrad [5] is by using the Dirichlet series. Since one can turn a power series to a Dirichlet series and vice versa, the two approaches are equivalent.

The key of the Golomb method is the following identity of Golomb.

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Lemma 6 ([6,7]). *Let $a_i > 1$ and $(a_i, a_j) = 1$ for $i \neq j$ and let $A = a_1 a_2 \cdots a_k$. Then we have*

$$\Lambda(a_1)\Lambda(a_2)\cdots\Lambda(a_k) = \frac{(-1)^k}{k!}\sum_{d|A}\mu(d)\log^k d. \tag{8}$$

Golomb used this identity to study the twin prime conjecture by a way analogous to Wiener's proof of the prime number theorem. That is, let $n > 2$ be even, then by (8) we have

$$2\Lambda(n-1)\Lambda(n+1) = \sum_{d|(n^2-1)}\mu(d)\log^2 d. \tag{9}$$

For $|z| < 1$ let

$$G(z) = 2\sum_{n\geq 1}\Lambda(n-1)\Lambda(n+1)z^n,$$

then we have [6,7]

$$(1-z)G(z) = \sum_{\substack{d=1\\(d,2)=1}}^{\infty}\frac{(1-z)\mu(d)\log^2 d}{1-z^{2d}}\sum_{i=1}^{2^{\omega(d)}} z^{a_i} \tag{10}$$

where the a_i are the $2^{\omega(d)}$ even roots of the congruence $a^2 \equiv 1 (\mathrm{mod} d)$ between 0 and $2d$. If the termwise limit could be justified, then we would have

$$\lim_{z\to 1^-}(1-z)G(z) = \sum_{\substack{d=1\\(d,2)=1}}^{\infty}\frac{\mu(d)\log^2 d\cdot 2^{\omega(d)}}{2d} = 4\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right), \tag{11}$$

which would lead to a proof of the Bateman-Horn conjecture for the case of twin primes.

For the general Bateman-Horn conjecture the identity (8) requires that $(a_i, a_j) = 1$ for $i \neq j$, for this Hindry and Rivoal [9] introduced the hypothesis F: for all integers $n \geq 1$, $(f_i(n), f_j(n)) = 1$ for $1 \leq i \neq j \leq k$. They proved the following result.

Lemma 7 ([9], Théorème 3). *Let $f_1, \cdots, f_k \in \mathbb{Z}[x]$ be as in Conjecture 2. If the Bateman-Horn conjecture 2 is true for all $f_1, \cdots, f_k$ that satisfies the hypothesis F, then it is true for all $f_1, \cdots, f_k \in \mathbb{Z}[x]$ that as in Conjecture 2.*

Note that Conrad [5] also addressed this coprime issue, see [5, Lemma 3, Theorem 4].

Henceforth in the following we let $f_1, \cdots, f_k \in \mathbb{Z}[x]$ be as in Conjecture 2 that satisfies the hypothesis F. Remember that $f = f_1 \cdots f_k$.

Now by (8) we have

$$\Lambda(f_1(n))\Lambda(f_2(n))\cdots\Lambda(f_k(n)) = \frac{(-1)^k}{k!}\sum_{d|f(n)}\mu(d)\log^k d. \tag{12}$$

Consider the absolutely convergent series for $|z| < 1$:

$$G_f(z) = (-1)^k k!\sum_{n\geq 1}\Lambda(f_1(n))\Lambda(f_2(n))\cdots\Lambda(f_k(n))z^n, \tag{13}$$

then as in Hindry and Rivoal [9] we have

$$G_f(z) = \sum_{d\geq 1}\frac{\mu(d)\log^k d}{1-z^d}\sum_{\substack{n=1\\d|f(n)}}^{d} z^n. \tag{14}$$

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Let

$$N_f(d) := \sum_{\substack{n=1 \\ d|f(n)}}^{d} 1 \tag{15}$$

be the number of solutions of the congruence $f(n) \equiv 0 (\bmod d)$. If the termwise limit could be justified, then we would have

$$\lim_{z\to 1^-} (1-z)G_f(z) = \sum_{d\geq 1} \mu(d) \log^k d \lim_{z\to 1^-} \frac{1-z}{1-z^d} \sum_{\substack{n=1 \\ d|f(n)}}^{d} z^n = \sum_{d\geq 1} \mu(d) \log^k d \frac{N_f(d)}{d}. \tag{16}$$

Furthermore Conrad proved the following result, in a slightly different but equivalent form.

Proposition 8 ([5], Theorem 7). *The right side of* (16) *converges and*

$$\sum_{d\geq 1} \mu(d) \log^k d \frac{N_f(d)}{d} = c(f) > 0$$

where $c(f)$ *is as in* (2).

Thus if the termwise limit could be justified, the Bateman-Horn conjecture would be proved. The termwise limit can be done for the case $f(x) = x$ of the prime number theorem. Hindry and Rivoal [9] also proved Dirichlet's theorem on primes in arithmetic progressions by using Golomb's method and thus the termwise limit can be done for linear polynomials. However for nonlinear polynomials the termwise limit is elusive.

By (14) and (16) we see that the Bateman-Horn conjecture is equivalent to

$$\lim_{z\to 1^-} \sum_{d=1}^{\infty} \left(\frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^{d} z^n \right) \mu(d) \log^k d \frac{N_f(d)}{d} = \sum_{d=1}^{\infty} \mu(d) \log^k d \frac{N_f(d)}{d}. \tag{17}$$

3. Proof of Theorem 1.2

Let the hypotheses of Theorem 1.2 be satisfied. In order to prove the interchange of limit and infinite summation of (4) we use the well known monotone convergence theorem, which is as follows.

Lemma 9 (Monotone convergence theorem). *Let* d *be positive integer,* x *be real and* $f(d,x)$ *be nonnegative. If for each fixed* d *we have* $f(d,x) \leq f(d,x')$ *whenever* $x \leq x'$*, then*

$$\lim_{x} \sum_{d=1}^{\infty} f(d,x) = \sum_{d=1}^{\infty} \lim_{x} f(d,x).$$

Recall that the (4) we want to prove is

$$\lim_{z\to 1^-} \sum_{d=1}^{\infty} \left| \frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^{d} z^n - 1 \right| = 0.$$

In general for each fixed d the summand

$$\left| \frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^{d} z^n - 1 \right|$$

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is not a monotone function in z in the interval $(0,1]$ since the interval is too large, so we consider the interval $(1-\varepsilon,1]$ for small $\varepsilon>0$. But now since

$$\left|\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1\right|\geq 0$$

and it tends to 0 as $z\to 1^-$, it is a decreasing function in z in the interval $(1-\varepsilon,1]$ and thus does not meet the condition of Lemma 9. To overcome these difficulties we consider

$$-\left|\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1\right|+\frac{1}{d^2}$$

and we have

Theorem 3.1. *Let the notations be as in Theorem 1.2. Then*

$$\lim_{z\to 1^-}\sum_{d=1}^{\infty}\left(-\left|\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1\right|+\frac{1}{d^2}\right)=\sum_{d=1}^{\infty}\frac{1}{d^2}=\frac{\pi^2}{6}. \tag{18}$$

Proof. For each fixed d, the function

$$\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1 \tag{19}$$

is a rational function in z and thus its stationary points are discrete. Therefore there exists sufficiently small $\varepsilon(d)$ such that

$$-\left|\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1\right|+\frac{1}{d^2} \tag{20}$$

is monotonic in the interval $(1-\varepsilon(d),1]$. Since

$$-\left|\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1\right|\leq 0$$

and it tends to 0 as $z\to 1^-$, we see that

$$-\left|\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1\right|+\frac{1}{d^2} \tag{21}$$

is increasing in the interval $z\in(1-\varepsilon(d),1]$. Also we can make $\varepsilon(d)$ small enough such that (21) is positive in $(1-\varepsilon(d),1]$. Now the summand (21) meets the conditions of Lemma (9) and thus we have

$$\lim_{z\to 1^-}\sum_{d=1}^{\infty}\left(-\left|\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1\right|+\frac{1}{d^2}\right) \tag{22}$$

$$=\sum_{d=1}^{\infty}\lim_{z\to 1^-}\left(-\left|\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1\right|+\frac{1}{d^2}\right)=\sum_{d=1}^{\infty}\frac{1}{d^2}=\frac{\pi^2}{6}. \tag{23}$$

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□

As a consequence we deduce Theorem 1.2.

Proof of Theorem 1.2. By (18) we have

$$\lim_{z\to 1^-}\sum_{d=1}^{\infty}\left(-\left|\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1\right|\right)=0 \tag{24}$$

and thus

$$\lim_{z\to 1^-}\sum_{d=1}^{\infty}\left|\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1\right|=0. \tag{25}$$

The proof of Theorem 1.2 is complete. □

4. Proof of Theorem 1.1

To complete the proof of Theorem 1.1 the following result of Agnew is crucial.

Lemma 10. *Suppose s_n is a bounded real or complex sequence. Let $c_d(x)$ be a sequence of functions defined over $0<x<1$ and satisfying*

$$\lim_{x\to 1^-} c_d(x)=0, \qquad d=1,2,3,\cdots \tag{26}$$

$$\limsup_{x\to 1^-}\sum_{d=1}^{\infty}|c_d(x)|=M<+\infty. \tag{27}$$

Then

$$\limsup_{x\to 1^-}\left|\sum_{d=1}^{\infty}c_d(x)s_d\right|\le M\limsup_{n\to\infty}|s_n|. \tag{28}$$

Proof. See [11, Lemma 5] or Agnew [1, Lemma 3.1]. □

Lemma 10 is often used for equiconvergence and Tauberian constants in the theory of summability of sequences and series, for more of its applications see [10] and the references therein.

As we said in (7) we now prove the following.

Theorem 4.1. *We have*

$$\lim_{z\to 1^-}\left|\sum_{d=1}^{\infty}\left(\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1\right)\mu(d)\log^k d\frac{N_f(d)}{d}\right|=0. \tag{29}$$

Proof. We take

$$s_n:=\mu(n)\log^k n\frac{N_f(n)}{n}, \qquad c_d(z):=\left(\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1\right) \tag{30}$$

in Lemma 10. Since $N_f(n)=o(n)$ it is clear s_n is bounded. The condition (26) is clearly met when $z\to 1^-$. By Theorem 1.2,

$$\lim_{z\to 1^-}\sum_{d=1}^{\infty}\left|\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n-1\right|=0,$$

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and thus

$$\limsup_{z\to 1^-}\sum_{d=1}^{\infty}\left|\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d}z^n-1\right|=0:=M,$$

hence the condition (27) is met. By Lemma 10 and by noting that $\limsup_{n\to\infty}|s_n|<\infty$ since s_n is bounded, we deduce that

$$\limsup_{z\to 1^-}\left|\sum_{d=1}^{\infty}\left(\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d}z^n-1\right)\mu(d)\log^k d\frac{N_f(d)}{d}\right|\leq 0. \tag{31}$$

On the other hand we have

$$0\leq\liminf_{z\to 1^-}\left|\sum_{d=1}^{\infty}\left(\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d}z^n-1\right)\mu(d)\log^k d\frac{N_f(d)}{d}\right|,$$

and thus we conclude (29). □

The next step is to deduce (6), which follows from the well known fact in calculus.

Lemma 11. *Let $f(z)$ be a real-valued function where $z\in\mathbb{R}$ and let $c\in\mathbb{R}$. If $\lim_{z\to c}|f(z)|=0$, then $\lim_{z\to c}f(z)=0$.*

Proof. This is a well known fact. Indeed it follows from

$$-|f(z)|\leq f(z)\leq|f(z)|$$

and $\lim_{z\to c}(-|f(z)|)=0$. □

We can now prove Theorem 1.1.

Proof of Theorem 1.1. By (29) and Lemma 11 we have

$$\lim_{z\to 1^-}\sum_{d=1}^{\infty}\left(\frac{d(1-z)}{N_f(d)(1-z^d)}\sum_{\substack{n=1\\ d|f(n)}}^{d}z^n-1\right)\mu(d)\log^k d\frac{N_f(d)}{d}=0. \tag{32}$$

Thus (6) and hence Theorem 1.1 is proved. □

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