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Article

# Regular Alena-Urbantke Geometries and Local Gauge-Sector Equivalence

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## Abstract

A geometric normal form for the visible sector of Alena-type energy-momentum tensors in four-dimensional Lorentzian geometry is established. The traceless Ricci-type content of such tensors is encoded, via a curvature-type lift and self-dual reduction, by a Hermitian endomorphism of the complex self-dual bundle. A regularity condition (positivity, full rank, and simple spectrum of that endomorphism) selects a class of structures termed regular Alena-Urbantke geometries, in which a canonical ordered spectral flag and a natural chiral carrier geometry emerge. The main result is a local equivalence theorem: regular gauge-sector representations of the visible tensor correspond precisely to regular Alena-Urbantke geometries, and each determines the other locally. This formulation separates the visible self-dual block from the remaining hidden curvature data, clarifies the role of the Urbantke construction as a carrier geometry distinct from the induced metric variable, and identifies the degeneration loci at which the regular structure breaks down. Global existence of a regular visible sector is shown to require a splitting of the self-dual bundle with characteristic-class consequences. To support explicit computation, a Mathematica tool for spectral and regularity diagnostics of the visible Hermitian block is provided, together with an example report.

**Keywords:** Lorentzian geometry; self-dual bundle; Hermitian endomorphism; Urbantke geometry; transition tensor; Alena tensor

## 1. Introduction

The Urbantke construction provides a canonical route from a nondegenerate triple of two-forms to a spacetime metric, or more precisely to its conformal class, and has long been part of the standard toolkit of chiral and BF-type formulations of gravity [1,2]. In particular, it underlies the geometric interpretation of Plebanski-type variables [3] and continues to play an important role in gauge-theoretic formulations of general relativity [4], in analyses of the relation between gravity and BF theory [3], and in more recent extensions of the Plebanski framework [5,6].

The Alena Tensor is a recently proposed class of energy-momentum tensors and, at present, still lacks a dedicated geometric formulation. In previous works, a dual description of matter-field systems was developed in which the metric tensor is treated not as a fundamental spacetime structure, but as an element of a chosen geometric representation of the system. The original motivation was to establish a smooth link between a curvilinear description compatible with general relativity [7] and a flat-space description, both classical and quantum [8], first in simple dust models [9] and subsequently in systems involving electromagnetic fields [10]. In the generalized setting introduced in [11], an Alena-type transition tensor is related algebraically to a Lorentzian metric variable  $k_{\mu\nu}$  through Eq. (1), with associated scalar invariant  $p_\Lambda$ . In the present work, the geometric content of this structure is examined and an appropriate geometric framework for its analysis is sought. The visible tensor is studied through the geometric structures associated with it and through the constraints imposed by the Alena relation on the corresponding metric data.

The basic observation is that the visible tensor may be placed naturally in a curvature-type setting. Its traceless Ricci-type content is then read through the standard self-dual splitting of complexified two-

forms in four dimensions [12–14]. This places the problem in the familiar four-dimensional framework in which self-dual decomposition, spinorial curvature data, and chiral organization are central [15–20]. In the present setting, the visible self-dual block is encoded by a Hermitian endomorphism of the complex self-dual bundle  $\Lambda_+^2(\eta)$ , and the visible tensor is recovered from it by a canonical representation map. Through (1), the same object determines the traceless part of the metric variable.

The regular sector is selected by positivity, full rank, and simple spectrum of that Hermitian endomorphism. On this locus, a canonical spectral flag is obtained, and the visible self-dual data determine a natural chiral carrier geometry of Urbantke type [21,22]. The role of that geometry is structural: it shows that the regular visible sector carries genuine self-dual geometric content, but it is not identified directly with the Alena metric variable. Related emphasis on chiral organization has also appeared in broader twistor-motivated settings [23,24]. More recent appearances of self-dual and chiral sectors in gauge and gravity constructions provide further context for treating such structures as primary organizing data [25–28].

The main point established below is local equivalence. A regular gauge-sector representation of the visible tensor determines a regular Hermitian self-dual endomorphism, while a regular Hermitian self-dual endomorphism determines, locally, a regular gauge-sector representation. This leads to the notion of regular Alena-Urbantke geometry, in which the visible tensor and the traceless metric deformation are induced from regular self-dual Hermitian data. In this sense, the geometry is organized through an internal chiral carrier rather than through an a priori external gauge group [29]. Although the paper is not formulated in celestial terms, the focus on distinguished chiral sectors fits quite well a broader contemporary pattern in twistor and celestial geometry [30–32].

The paper is organized as follows. Section 2.1 introduces the curvature-type lift of the visible tensor and its self-dual reduction. Section 2.2 defines regular Alena-Urbantke geometry, derives the canonical spectral flag, and places the construction in the Urbantke setting. Section 2.3 proves the local equivalence theorem and records an explicit regular local model together with the interpretation of the hidden curvature data. The later sections then study the internal organization of the regular visible sector, including local classification, structural reduction, model geometric dynamics, degeneration mechanisms, and first global bundle-theoretic constraints. A supplementary computational tool is also included in order to make the spectral organization of the visible Hermitian block and its regularity and degeneration loci directly inspectable in explicit metric examples.

No derivation of a full Yang-Mills dynamics is claimed. What is isolated is a regular geometric class in which Alena-type transition tensors, the associated visible metric data, and their visible curvature-type block may be treated in a canonical self-dual Hermitian form, with a supplementary computational tool provided for direct spectral and regularity diagnostics in explicit examples.

## 2. Regular Alena-Urbantke Geometries

### 2.1. Curvature-Type Lift and Self-Dual Reduction of the Visible Tensor

The visible tensor is first placed in a curvature-type setting. Its self-dual reduction is then identified with a positive Hermitian endomorphism on the complex self-dual bundle. Only the local Lorentzian background geometry  $(U, \eta)$  is used. No regularity assumption is imposed at the outset.

The Alena relation (1) is equivalent to

$$\hat{k}_{\mu\nu} := k_{\mu\nu} - \frac{k}{4} \eta_{\mu\nu} = \frac{k}{4p_\Lambda} Y_{\mu\nu}, \quad p_\Lambda = p_o k^2. \quad (1)$$

where  $p_o$  is some constant or just invariant of the metric. It is therefore natural to associate to  $Y$  the curvature-like tensor

$$H_{\mu\nu\alpha\beta}^{(0)} := \frac{2p_\Lambda}{k} (\hat{k} \otimes \eta)_{\mu\nu\alpha\beta}, \quad (2)$$

where  $\otimes$  denotes the Kulkarni-Nomizu product. By construction,  $H^{(0)}$  is the canonical curvature-type tensor whose traceless Ricci part is determined by  $Y$  [12].

In four dimensions, after complexification and the decomposition

$$\Lambda^2 T^*U \otimes \mathbb{C} = \Lambda_+^2(\eta)|_U \oplus \Lambda_-^2(\eta)|_U, \quad (3)$$

the traceless Ricci part of a curvature operator is represented by the off-diagonal blocks, while the diagonal blocks carry the Weyl and scalar data [12–14]. The visible tensor should therefore be read, on the self-dual side, through the corresponding self-dual Ricci-type block.

Let

$$F^{(a)} \in \Omega^2(U), \quad a = 1, \dots, N, \quad (4)$$

be a finite family of real 2-forms, and let  $h_{ab}$  be a real symmetric positive definite matrix on the sector labels. The corresponding traceless symmetric tensor is taken in the form

$$Y_{\mu\nu} = h_{ab} \left( F_{\mu\alpha}^{(a)} F^{(b)\alpha}{}_{\nu} - \frac{1}{4} \eta_{\mu\nu} F_{\alpha\beta}^{(a)} F^{(b)\alpha\beta} \right). \quad (5)$$

The standard self-dual decomposition of two-forms in four dimensions is then used [13,14]. Let

$$\Phi^{(a)} := (F^{(a)})_+ = \frac{1}{2} (F^{(a)} - i * F^{(a)}) \in \Gamma(\Lambda_+^2(\eta)|_U) \quad (6)$$

denote the complex self-dual part of  $F^{(a)}$ .

**Proposition 1.** *The tensor (5) may be written in the form*

$$Y_{\mu\nu} = 2 h_{ab} \Phi_{\mu\alpha}^{(a)} \overline{\Phi_{\nu}^{(b)\alpha}}. \quad (7)$$

**Proof.** The standard decomposition of a real two-form into self-dual and anti-self-dual parts is applied. In four dimensions, the Maxwell-type traceless symmetric tensor depends only on the Hermitian square of the self-dual part, equivalently on the corresponding rank-two spinor bilinear [13,14]. Applying this identity to each  $F^{(a)}$  and contracting with the real symmetric matrix  $h_{ab}$  gives (7).  $\square$

Let  $h_+$  denote a fixed positive Hermitian metric on  $\Lambda_+^2(\eta)$ , taken as part of the background chiral data associated with the Lorentzian geometry [13,14]. All adjoints, positivity conditions, and orthonormal eigenbases below are taken with respect to  $h_+$ . No later construction depends on a further choice of Hermitian structure on  $\Lambda_+^2(\eta)$ .

It is convenient to introduce the polarized Maxwell-type bilinear form

$$T(F, G)_{\mu\nu} := F_{\mu\alpha} G_{\nu}{}^{\alpha} - \frac{1}{4} \eta_{\mu\nu} F_{\alpha\beta} G^{\alpha\beta}, \quad T(F)_{\mu\nu} := T(F, F)_{\mu\nu}, \quad (8)$$

so that (5) may be written as  $Y_{\mu\nu} = h_{ab} T(F^{(a)}, F^{(b)})_{\mu\nu}$ . The self-dual visible block is encoded by the Hermitian endomorphism  $Q_F$  defined by

$$h_+(Q_F \alpha, \beta) = h_{ab} h_+(\alpha, \Phi^{(a)}) \overline{h_+(\beta, \Phi^{(b)})}, \quad \alpha, \beta \in \Gamma(\Lambda_+^2(\eta)|_U). \quad (9)$$

Hermiticity and positive semidefiniteness follow directly from (9). Since  $\Lambda_+^2(\eta)$  has complex rank 3, one also has  $\text{rank } Q_F \leq 3$ .

A canonical linear map

$$\mathcal{R} : \{A \in \text{End}(\Lambda_+^2(\eta)|_U) : A^\dagger = A\} \rightarrow \text{Sym}_0^2 T^*U \quad (10)$$

is used to pass from Hermitian self-dual data to visible tensor data. Its spinorial meaning is standard [13,14]; a local formula is recorded in Appendix A.1. With this notation, (7) is read as

$$Y = \mathcal{R}(Q_F). \quad (11)$$

The Hermitian endomorphism  $Q_F$  is therefore the self-dual visible Ricci-type block associated with the curvature-type lift of  $Y$ .

By (1), one also has

$$\hat{k}_{\mu\nu} = \frac{k}{4p_\Lambda} \mathcal{R}(Q_F)_{\mu\nu}. \quad (12)$$

**Remark 1.** *The gauge-sector data therefore determine not only the visible tensor  $Y_{\mu\nu}$  but also the traceless part of the Alena metric variable  $k_{\mu\nu}$  and the corresponding visible curvature-type block. The primary geometric output of the gauge-sector input is the positive Hermitian endomorphism  $Q_F$  on  $\Lambda_+^2(\eta)$ .*

## 2.2. Geometry Selection

The geometric class used in the sequel is now introduced. The guiding principle is that the Hermitian self-dual endomorphism produced by a regular gauge-sector representation should itself be treated as primary visible geometric data. The corresponding regularity condition is spectral.

Let

$$Q \in \Gamma(\text{End}(\Lambda_+^2(\eta)|_U)), \quad Q^\dagger = Q. \quad (13)$$

Its spectral invariants are written as

$$I_1 := \text{tr } Q, \quad I_2 := \frac{1}{2}((\text{tr } Q)^2 - \text{tr}(Q^2)), \quad I_3 := \det Q. \quad (14)$$

The discriminant of the characteristic polynomial of  $Q$  is denoted by

$$\Delta(Q) = I_1^2 I_2^2 - 4I_2^3 - 4I_1^3 I_3 - 27I_3^2 + 18I_1 I_2 I_3. \quad (15)$$

**Definition 1.** *The Hermitian endomorphism  $Q$  will be called regular on  $U$  if*

$$Q \geq 0, \quad \det Q > 0, \quad \Delta(Q) > 0 \quad (16)$$

*holds on  $U$ .*

**Lemma 1.** *The regularity condition (16) is equivalent to the statement that  $Q$  has three distinct positive eigenvalues*

$$\lambda_1 > \lambda_2 > \lambda_3 > 0 \quad (17)$$

*at every point of  $U$ .*

**Proof.** Positivity and nonvanishing determinant imply positivity and full rank. Positivity of the discriminant is equivalent to the absence of repeated roots in the characteristic polynomial. Since  $Q$  is Hermitian, all roots are real. Hence (17) follows.  $\square$

**Definition 2.** *A regular Alena-Urbantke geometry on  $U$  is a pair  $(Q, k)$  such that*

- $Q \in \Gamma(\{A \in \text{End}(\Lambda_+^2(\eta)|_U) : A^\dagger = A\})$  is regular,
- the visible tensor is defined by

$$Y := \mathcal{R}(Q), \quad (18)$$

- the Lorentzian symmetric tensor  $k_{\mu\nu}$  satisfies

$$\hat{k}_{\mu\nu} = \frac{k}{4p_\Lambda} \mathcal{R}(Q)_{\mu\nu}. \quad (19)$$

**Proposition 2.** Let  $(Q, k)$  satisfy (19). Define the mixed tensor

$$M^\mu{}_\nu := \delta^\mu{}_\nu + \frac{1}{p_\Lambda} \eta^{\mu\alpha} \mathcal{R}(Q)_{\alpha\nu}. \quad (20)$$

Then (19) is equivalent to

$$k_{\mu\nu} = \frac{k}{4} \eta_{\mu\alpha} M^\alpha{}_\nu. \quad (21)$$

Consequently,  $k_{\mu\nu}$  is nondegenerate if and only if  $\det M \neq 0$ . In that case,

$$k^{\mu\nu} = \frac{4}{k} (M^{-1})^\mu{}_\alpha \eta^{\alpha\nu}, \quad (22)$$

and therefore

$$k^{\mu\nu} \eta_{\mu\nu} = \frac{4}{k} \operatorname{tr}(M^{-1}), \quad \det(\eta^{\mu\alpha} k_{\alpha\nu}) = \left(\frac{k}{4}\right)^4 \det(M). \quad (23)$$

The Lorentzian condition on  $k_{\mu\nu}$  is thus an additional restriction on the tensor (20). It is not implied by regularity of  $Q$  alone.

**Proof.** By adding  $\frac{k}{4} \eta_{\mu\nu}$  to both sides of (19), (21) is obtained. Since  $\eta$  is nondegenerate, the tensor  $k_{\mu\nu}$  is nondegenerate if and only if the mixed tensor  $M^\mu{}_\nu$  is invertible. Formula (22) follows from (21) and  $\eta^{\mu\alpha} \eta_{\alpha\nu} = \delta^\mu{}_\nu$ . The identities (23) are immediate.  $\square$

**Remark 2.** Once (21) is nondegenerate, the identities  $k^{\mu\alpha} k_{\alpha\nu} = \delta^\mu{}_\nu$  and  $k^{\mu\nu} k_{\mu\nu} = 4$  are automatic. The nontrivial restriction is the requirement that (21) be Lorentzian.

**Remark 3.** This definition is primary. It is not formulated through fluctuations, field equations, or external gauge groups. The regular self-dual Hermitian data are taken as fundamental, and the visible tensor  $Y$ , the traceless metric deformation, and the visible curvature-type block are induced from them.

Let

$$Q = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 \quad (24)$$

be the spectral decomposition of a regular  $Q$ , with  $P_i$  the rank-one spectral projectors corresponding to (17). Define

$$L := \operatorname{Im} P_1, \quad E := \operatorname{Im}(P_1 + P_2). \quad (25)$$

Then a canonical spectral flag

$$L \subset E \subset \Lambda_+^2(\eta)|_U, \quad \dim_{\mathbb{C}} L = 1, \quad \dim_{\mathbb{C}} E = 2 \quad (26)$$

is obtained. The ordering is canonical because the eigenvalues are distinct.

The purpose of this section is to clarify the geometric meaning of regularity. A regular Hermitian endomorphism on the self-dual bundle determines a regular triple of self-dual two-forms, and hence a natural Urbantke-type chiral geometry on the carrier. The role of that geometry is, however, that of a carrier geometry. It is attached to the visible self-dual block and is not identified directly with the Alena metric variable  $k_{\mu\nu}$ .

Let  $e_i$  be a local  $h_+$ -orthonormal eigenbasis corresponding to the ordered eigenvalues (17). Define

$$\Sigma^i := \sqrt{\lambda_i} e_i, \quad i = 1, 2, 3. \quad (27)$$

Since the eigenvalues are positive and the eigenvectors form a basis, the triple  $(\Sigma^1, \Sigma^2, \Sigma^3)$  is pointwise linearly independent in  $\Lambda_+^2(\eta)$ . A regular Urbantke-type chiral geometry is therefore determined on the self-dual carrier [1,21,22].

The role of this construction in the present setting is limited and should be stated precisely.

**Proposition 3.** *The Urbantke conformal structure reconstructed from the triple (27) is the conformal structure naturally associated with the chiral carrier determined by  $\Lambda_+^2(\eta)$ . In general it is not the metric variable  $k_{\mu\nu}$  determined by (19).*

**Proof.** The forms  $\Sigma^i$  are, by construction, sections of the background self-dual bundle  $\Lambda_+^2(\eta)$ . The Urbantke construction therefore reconstructs the conformal geometry attached to that self-dual carrier [1,21,22]. The metric variable  $k_{\mu\nu}$  is instead determined by the Alena relation through  $\mathcal{R}(Q)$ . The latter depends on the full Hermitian endomorphism  $Q$ , including its spectral weights, and is not fixed by the self-dual span alone. Hence the Urbantke conformal structure should be read as the geometry of the chiral carrier and not as a direct reconstruction of  $k_{\mu\nu}$ .  $\square$

**Remark 4.** *The role of the Urbantke construction is therefore structural rather than identificatory. It shows that regularity of  $Q$  carries genuine chiral geometry and is not merely a spectral algebraic condition.*

**Definition 3.** *A regular Hermitian endomorphism  $Q$  will be called Urbantke-compatible if the Urbantke conformal class determined by the triple (27) agrees with the conformal class of a metric representative extracted from the Alena-side relation (19).*

**Proposition 4.** *No regular Hermitian endomorphism  $Q$  is Urbantke-compatible in the sense of Definition 3.*

**Proof.** By construction, the triple (27) consists of sections of the fixed background self-dual bundle  $\Lambda_+^2(\eta)$ . Its Urbantke conformal structure is therefore the conformal structure attached to that carrier, namely the background conformal class  $[\eta]$ ; cf. Proposition 3. If  $Q$  were Urbantke-compatible in the sense of Definition 3, the conformal class extracted from (19) would have to agree with  $[\eta]$ . Hence the corresponding metric representative would be conformal to  $\eta$ , and its traceless part with respect to  $\eta$  would vanish:  $\hat{k}_{\mu\nu} = 0$ . By (19), this implies  $\mathcal{R}(Q) = 0$ . By Appendix A.1, the map  $\mathcal{R}$  is a real vector bundle isomorphism. One then has  $Q = 0$ , which is incompatible with regularity.  $\square$

**Remark 5.** *Proposition 4 shows that the carrier geometry selected by the regular self-dual triple and the Alena metric variable remain strictly distinct on the regular locus. An explicit counterexample is therefore no longer logically required, although a computed nonconstant example may still be useful for illustration.*

### 2.3. Local Equivalence with Regular Gauge-Sector Representations

The central result of the paper is established in this section. It is shown that regular gauge-sector representations and regular Alena-Urbantke geometries are locally equivalent. In the present formulation, the equivalence is read at the level of the visible self-dual Ricci-type block of the curvature-type lift.

Let  $Y$  admit a local gauge-sector representation of the form (5). If the associated Hermitian endomorphism  $Q_F$  is regular, then (11) and (12) show that  $(Q_F, k)$  defines a regular Alena-Urbantke geometry.

**Theorem 1.** Let  $(Q, k)$  be a regular Alena-Urbantke geometry on  $U$ . Then, locally on  $U$ , there exist three real two-forms  $F_1, F_2, F_3$  such that, for

$$\Phi_i := (F_i)_+, \quad (28)$$

the endomorphism  $Q$  admits the rank-one decomposition

$$Q = \sum_{i=1}^3 \Phi_i \otimes \overline{\Phi}_i, \quad (29)$$

where  $\Phi_i \otimes \overline{\Phi}_i$  denotes the Hermitian rank-one endomorphism defined by

$$h_+((\Phi_i \otimes \overline{\Phi}_i)\alpha, \beta) = h_+(\alpha, \Phi_i) \overline{h_+(\beta, \Phi_i)}. \quad (30)$$

Moreover,

$$Y = \sum_{i=1}^3 T(F_i), \quad (31)$$

where  $T(F_i) = T(F_i, F_i)$  with  $T(F, G)$  defined by (8). The reconstruction is local and depends on the local choice of eigenvectors of  $Q$ ; only  $Q$  and the resulting tensor  $Y$  are canonical.

**Proof.** Let

$$Q = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 \quad (32)$$

be the spectral decomposition of  $Q$ . Choose local  $h_+$ -orthonormal eigenvectors  $e_i$  such that  $P_i = e_i \otimes \bar{e}_i$ . Define

$$\Phi_i := \sqrt{\lambda_i} e_i, \quad F_i := \Phi_i + \overline{\Phi}_i. \quad (33)$$

Then (29) follows directly from (32) and (30). The choice of the local eigenvectors  $e_i$  is not canonical, since each  $e_i$  may be multiplied by a local phase, but the resulting endomorphism  $Q$  and tensor  $Y$  are unchanged. By linearity of  $\mathcal{R}$  and (7), one obtains

$$Y = \mathcal{R}(Q) = \sum_{i=1}^3 \mathcal{R}(\Phi_i \otimes \overline{\Phi}_i) = \sum_{i=1}^3 T(F_i), \quad (34)$$

which gives (31).  $\square$

**Theorem 2.** Let  $U \subset M$  be an open set. The following are locally equivalent:

- (i)  $Y$  admits a regular gauge-sector representation, i.e. a representation of the form (5) whose associated Hermitian endomorphism  $Q_F$  is regular;
- (ii) there exists a regular Alena-Urbantke geometry  $(Q, k)$  on  $U$  such that

$$Y = \mathcal{R}(Q), \quad \hat{k} = \frac{k}{4p_\Lambda} \mathcal{R}(Q). \quad (35)$$

**Proof.** The implication (i) $\Rightarrow$ (ii) is the observation preceding Theorem 1. The implication (ii) $\Rightarrow$ (i) is Theorem 1.  $\square$

**Remark 6.** Theorem 2 is the principal result of the paper. It identifies regular Alena-Urbantke geometry as the local geometric form of the regular gauge-sector problem attached to the Alena transition tensor and, at the same time, as the regular visible self-dual reduction of the curvature-type lift.

The regular spectral flag determines natural carrier spaces. Several explicit local models may also be written down. The first one isolates the diagonal non-rotating sector, while the next examples

record minimal regular rotating sectors and the first stationary-axisymmetric ansatz within the same Hermitian framework.

**Example 1.** Let  $(U, \eta)$  be a local Minkowski chart, and let  $(e_1, e_2, e_3)$  be a local  $h_+$ -orthonormal frame of  $\Lambda_+^2(\eta)|_U$ . Fix constants

$$\lambda_1 > \lambda_2 > \lambda_3 > 0. \quad (36)$$

Define

$$\Phi_i := \sqrt{\lambda_i} e_i, \quad F_i := \Phi_i + \overline{\Phi}_i, \quad i = 1, 2, 3. \quad (37)$$

Then

$$Q = \sum_{i=1}^3 \Phi_i \otimes \overline{\Phi}_i = \sum_{i=1}^3 \lambda_i P_i, \quad (38)$$

with  $P_i = e_i \otimes \bar{e}_i$ . Hence  $Q$  is positive, has full rank, and has simple spectrum, so it is regular by Definition 1. The associated visible tensor is therefore given by

$$Y = \sum_{i=1}^3 T(F_i), \quad (39)$$

by Theorem 1, and the corresponding traceless metric deformation is determined by (19). In this way, a regular gauge-sector realization is obtained in explicit local form.

**Corollary 1.** In the setting of Example 1, work in the standard local self-dual frame of Appendix A.1, in which  $Q = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$  with (36). Then the metric variable determined by (19) takes the local diagonal form

$$k_{\mu\nu} = \frac{k}{4} \text{diag}\left(-1 + \frac{\lambda_1 + \lambda_2 + \lambda_3}{p_\Lambda}, 1 + \frac{-\lambda_1 + \lambda_2 + \lambda_3}{p_\Lambda}, 1 + \frac{\lambda_1 - \lambda_2 + \lambda_3}{p_\Lambda}, 1 + \frac{\lambda_1 + \lambda_2 - \lambda_3}{p_\Lambda}\right). \quad (40)$$

If  $p_\Lambda > 0$ , the Lorentzian branch connected to the background metric is characterized by

$$p_\Lambda > \lambda_1 + \lambda_2 + \lambda_3 = \text{tr } Q. \quad (41)$$

If  $\lambda_1 > \lambda_2 + \lambda_3$ , a second Lorentzian branch is present and is characterized by

$$0 < p_\Lambda < \lambda_1 - \lambda_2 - \lambda_3. \quad (42)$$

At

$$p_\Lambda = \lambda_1 + \lambda_2 + \lambda_3 \quad \text{or} \quad p_\Lambda = \lambda_1 - \lambda_2 - \lambda_3 > 0, \quad (43)$$

the metric variable is degenerate.

**Proof.** The local diagonal expression follows from (19) and the local formula for  $\mathcal{R}$  in Appendix A.1. For  $p_\Lambda > 0$ , the last two spatial entries in (40) are always positive. The sign changes are therefore controlled by the first entry and, when  $\lambda_1 > \lambda_2 + \lambda_3$ , by the second one. The stated Lorentzian branches and degeneration values follow.  $\square$

**Remark 7.** The boundaries (43) concern the induced metric variable and are distinct from the carrier-degeneration loci (56). In particular,  $Q$  may remain regular while the tensor  $k_{\mu\nu}$  becomes degenerate or ceases to be Lorentzian.

**Remark 8.** The model of Example 1 also makes the degeneration loci explicit. If  $\lambda_3 \rightarrow 0^+$ , then  $\det Q \rightarrow 0$  and full rank is lost. If  $\lambda_1 \rightarrow \lambda_2$  with all  $\lambda_i$  remaining positive, then  $\Delta(Q) \rightarrow 0$  and the ordered spectral resolution ceases to be simple. The loci (56) may therefore be read as concrete degenerations of the regular visible carrier geometry.

**Remark 9.** Example 1 lies in the real-symmetric subcone of  $\{A \in \text{End}(\Lambda_+^2(\eta)|_U) : A^\dagger = A\}$ . In the local formula (A109), the real off-diagonal part of  $Q$  contributes to the spatial off-diagonal entries of  $Y$ , whereas the imaginary off-diagonal part contributes to the mixed time-space sector. A rotating visible sector is therefore already obtained at the level of a constant Hermitian endomorphism with nonzero imaginary off-diagonal part.

**Example 2.** Let  $(U, \eta)$  be a local Minkowski chart, and let  $(e_1, e_2, e_3)$  be a local  $h_+$ -orthonormal frame of  $\Lambda_+^2(\eta)|_U$ . Fix constants

$$\lambda_1, \lambda_2, \lambda_3 > 0, \quad \omega \in \mathbb{R}, \quad (44)$$

and define the Hermitian endomorphism

$$Q_{\text{rot}} := \begin{pmatrix} \lambda_1 & i\omega & 0 \\ -i\omega & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \quad (45)$$

Then

$$\text{tr } Q_{\text{rot}} = \lambda_1 + \lambda_2 + \lambda_3, \quad \det Q_{\text{rot}} = \lambda_3(\lambda_1\lambda_2 - \omega^2), \quad (46)$$

and the eigenvalues are

$$\mu_{\pm} = \frac{\lambda_1 + \lambda_2}{2} \pm \frac{1}{2} \sqrt{(\lambda_1 - \lambda_2)^2 + 4\omega^2}, \quad \mu_3 = \lambda_3. \quad (47)$$

Hence positivity and full rank hold whenever

$$\lambda_3 > 0, \quad \lambda_1\lambda_2 > \omega^2, \quad (48)$$

and regularity in the sense of Definition 1 is obtained once, in addition,  $\Delta(Q_{\text{rot}}) > 0$ . The associated visible tensor is again defined by

$$Y = \mathcal{R}(Q_{\text{rot}}). \quad (49)$$

In contrast with Example 1, the mixed time-space sector is now nontrivial: by (A109), the nonzero imaginary off-diagonal entry of  $Q_{\text{rot}}$  produces a nonzero mixed component of  $Y$ , proportional to  $\omega$ . The metric variable determined by (19) is therefore generally non-diagonal. A minimal regular rotating model is thus obtained.

**Remark 10.** The model of Example 2 differs from Example 1 only by the presence of an imaginary off-diagonal sector. No change in the definition of regular Alena-Urbantke geometry is required. Rotation is already encoded at the level of a regular Hermitian visible block. The example may therefore be used as a local carrier-level prototype for later comparison with stationary rotating metrics, without identifying it with any particular spacetime solution.

A further local model may also be recorded, in which the rotating visible block is allowed to vary over a stationary-axisymmetric chart. Its role is to pass from the constant rotating sector to a minimal  $(r, \theta)$ -dependent rotating ansatz.

**Example 3.** Let  $(U, \eta)$  be a local Minkowski chart, and let  $(e^0, e^1, e^2, e^3)$  be a local orthonormal coframe adapted to a stationary-axisymmetric splitting, with  $e^0$  timelike and  $e^3$  aligned with the axial direction. Let  $(\Sigma_1, \Sigma_2, \Sigma_3)$  be the associated standard self-dual frame of Appendix A.1.

Consider the Hermitian endomorphism

$$Q_{\text{axi}}(r, \theta) = \begin{pmatrix} \lambda_1(r, \theta) & i\omega(r, \theta) & 0 \\ -i\omega(r, \theta) & \lambda_2(r, \theta) & 0 \\ 0 & 0 & \lambda_3(r, \theta) \end{pmatrix}, \quad (50)$$

where  $\lambda_1, \lambda_2, \lambda_3, \omega$  are real functions of  $(r, \theta)$ . If

$$\lambda_i(r, \theta) > 0, \quad \lambda_1(r, \theta)\lambda_2(r, \theta) - \omega(r, \theta)^2 > 0, \quad \Delta(Q_{\text{axi}}) > 0, \quad (51)$$

then  $Q_{\text{axi}}$  is regular on the locus under consideration. In particular, the visible block remains positive, full rank, and of simple spectrum.

**Proposition 5.** In the standard local self-dual frame of Appendix A.1, let

$$Q = \begin{pmatrix} q_{11} & a_{12} + ib_{12} & a_{13} + ib_{13} \\ a_{12} - ib_{12} & q_{22} & a_{23} + ib_{23} \\ a_{13} - ib_{13} & a_{23} - ib_{23} & q_{33} \end{pmatrix}, \quad (52)$$

with  $q_{11}, q_{22}, q_{33}, a_{12}, a_{13}, a_{23}, b_{12}, b_{13}, b_{23} \in \mathbb{R}$ . Then the visible tensor  $Y = \mathcal{R}(Q)$  is given by

$$Y = \begin{pmatrix} q_{11} + q_{22} + q_{33} & -2b_{23} & 2b_{13} & -2b_{12} \\ -2b_{23} & -q_{11} + q_{22} + q_{33} & -2a_{12} & -2a_{13} \\ 2b_{13} & -2a_{12} & q_{11} - q_{22} + q_{33} & -2a_{23} \\ -2b_{12} & -2a_{13} & -2a_{23} & q_{11} + q_{22} - q_{33} \end{pmatrix}. \quad (53)$$

In particular, for the ansatz (50),

$$Y_{\text{axi}} = \begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 & 0 & 0 & -2\omega \\ 0 & -\lambda_1 + \lambda_2 + \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_1 - \lambda_2 + \lambda_3 & 0 \\ -2\omega & 0 & 0 & \lambda_1 + \lambda_2 - \lambda_3 \end{pmatrix}. \quad (54)$$

Hence the corresponding Alena-side traceless metric deformation satisfies

$$\hat{k}_{03} = -\frac{k}{2p_\Lambda} \omega, \quad (55)$$

and is therefore nonzero whenever  $\omega \neq 0$ .

**Proof.** Formula (53) is obtained by direct evaluation of (A109) in the standard self-dual frame. The specialized form (54) follows by substitution of (50). Equation (55) is then immediate from (19).  $\square$

**Remark 11.** The ansatz (50) is not a Kerr reconstruction. Its role is only to isolate the minimal regular Hermitian mechanism by which a mixed time-space visible sector is produced. In an adapted stationary-axisymmetric frame, the quantity  $\omega(r, \theta)$  generates a nonvanishing 0-3 component of  $Y$  and hence of  $\hat{k}$ . This is of the same kinematical type as the rotating  $t$ - $\phi$  sector in stationary axisymmetric metrics.

**Remark 12.** The example shows that the purely real symmetric sector of  $Q$  is too restrictive for rotating visible deformations. Once a nontrivial imaginary off-diagonal Hermitian part is allowed, the regular visible block already admits a stationary-axisymmetric mixed sector without any change in the basic definitions of regular Alena-Urbantke geometry.

The curvature-type lift introduced in Section 2.1 contains more information than the visible tensor alone. In the self-dual decomposition (3), the visible part is encoded by the self-dual Ricci-type block  $Q$ . The remaining curvature data are carried by the diagonal Weyl and scalar blocks [12–14]. They are not determined by  $Y$ , by the Alena relation, or by the regularity condition on  $Q$ .

The present paper is restricted to the visible regular block and to the geometry induced by it. In this sense, the hidden curvature data are those parts of the curvature-type completion that remain free after the visible self-dual block has been fixed.

At the level of the visible carrier block, regularity fails precisely when one of the regularity conditions fails, namely on the loci

$$\det Q = 0 \quad \text{or} \quad \Delta(Q) = 0. \quad (56)$$

If  $\det Q = 0$ , full rank is lost. Equivalently, the visible Ricci-type block ceases to have rank three. If  $\Delta(Q) = 0$ , the ordered simple spectrum is lost. Equivalently, the canonical spectral resolution of the visible block breaks down.

**Remark 13.** *The reduced sectors associated with such loci are best viewed as degenerations of the regular visible carrier geometry. Their possible geometric interpretation at the level of the carrier flag is recorded in Appendix A.4.*

### 3. Classification, Structural Reduction, and Geometric Dynamics

The regular sector determined in Sections 2.2 and 2.3 admits a more intrinsic organization. On the one hand, a regular Hermitian endomorphism  $Q$  determines an ordered spectral flag in the self-dual bundle and hence a reduction of the carrier geometry. On the other hand, smooth families of regular endomorphisms admit a natural decomposition into spectral and flag-moving parts, as in Appendix A.3. The purpose of the present section is to record these structural consequences, to isolate a first local classification statement, and to indicate natural dynamical and global questions in a form adapted to the regular visible sector.

Only the regular visible block is used throughout. No Yang-Mills field equation is imposed, and no curvature completion beyond the visible self-dual Ricci-type block is fixed.

#### 3.1. Local Classification and Structural Reduction of the Regular Visible Sector

The regularity condition of Definition 1 implies that the eigenvalues of  $Q$  are positive and simple, hence ordered as in (17). It follows that the spectral projectors  $P_i$  and the associated eigenline subbundles are locally smooth. The corresponding spectral flag was already recorded in (26). The point to be added here is that, locally, the regular visible sector is exhausted by the ordered spectrum together with that flag.

Let  $U \subset M$  be an open set on which  $\Lambda_+^2(\eta)|_U$  is equipped with the fixed Hermitian metric  $h_+$ . Two regular Hermitian endomorphisms

$$Q, Q' \in \Gamma(\{A \in \text{End}(\Lambda_+^2(\eta)|_U) : A^\dagger = A\}) \quad (57)$$

will be called *locally carrier-equivalent* if there exists a local unitary bundle automorphism

$$U \in \Gamma(\text{Aut}(\Lambda_+^2(\eta)|_U, h_+)) \quad (58)$$

such that

$$Q' = UQU^{-1}. \quad (59)$$

This relation preserves the ordered spectrum, the positivity properties, and the induced spectral flag.

**Proposition 6.** *Let  $Q$  be regular on  $U$ . Then, locally on  $U$ , there exists a unitary frame of  $\Lambda_+^2(\eta)|_U$  in which*

$$Q = U \text{diag}(\lambda_1, \lambda_2, \lambda_3)U^{-1}, \quad (60)$$

with  $\lambda_1 > \lambda_2 > \lambda_3 > 0$  as in (17). In particular, the local carrier-equivalence class of  $Q$  is determined by the ordered eigenvalue triple together with the induced full spectral flag.

**Proof.** By Lemma 1, the spectrum of  $Q$  is pointwise simple and positive. Local smoothness of the spectral projectors for simple Hermitian eigenvalues is standard. A local  $h_+$ -orthonormal eigenframe

may therefore be chosen, and in that frame  $Q$  is diagonal with entries  $\lambda_1, \lambda_2, \lambda_3$ . The flag is then exactly the flag determined by the ordered eigenspaces. The converse local reconstruction is immediate from the same diagonal form.  $\square$

**Remark 14.** *Proposition 6 is only a local statement. No global splitting of  $\Lambda_+^2(\eta)$  is implied at this stage. The global problem is deferred to Subsection 3.3.*

The representation map  $\mathcal{R}$  then shows that the visible tensor

$$Y = \mathcal{R}(Q) \quad (61)$$

depends on both parts of the data: the ordered spectral functions and the local position of the corresponding flag inside  $\Lambda_+^2(\eta)$ . In particular, the visible tensor is not determined by the unordered eigenvalue set alone.

**Remark 15.** *The trace  $\text{tr } Q = \lambda_1 + \lambda_2 + \lambda_3$  plays the role of the total visible spectral intensity of the regular block. The channel weights (64) therefore separate the relative channel distribution from the overall spectral scale.*

The self-dual bundle  $\Lambda_+^2(\eta)$  is a complex rank-3 Hermitian vector bundle once  $h_+$  has been fixed. Its Hermitian structure group is therefore reduced to  $U(3)$ . A regular Hermitian endomorphism  $Q$  reduces that structure further. Indeed, by (24), a regular  $Q$  determines three pairwise orthogonal rank-one spectral projectors. Equivalently, it determines an ordered decomposition

$$\Lambda_+^2(\eta)|_U = L_1 \oplus L_2 \oplus L_3, \quad (62)$$

locally on  $U$ , where  $L_i := \text{Im } P_i$ . The flag (26) is then the partial form of the same ordered splitting. It follows that the regular visible sector may be viewed as a reduction from the Hermitian bundle  $\Lambda_+^2(\eta)$  to an ordered eigenline geometry.

This viewpoint may be stated more invariantly. Let

$$\text{Flag}(\Lambda_+^2(\eta)|_U) \quad (63)$$

denote the bundle of complete Hermitian flags in  $\Lambda_+^2(\eta)|_U$ . Then a regular  $Q$  defines a canonical local section of that bundle, namely the section whose value at each point is the ordered eigenflag of  $Q$ .

**Proposition 7.** *A regular Hermitian endomorphism  $Q$  on  $U$  determines canonically a local section of the full flag bundle of  $\Lambda_+^2(\eta)|_U$ . Conversely, a local Hermitian flag together with three smooth functions  $\lambda_1 > \lambda_2 > \lambda_3 > 0$  determines a unique regular Hermitian endomorphism of the form (24).*

**Proof.** The first statement follows from the existence and smoothness of the ordered spectral projectors in the regular case. For the second statement, the rank-one orthogonal projectors associated with the three flag lines are inserted into (24). Hermiticity, positivity, full rank, and simplicity of the spectrum are immediate from the assumptions on the  $\lambda_i$ .  $\square$

**Remark 16.** *The regular visible block is therefore not only a field of Hermitian endomorphisms but also a field of ordered internal carrier data. In this sense, the regularity assumption selects a flag geometry internal to the self-dual bundle.*

**Definition 4.** Let  $Q$  be regular on  $U$ , with ordered positive eigenvalues  $\lambda_1 > \lambda_2 > \lambda_3 > 0$ . The corresponding channel weights are defined by

$$w_i(Q) := \frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{\lambda_i}{\text{tr } Q}, \quad i = 1, 2, 3. \quad (64)$$

They satisfy  $w_i > 0$ ,  $w_1 + w_2 + w_3 = 1$ . The associated channel entropy is defined by

$$S(Q) := - \sum_{i=1}^3 w_i \log w_i. \quad (65)$$

**Remark 17.** The functions (64) are dimensionless spectral invariants of the regular visible block. They encode the relative distribution of the ordered visible intensities carried by the three spectral channels of  $Q$ , independently of the overall scale  $\text{tr } Q$ . In particular,  $w_1, w_2, w_3$  may be read as normalized carrier-channel fractions attached to the regular self-dual block.

**Remark 18.** The entropy (65) provides a scalar measure of spectral mixing in the regular visible sector. It is minimized when one channel dominates strongly and increases as the three channel weights become more evenly distributed. In this sense, it gives a canonical local indicator of how concentrated or how mixed the visible carrier content is.

**Remark 19.** At the level of local unitary frames, the ordered decomposition (62) reduces the carrier data from  $U(3)$  to the diagonal unitary subgroup  $U(1)^3$ , which preserves the three ordered eigenlines separately. If the ordering is forgotten and only the partial flag (26) is retained, the residual symmetry enlarges accordingly. In the present formulation, however, the ordered spectral splitting is primary.

The map  $\mathcal{R}$  introduced in (10) was used above as the passage from Hermitian self-dual data to visible symmetric tensor data. Its relation to the standard spinorial decomposition of the traceless Ricci part may be stated somewhat more explicitly.

By Appendix A.1, one has locally the standard identification

$$\Lambda_+^2(\eta) \cong \text{Sym}^2 S, \quad (66)$$

where  $S$  is the rank-two Weyl spinor bundle. A Hermitian endomorphism of  $\Lambda_+^2(\eta)$  is thus represented by a Hermitian endomorphism of  $\text{Sym}^2 S$ , as in (A107). On the other hand, the traceless Ricci sector is represented spinorially by a real symmetric rank-two object of mixed primed and unprimed type in the standard four-dimensional decomposition of curvature; see [13,14]. The map  $\mathcal{R}$  is precisely the bundle-level realization of this passage from Hermitian self-dual endomorphism data to real traceless symmetric rank-two tensor data.

In particular, the visible tensor

$$Y = \mathcal{R}(Q) \quad (67)$$

should be read as the real tensor representative of the Hermitian self-dual Ricci-type block encoded by  $Q$ . The relation (12) then shows that the traceless part of the Alena metric variable is induced from exactly that block. No additional visible degrees of freedom are inserted at this stage.

**Lemma 2.** The map  $\mathcal{R} : \{A \in \text{End}(\Lambda_+^2(\eta)|_U) : A^\dagger = A\} \rightarrow \text{Sym}_0^2 T^*U$  is a real vector bundle isomorphism.

**Proof.** By Appendix A.1, one has locally the standard identification  $\Lambda_+^2(\eta) \cong \text{Sym}^2 S$ , where  $S$  is the Weyl spinor bundle. A Hermitian endomorphism of  $\Lambda_+^2(\eta)$  is therefore identified with a Hermitian endomorphism of  $\text{Sym}^2 S$ , equivalently with a mixed spinor object of type  $Q_{AB A'B'}$  which is symmetric in  $AB$  and in  $A'B'$ . This is precisely the standard spinorial form of a real traceless symmetric rank-two

tensor in four dimensions; see [13,14]. The local formula (A109) is the corresponding tensor realization in a self-dual frame. Hence  $\mathcal{R}$  is the induced real-linear bundle isomorphism.  $\square$

**Remark 20.** *The relation of  $\mathcal{R}(Q)$  to the standard curvature decomposition may also be stated explicitly. After complexification and the splitting  $\Lambda^2 T^*U \otimes \mathbb{C} = \Lambda_+^2(\eta)|_U \oplus \Lambda_-^2(\eta)|_U$ , the curvature operator takes the usual block form in which the diagonal blocks carry the self-dual and anti-self-dual Weyl parts together with the scalar term, while the off-diagonal blocks carry the traceless Ricci data [12–14]. Under the spinorial identification used above, the Hermitian endomorphism  $Q$  is exactly the self-dual representative of that visible Ricci-type block, and  $\mathcal{R}(Q)$  is its realization as a real traceless symmetric rank-two tensor. In this sense, the map  $\mathcal{R}$  passes from the visible self-dual Ricci-type datum to the corresponding tensor representative, while the Weyl and scalar blocks remain hidden curvature data.*

### 3.2. Geometric Dynamics of Regular Hermitian Sectors

The perturbative decomposition recorded in Appendix A.3 suggests a natural geometric language for smooth families of regular visible blocks. Let

$$t \mapsto Q(t) \in \Gamma(\{A \in \text{End}(\Lambda_+^2(\eta)|_U) : A^\dagger = A\}) \quad (68)$$

be a smooth family which remains regular on a time interval. Then, after local smooth choice of spectral data, one has

$$\dot{Q} = \sum_{i=1}^3 \dot{\lambda}_i P_i + [A, Q], \quad A^\dagger = -A, \quad (69)$$

as in (A112).

The first term in (69) changes the ordered spectrum while preserving the instantaneous eigenspaces. The second term preserves the spectrum and moves the spectral flag. The two contributions may therefore be read as spectral and flag-moving parts of the regular visible dynamics. Through (18) and (19), they induce corresponding variations of the visible tensor and of the traceless metric deformation.

It is convenient to distinguish two model cases. A first natural possibility is to place the regular Hermitian sector under a variational evolution. Let  $\nabla^+$  denote the Hermitian connection induced on  $\Lambda_+^2(\eta)$  by the background Lorentzian geometry. Then a formal energy functional of the form

$$\mathcal{E}[Q] = \int_U (\alpha |\nabla^+ Q|^2 + V(I_1, I_2, I_3)) \text{vol}_\eta \quad (70)$$

may be considered, where  $\alpha > 0$  is constant and  $V$  is a smooth real function of the spectral invariants (14).

The associated Euler-Lagrange equation, or the corresponding gradient-type flow, would govern a dynamics in which both spectral and flag data are allowed to evolve. Since  $V$  depends only on the spectral invariants, it separates naturally the scalar part of the visible Hermitian data from the bundle-geometric part encoded by the eigenflag. The decomposition (69) is then compatible with the distinction between spectral deformation and flag motion.

No field-theoretic interpretation is imposed here. The functional (70) is only meant to indicate that the regular visible sector carries a natural geometric variational structure already at the level of the Hermitian self-dual block.

Let  $\langle \cdot, \cdot \rangle$  denote the fiberwise Hilbert-Schmidt pairing on  $\{A \in \text{End}(\Lambda_+^2(\eta)|_U) : A^\dagger = A\}$  induced by  $h_+$ , and let  $|\nabla^+ Q|^2 := \langle \nabla^+ Q, \nabla^+ Q \rangle$ . Variations are taken within the Hermitian sector and, for simplicity, with compact support in  $U$ . Then

$$\delta I_1 = \text{tr}(\delta Q), \quad \delta I_2 = (\text{tr} Q) \text{tr}(\delta Q) - \text{tr}(Q \delta Q), \quad \delta I_3 = \text{tr}(\text{adj}(Q) \delta Q), \quad (71)$$

where  $\text{adj}(Q)$  denotes the adjugate endomorphism. Equivalently,

$$\delta I_1 = \langle \text{id}, \delta Q \rangle, \quad \delta I_2 = \langle I_1 \text{id} - Q, \delta Q \rangle, \quad \delta I_3 = \langle \text{adj}(Q), \delta Q \rangle. \quad (72)$$

If

$$V_a := \frac{\partial V}{\partial I_a}, \quad a = 1, 2, 3, \quad (73)$$

then the first variation of (70) is

$$\delta \mathcal{E}[Q] = \int_U \langle -2\alpha (\nabla^+)^* \nabla^+ Q + V_1 \text{id} + V_2(I_1 \text{id} - Q) + V_3 \text{adj}(Q), \delta Q \rangle \text{vol}_\eta. \quad (74)$$

Here  $(\nabla^+)^*$  denotes the formal adjoint of  $\nabla^+$  with respect to the induced  $L^2$  pairing. The corresponding formal Euler-Lagrange equation is therefore

$$2\alpha (\nabla^+)^* \nabla^+ Q = V_1 \text{id} + V_2(I_1 \text{id} - Q) + V_3 \text{adj}(Q). \quad (75)$$

The associated negative  $L^2$ -gradient flow is

$$\partial_\tau Q = 2\alpha (\nabla^+)^* \nabla^+ Q - V_1 \text{id} - V_2(I_1 \text{id} - Q) - V_3 \text{adj}(Q). \quad (76)$$

**Remark 21.** The right-hand sides of (75) and (76) are expressed entirely in terms of the Hermitian self-dual block  $Q$  and its spectral invariants. In this sense, the variational dynamics closes on the regular visible sector itself.

A more detailed analytic treatment of (75)-(76), including boundary conditions and distinguished choices of the potential  $V(I_1, I_2, I_3)$ , is left for later work.

**Proposition 8.** The formal gradient flow (76) preserves the Hermitian sector. More precisely, if  $Q(\tau)$  is Hermitian at some time  $\tau_0$ , then the right-hand side of (76) is Hermitian at  $\tau_0$  as well.

**Proof.** If  $Q$  is Hermitian, then the spectral invariants  $I_1, I_2, I_3$  are real. The identity endomorphism  $\text{id}$  is Hermitian, and so are  $Q$  and  $\text{adj}(Q)$ . Since  $\nabla^+$  is the Hermitian connection induced by  $h_+$ , the rough Laplace-type term  $(\nabla^+)^* \nabla^+ Q$  is Hermitian whenever  $Q$  is Hermitian. Hence every term on the right-hand side of (76) is Hermitian.  $\square$

**Remark 22.** Proposition 8 shows that the variational dynamics is formally closed on the real bundle  $\{A \in \text{End}(\Lambda_+^2(\eta)|_U) : A^\dagger = A\}$ . The genuinely nontrivial question is therefore not Hermiticity but preservation of positivity, and hence of regularity.

Preservation of positivity, and hence of regularity, depends on further assumptions on the potential and on the analytic setting. This question is not pursued here.

A symbolic implementation of the finite-dimensional and isospectral test models has been carried out in Mathematica, and the corresponding notebook is provided as supplementary material.

The second model case is obtained by imposing the purely commutator form

$$\dot{Q} = [A, Q], \quad A^\dagger = -A. \quad (77)$$

This is the intrinsic flag-moving part of (69). The spectrum is then preserved, while the eigenspaces move unitarily.

**Proposition 9.** Let  $Q(t)$  satisfy (77) on a time interval and assume that  $Q(0)$  is regular. Then  $Q(t)$  remains regular for all times for which the solution exists. Moreover, the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  and the invariants  $I_1, I_2, I_3, \Delta(Q)$  are constant along the flow.

**Proof.** Equation (77) is integrated by a unitary propagator. More precisely, if  $U(t)$  solves

$$\dot{U} = AU, \quad U(0) = \text{id}, \quad (78)$$

then  $U(t)$  is unitary and

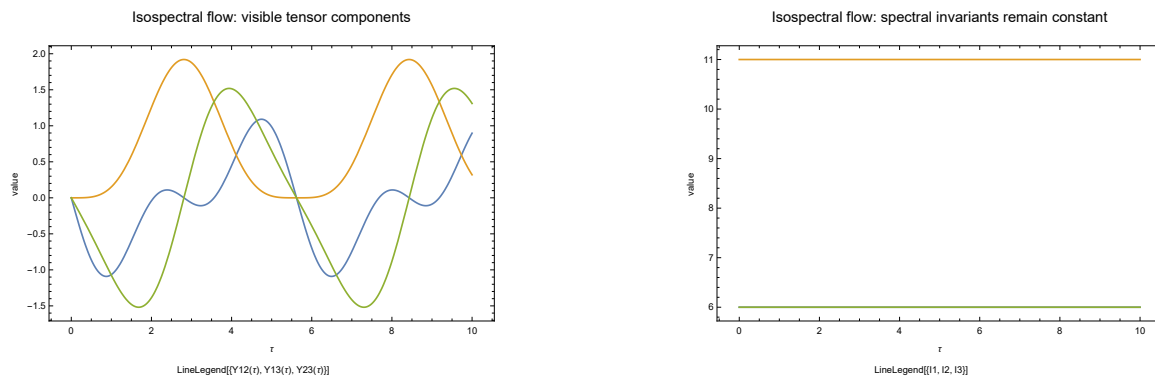
$$Q(t) = U(t)Q(0)U(t)^{-1}. \quad (79)$$

Hence  $Q(t)$  is unitarily conjugate to  $Q(0)$  for all  $t$ , so the spectrum is preserved. Since regularity is equivalent to positivity and simple positive spectrum by Lemma 1, regularity is preserved as well. Constancy of the spectral invariants and of the discriminant then follows immediately.  $\square$

**Remark 23.** Under the isospectral flow (77), the visible tensor evolves by

$$\dot{Y} = \mathcal{R}([A, Q]). \quad (80)$$

Thus the induced visible dynamics is entirely carried by the motion of the spectral flag, while the ordered spectral data remain fixed.



**Figure 1.** A finite-dimensional isospectral model for (77). Left: evolution of the visible tensor components  $Y_{12}(\tau)$ ,  $Y_{13}(\tau)$ , and  $Y_{23}(\tau)$ . Right: constancy of the spectral invariants  $I_1$ ,  $I_2$ , and  $I_3$  along the flow. The model illustrates that visible evolution is generated by flag motion at fixed ordered spectrum.

**Remark 24.** The solutions of (77) lie on the unitary conjugacy orbit of the initial datum  $Q(0)$ . The tangent space to that orbit at  $Q$  is therefore given by commutators  $[A, Q]$  with  $A^\dagger = -A$ . In this sense, the isospectral dynamics is the intrinsic motion along the carrier orbit determined by the fixed ordered spectrum.

**Remark 25.** The isospectral flow (77) may also be viewed as a canonical exploratory deformation of the regular visible sector. Since the ordered spectrum is preserved, the quantities  $\lambda_1, \lambda_2, \lambda_3, w_i = \frac{\lambda_i}{\text{tr} Q}$ ,  $I_1, I_2, I_3, \Delta(Q)$  remain fixed along the flow, while the spectral flag moves through the corresponding unitary orbit. The induced variation  $\dot{Y} = \mathcal{R}([A, Q])$  therefore isolates the visible effect of pure carrier motion at fixed spectral content.

**Remark 26.** This makes the isospectral family a natural probe of the regular visible geometry. For a fixed regular initial datum  $Q(0)$ , the orbit  $Q(\tau) = U(\tau)Q(0)U(\tau)^{-1}$ ,  $U(\tau) \in U(3)$ , may be used to compare visible tensors with identical ordered channel weights but different carrier flags. In this sense, isospectral deformations provide a distinguished testing family for separating spectral data from flag-dependent visible effects.

**Proposition 10.** For the nonconstant model (92), the visible tensor  $Y = \mathcal{R}(Q)$  takes the explicit form

$$Y = \begin{pmatrix} Q_{11} + Q_{22} + Q_{33} & 0 & 0 & 0 \\ 0 & -Q_{11} + Q_{22} + Q_{33} & -2Q_{12} & -2Q_{13} \\ 0 & -2Q_{12} & Q_{11} - Q_{22} + Q_{33} & -2Q_{23} \\ 0 & -2Q_{13} & -2Q_{23} & Q_{11} + Q_{22} - Q_{33} \end{pmatrix}, \quad (81)$$

where the coefficients  $Q_{ij}$  are given by (94)-(99). In particular,

$$Y_{00} = I_1 = \lambda_1 + \lambda_2 + \lambda_3 \quad (82)$$

is constant, and all time-space off-diagonal components vanish:

$$Y_{0i} = 0, \quad i = 1, 2, 3. \quad (83)$$

**Proof.** For the ansatz (89), the matrix  $Q$  is real symmetric. In a standard local self-dual frame on Minkowski space, the representation map  $\mathcal{R}$  sends a real symmetric Hermitian endomorphism

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{pmatrix} \quad (84)$$

to the tensor (81). Since  $Q$  is pointwise unitarily conjugate to  $\Lambda$ , its trace is constant and equal to  $\lambda_1 + \lambda_2 + \lambda_3$ , which gives (82). The vanishing of the mixed time-space components is immediate from (81).  $\square$

**Remark 27.** Proposition 10 shows that the nonconstant model separates two effects cleanly. The spectral invariants remain fixed, whereas the spatial off-diagonal entries of  $Y$  are generated entirely by the flag-moving terms  $Q_{12}, Q_{13}, Q_{23}$ . In this sense, the visible tensor records the carrier motion even though the ordered spectrum is unchanged.

The two-rotation model has been implemented symbolically. Selected plots are displayed below, and the full symbolic notebook is provided as supplementary material.

### 3.3. Dynamics, Degenerations, and Global Aspects of the Regular Visible Sector

Example 1 exhibits the regular sector in diagonal form, but the corresponding flag is fixed. A first nonconstant model is obtained by allowing the eigenframe to vary while the ordered spectrum remains fixed.

Let  $(U, \eta)$  be a local Minkowski chart, and let  $(e_1, e_2, e_3)$  be a local  $h_+$ -orthonormal frame of  $\Lambda_+^2(\eta)|_U$ . Fix constants  $\lambda_1 > \lambda_2 > \lambda_3 > 0$  and let

$$U : U \rightarrow U(3) \quad (85)$$

be a smooth unitary matrix-valued function. Define a new local orthonormal frame

$$\tilde{e}_i := U_i^j e_j \quad (86)$$

and set

$$Q = \sum_{i=1}^3 \lambda_i \tilde{e}_i \otimes \bar{\tilde{e}}_i. \quad (87)$$

Then  $Q$  is regular on  $U$ , has the same ordered eigenvalues as the constant model, and defines a generally nonconstant spectral flag. The associated visible tensor is

$$Y = \mathcal{R}(Q), \quad (88)$$

and the traceless metric deformation is again determined by (19).

This model is structurally different from Example 1: its spectral invariants are constant, but its carrier flag varies. It therefore isolates the flag-moving contribution without changing the spectral data.

A concrete choice of  $\mathcal{U}(x, y)$  may already be fixed at the level of the model. A convenient first ansatz is

$$\mathcal{U}(x, y) = R_{12}(\theta(x)) R_{23}(\phi(y)), \quad (89)$$

where

$$R_{12}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{23}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}. \quad (90)$$

With

$$\Lambda := \text{diag}(\lambda_1, \lambda_2, \lambda_3), \quad \lambda_1 > \lambda_2 > \lambda_3 > 0, \quad (91)$$

the corresponding regular Hermitian endomorphism

$$Q(x, y) = \mathcal{U}(x, y) \Lambda \mathcal{U}(x, y)^\dagger \quad (92)$$

takes the explicit form

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{pmatrix}, \quad (93)$$

where

$$Q_{11} = \lambda_1 \cos^2 \theta + \lambda_2 \sin^2 \theta \cos^2 \phi + \lambda_3 \sin^2 \theta \sin^2 \phi, \quad (94)$$

$$Q_{22} = \lambda_1 \sin^2 \theta + \lambda_2 \cos^2 \theta \cos^2 \phi + \lambda_3 \cos^2 \theta \sin^2 \phi, \quad (95)$$

$$Q_{33} = \lambda_2 \sin^2 \phi + \lambda_3 \cos^2 \phi, \quad (96)$$

$$Q_{12} = (\lambda_1 - \lambda_2 \cos^2 \phi - \lambda_3 \sin^2 \phi) \sin \theta \cos \theta, \quad (97)$$

$$Q_{13} = (\lambda_3 - \lambda_2) \sin \theta \sin \phi \cos \phi, \quad (98)$$

$$Q_{23} = (\lambda_2 - \lambda_3) \cos \theta \sin \phi \cos \phi. \quad (99)$$

In particular,  $Q$  is pointwise unitarily conjugate to  $\Lambda$ , so its ordered spectrum is constant and equal to  $(\lambda_1, \lambda_2, \lambda_3)$ , while the corresponding spectral flag varies whenever  $\theta$  and  $\phi$  are nonconstant. The simplest realization is obtained by taking

$$\theta(x) = ax, \quad \phi(y) = by, \quad (100)$$

with real constants  $a, b$ .

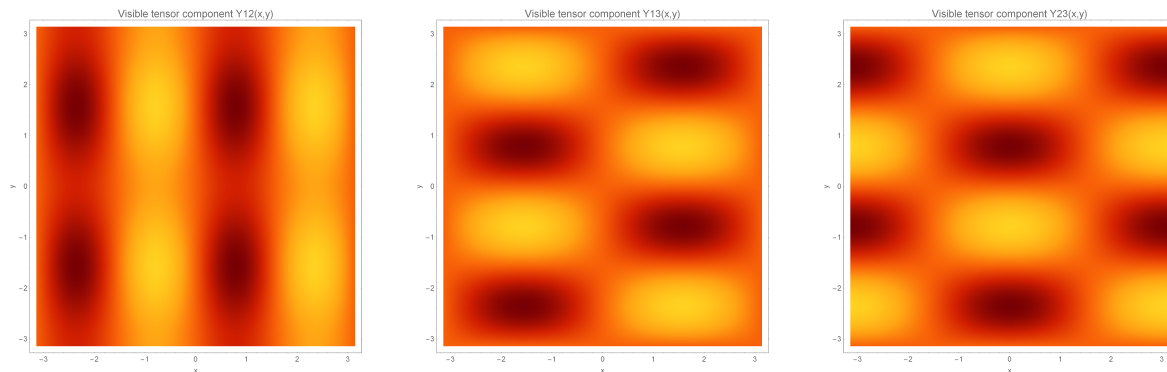
For the linear-angle choice (100), the visible tensor has been evaluated explicitly. The resulting off-diagonal spatial components are shown in Figure 2.

A second example from a curved background known in the self-dual literature is left for later work. At the present stage, the nonconstant Minkowski model already isolates the geometric point needed here, namely a regular sector with fixed ordered spectrum and nontrivial flag motion. The compatibility question between the Urbantke carrier geometry and the Alena-side metric variable has already been settled in Section 2.2. The present subsection is therefore restricted to dynamical, degenerative, and global aspects of the regular visible sector.

The regularity conditions (16) define an open condition in the Hermitian sector. This follows immediately from continuity of the eigenvalues, or equivalently from continuity of the invariants (14) and of the discriminant (15). The degeneration loci were already recorded in (56). The present point is that, under a smooth evolution, these loci describe the precise mechanisms by which the regular visible carrier can fail.

If  $\det Q \rightarrow 0$ , the visible block loses full rank and the rank-three carrier decomposition collapses. If  $\Delta(Q) \rightarrow 0$ , eigenvalue collision occurs and the canonical ordered spectral splitting ceases to be

simple. These two mechanisms are geometrically distinct: the first destroys the full carrier rank, while the second destroys the ordered flag structure while keeping positivity possible.



**Figure 2.** Density plots of the off-diagonal spatial components  $Y_{12}(x, y)$ ,  $Y_{13}(x, y)$ , and  $Y_{23}(x, y)$  for the two-rotation model (92) with (100), evaluated for  $\lambda_1 = 3$ ,  $\lambda_2 = 2$ ,  $\lambda_3 = 1$  and  $a = b = 1$ . The ordered spectrum remains fixed, while the visible tensor varies through the motion of the spectral flag.

For the isospectral dynamics (77), regularity is preserved by Proposition 9. For more general evolutions of the form (69), the same conclusion need not hold globally, but the openness of the regular locus implies that regularity is preserved at least for sufficiently short time so long as neither degeneration condition is reached.

**Lemma 3.** *Let  $t \mapsto Q(t) \in \Gamma(\{A \in \text{End}(\Lambda_+^2(\eta)|_U) : A^\dagger = A\})$  be a smooth family on an interval  $I$ , and let  $t_0 \in I$ . If  $Q(t_0)$  is regular, then there exists  $\varepsilon > 0$  such that  $Q(t)$  is regular for all  $t \in I$  with  $|t - t_0| < \varepsilon$ .*

**Proof.** Since  $Q(t)$  is Hermitian for all  $t$ , its ordered eigenvalues  $\lambda_1(t) \geq \lambda_2(t) \geq \lambda_3(t)$  depend continuously on  $t$ . If  $Q(t_0)$  is regular, then by Lemma 1 one has  $\lambda_1(t_0) > \lambda_2(t_0) > \lambda_3(t_0) > 0$ . By continuity, these strict inequalities remain valid for all  $t$  sufficiently close to  $t_0$ . Hence  $Q(t)$  continues to have simple positive spectrum on a possibly smaller interval about  $t_0$ . Another application of Lemma 1 gives regularity.  $\square$

**Remark 28.** *Lemma 3 shows that a smooth Hermitian evolution can leave the regular sector only by reaching one of the degeneration loci (56). In particular, the loss of regularity is necessarily detected by either loss of positivity or collision of eigenvalues.*

**Remark 29.** *A natural way to monitor possible loss of regularity along the variational flow (76) is to use the barrier functional*

$$\mathcal{B}(Q) := -\log \det Q - \log \Delta(Q), \quad (101)$$

*defined on the regular locus. Since  $\det Q > 0$  and  $\Delta(Q) > 0$  there, the function  $\mathcal{B}$  is smooth on the regular sector and diverges as  $Q$  approaches either degeneration locus in (56). Consequently, any a priori upper bound for  $\mathcal{B}(Q(\tau))$  along the flow prevents finite-time approach to the boundary of the regular sector.*

The preceding discussion has been local. On a general four-manifold, the existence of a global regular Hermitian endomorphism on  $\Lambda_+^2(\eta)$  is more restrictive. The reason is that a globally regular  $Q$  would produce globally defined simple spectral projectors and hence a global decomposition of the self-dual bundle.

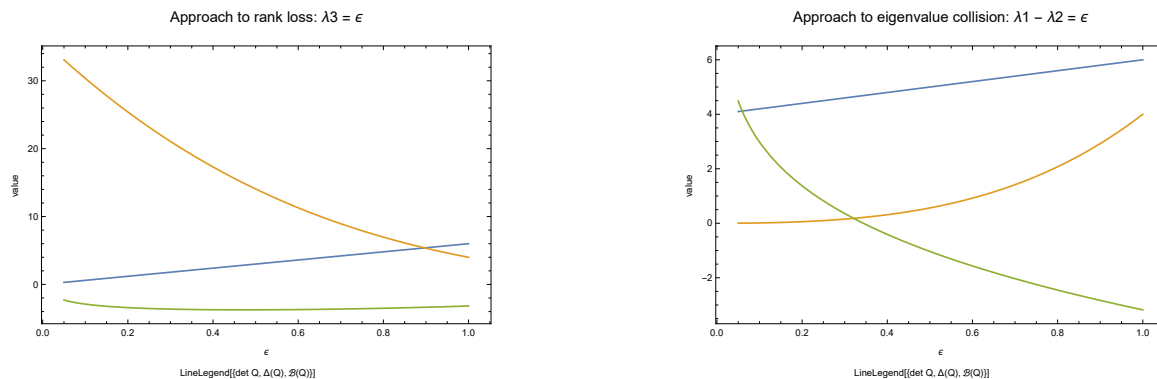
More precisely, let  $(M, \eta)$  be an oriented four-dimensional Lorentzian manifold for which the complex self-dual bundle  $\Lambda_+^2(\eta)$  is defined globally. Suppose that

$$Q \in \Gamma(\{A \in \text{End}(\Lambda_+^2(\eta)) : A^\dagger = A\}) \quad (102)$$

is globally regular. Then the ordered spectral projectors  $P_i$  are globally defined and smooth, and one obtains

$$\Lambda_+^2(\eta) = L_1 \oplus L_2 \oplus L_3, \quad L_i := \text{Im } P_i, \tag{103}$$

as a decomposition into Hermitian eigenline subbundles.



**Figure 3.** Two model approaches to the boundary of the regular sector. Left: approach to rank loss with  $\lambda_3 = \epsilon$ . Right: approach to eigenvalue collision with  $\lambda_1 - \lambda_2 = \epsilon$ . In both cases, the determinant, the discriminant, and the barrier functional (101) are shown. The plots illustrate the two distinct degeneration mechanisms recorded in (56).

**Proposition 11.** *If a global regular Alena-Urbantke geometry exists on  $(M, \eta)$ , then the complex self-dual bundle  $\Lambda_+^2(\eta)$  admits a global splitting into three Hermitian line subbundles as in (103).*

**Proof.** The statement follows directly from the global spectral decomposition of a Hermitian endomorphism with everywhere simple spectrum.  $\square$

**Remark 30.** *Proposition 11 is only a necessary condition. It reduces the global existence problem to a bundle-splitting problem for  $\Lambda_+^2(\eta)$  and hence to standard topological constraints on complex rank-3 bundles.*

At the level of characteristic classes, the splitting (103) implies that the total Chern class of  $\Lambda_+^2(\eta)$  factorizes as

$$c(\Lambda_+^2(\eta)) = \prod_{i=1}^3 (1 + c_1(L_i)). \tag{104}$$

Thus any obstruction to such a factorization is automatically an obstruction to the existence of a global regular visible sector.

**Remark 31.** *For the global statements of the present section, no spin or spin<sup>c</sup> assumption is required. It is enough to work on an oriented four-dimensional Lorentzian manifold  $(M, \eta)$ . Indeed, after complexification, the Hodge operator on 2-forms satisfies  $*^2 = -\text{id}$  and therefore determines the global eigenbundle decomposition  $\Lambda^2 T^* M \otimes \mathbb{C} = \Lambda_+^2(\eta) \oplus \Lambda_-^2(\eta)$ ,  $\Lambda_\pm^2(\eta) := \ker(* \mp i \text{id})$ . Thus the complex self-dual bundle  $\Lambda_+^2(\eta)$  is globally defined as soon as the oriented Lorentzian background is fixed. Spinorial language is only needed later, when the identification  $\Lambda_+^2(\eta) \cong \text{Sym}^2 S$  is invoked.*

**Proposition 12.** *If a global regular Alena-Urbantke geometry exists on  $(M, \eta)$ , then there exist classes  $x_i := c_1(L_i) \in H^2(M; \mathbb{Z})$ ,  $i = 1, 2, 3$ , such that*

$$c_1(\Lambda_+^2(\eta)) = x_1 + x_2 + x_3, \tag{105}$$

and

$$c_2(\Lambda_+^2(\eta)) = x_1 x_2 + x_1 x_3 + x_2 x_3. \tag{106}$$

In particular, failure of the Chern classes of  $\Lambda_+^2(\eta)$  to admit such a decomposition is an obstruction to the existence of a global regular visible sector.

**Proof.** By Proposition 11, a global regular visible sector determines a splitting  $\Lambda_+^2(\eta) = L_1 \oplus L_2 \oplus L_3$  into Hermitian line subbundles. The Whitney product formula therefore gives  $c(\Lambda_+^2(\eta)) = c(L_1)c(L_2)c(L_3) = \prod_{i=1}^3(1 + c_1(L_i))$ . Expanding by degree yields (105) and (106). Since  $M$  is four-dimensional, the degree-six class  $c_3$  carries no further information in the present setting.  $\square$

**Remark 32.** Proposition 12 is only a first obstruction statement. It does not by itself decide whether a compatible splitting exists, but it reduces the global problem to a concrete factorization condition on the low-degree Chern data of  $\Lambda_+^2(\eta)$ .

**Remark 33.** For the purposes of the present paper, the splitting formulation is taken to be primary. A flag-bundle formulation is also available: global existence of a regular  $Q$  may equally be viewed as existence of a global section of the full flag bundle of  $\Lambda_+^2(\eta)$  together with a globally ordered positive eigenvalue triple. The splitting form is retained here because it leads more directly to the characteristic-class constraints.

A concrete four-manifold example has not been analyzed in the present work. Such a case study would be a natural next step for the global splitting problem.

The regular visible sector is sufficiently explicit to admit direct symbolic experimentation. In particular, the following objects are computable once a local self-dual frame has been fixed:

- the Hermitian endomorphism  $Q$ ,
- the spectral invariants  $I_1, I_2, I_3$  and the discriminant  $\Delta(Q)$ ,
- the spectral projectors  $P_i$  and the induced flag,
- the visible tensor  $Y = \mathcal{R}(Q)$ ,
- the degeneration indicators  $\det Q$  and  $\Delta(Q)$ ,
- the induced dynamics under (69), (70), or (77).

This makes the regular sector suitable for a symbolic notebook treatment. The symbolic and visualization modules used for the two-rotation model, the isospectral flow, and the degeneration indicators are provided in the supplementary Mathematica notebook. In this way, the nonconstant regular sector may be examined concretely before any stronger analytic or global statement is attempted.

### 3.4. Computational Visualization Tool

In order to facilitate work with Alena-Urbantke geometry and to support further analytical and numerical study of the regular visible sector, a dedicated Mathematica script, `Metric2GaugeAlenaDashboard`, has been prepared. The tool is intended as an auxiliary research instrument rather than as part of the formal construction itself. Its role is to make the structures introduced in Secs. 2.1-3.3 directly inspectable for concrete metric backgrounds, and to provide a reproducible environment in which the behavior of the visible self-dual block may be explored on selected slices of spacetime.

For a given Lorentzian metric, the script evaluates the visible tensor, the associated curvature-type lift, the Hermitian visible block  $Q$ , its ordered eigenvalue data, the induced spectral weights, the regularity diagnostics, and the derived quantities discussed in the preceding sections. The resulting objects are then displayed on configurable two-dimensional slices, with the remaining coordinates fixed by user-selected rules. A library of 25 predefined metrics is included, covering flat, static, cosmological, rotating, radiative, and toy-model backgrounds. A user-defined metric may also be supplied directly.

The interface is organized into a compact set of analysis panels. The *Analysed metric* panel displays the currently selected metric, parameter values, the chosen slice, the effective value of  $p_\Lambda$ , basic regularity and admissibility information, the interpretation of the applied  $\Lambda$ -branch, and warning data relevant for the numerical or symbolic reconstruction. The remaining panels are devoted to derived visual diagnostics:

1. *Mathematical.* Spectral data associated with  $Q$  are displayed, including the eigenvalue maps, the invariant maps for  $I_1, I_2, I_3$ , and  $\Delta(Q)$ , the barrier quantity based on  $\det Q$  and  $\Delta(Q)$ , the admissibility-ratio map, and configurable one-dimensional profiles extracted from the selected slice. Basic display controls are included in order to allow normalized, logarithmic, clipped, or shared-scale inspection when such comparisons are useful.
2. *Physical.* The normalized channel weights  $w_1, w_2, w_3$ , the corresponding entropy map, the RGB carrier composite, and the dominant-channel map are displayed. In addition, three carrier-archetype score maps are shown: single-channel concentration, two-channel pairing, and three-channel balance. These scores are intended to provide a compact geometric summary of how the local carrier state is positioned in the weight simplex.
3. *Isospectral Explorer.* A local family of isospectral deformations of the visible block is displayed, based on conjugation of the reference matrix  $Q_0$  by a one-parameter unitary family. In this way, spectral invariants may be compared with frame-sensitive quantities while the eigenvalue data are kept fixed.

The tool is intended to support in explicit examples. In particular, it may be used to examine the location of regular and degenerate loci, to compare different metric backgrounds at the level of the visible Hermitian block, to inspect the dependence of spectral data on the chosen slice and parameters, to test the behavior of carrier-channel decompositions in concrete geometries, and to explore how the regular visible sector changes under controlled deformations. In this sense, the script provides a practical bridge between the abstract local theory developed above and the detailed case-by-case analysis that may be required in further mathematical or physical investigations. The Mathematica script implementing the `Metric2GaugeAlenaDashboard`, together with a sample report produced for the Kerr-Newman metric, has been included with the present article as supplementary material.

#### 4. Conclusions and Discussion

A regular geometric class has been identified in which the visible sector of an Alena-type transition tensor is organized by a positive Hermitian endomorphism of the self-dual bundle of the background Lorentzian geometry. The starting point has been the curvature-type lift associated with the visible tensor. In four dimensions, its visible self-dual Ricci-type block is encoded by the Hermitian endomorphism  $Q$ , and the corresponding visible tensor is recovered through the representation map  $\mathcal{R}$ . The main result is that regular gauge-sector representations and regular Alena-Urbantke geometries are locally equivalent.

The geometric content of this equivalence may be stated succinctly. The visible tensor  $Y$  and the traceless part of the metric variable are both induced from the same regular self-dual Hermitian datum. On the regular locus, this datum carries more structure than positivity alone: a canonical ordered spectral flag is determined, and with it a natural carrier geometry internal to  $\Lambda_+^2(\eta)$ . The curvature-type viewpoint therefore separates the visible self-dual block from the remaining hidden curvature data and makes precise which part of the geometry is fixed by the visible sector and which part is not.

The role of the Urbantke construction is also clarified by the regular formulation. A regular  $Q$  determines a genuine chiral carrier geometry, but that geometry is not identified directly with the Alena metric variable. The latter is selected instead through the Alena-side relation and the tensor  $\mathcal{R}(Q)$ . In this sense, the carrier geometry and the induced visible metric deformation remain distinct objects, even though both are controlled by the same regular Hermitian endomorphism. In addition, regularity of the self-dual visible block does not by itself guarantee metric admissibility of the Alena-side tensor: nondegeneracy and Lorentzian signature impose an additional restriction on the induced metric variable.

From the structural point of view, the paper provides a local normal form for the regular visible sector. The regular problem is reduced to Hermitian endomorphisms with simple positive spectrum on  $\Lambda_+^2(\eta)$ , equivalently to ordered spectral data together with the induced flag. This gives a concrete

carrier-level organization of the regular sector and places the gauge-sector realization, the visible tensor, and the induced metric deformation in a single local framework. No claim has been made that a full Standard Model or Yang-Mills dynamics is thereby derived. What has been isolated is the regular visible geometric class itself. In parallel with this structural reduction, a supplementary Mathematica tool and an example report have been included in order to make the spectral organization of the visible Hermitian block and its associated regularity and degeneration loci directly testable on explicit metric backgrounds.

Several directions remain open, but they arise naturally from the structure already obtained. First, for smooth families of regular Hermitian endomorphisms, the perturbation formula recorded in Appendix A.3 separates variation into spectral and flag-moving parts. This suggests a natural proto-dynamical interpretation of the regular visible sector, in which the ordered spectrum and the motion of the spectral flag play distinct geometric roles. At the level of formal analogy, this separation is compatible with invariant-based descriptions in which intrinsic spectral data are distinguished from frame-dependent mixing data [33–35]. In that context, the supplementary script may be used as an auxiliary device for testing concrete metric backgrounds, tracking regular and degenerate loci on selected slices, resolving spectral transitions, and comparing how the visible Hermitian block changes under controlled deformations.

Second, the global existence problem is reduced to a bundle-splitting problem for  $\Lambda_+^2(\eta)$ . A globally regular endomorphism would determine globally defined eigenline subbundles and hence a global ordered splitting of the self-dual bundle. This brings the global theory into contact with standard bundle-theoretic and topological tools, including characteristic classes, elliptic and spectral methods, and the topology of families of bundles [36–40]. At that stage, the present local theory may be compared with more classical and more recent constraints on twistor and self-dual geometries, where curvature, integrability, and bundle structure are tightly linked [41,42].

Third, the regular visible block leaves the Weyl and scalar parts of a curvature-type completion undetermined. It is therefore natural to ask whether distinguished hidden-sector completions can be selected by compatibility with the ordered carrier flag or with the induced chiral geometry. A related question is whether the vacuum and vortex picture proposed in [11] may be used as a realization mechanism for the regular visible geometry identified here, rather than as part of its definition.

A further extension is suggested by comparison with other geometric reorganizations of gauge-theoretic data. In several settings, the primary role is shifted away from elementary fields toward more structured carriers or operator-valued data. The present regular Hermitian formulation appears compatible with that broader pattern, although no direct identification is assumed here [43–47].

The main point may therefore be stated as follows. The regular visible sector attached to an Alena-type transition tensor admits a canonical self-dual Hermitian description, and in that description its gauge-sector realizations, induced metric data, and carrier geometry can be treated within a single local framework. This does not exhaust the full geometry of the problem, but it isolates the regular visible part in a form suitable for further local, dynamical, and global study, while the accompanying supplementary tool and example report provide a direct route to explicit spectral and regularity diagnostics in concrete cases.

## Statements

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All data, symbolic computations, numerical evaluations, and plotting routines used in this article are contained in the accompanying Mathematica notebooks, provided as supplementary materials.

Declaration on the use of AI. During the preparation of this manuscript, the author used generative AI tools to assist with language editing, formatting, idea organization, and limited early drafting of selected passages. These tools were not used to generate research data, perform the main analysis, interpret the results, or draw the scientific conclusions. All content included in the manuscript,

including citations and factual statements, was reviewed and verified by the author, who takes full responsibility for the final manuscript.

**Supplementary Materials:** The following supporting information can be downloaded at the website of this paper posted on [Preprints.org](https://preprints.org).

## Appendix A. Additional Observations

### Appendix A.1. The Representation Map $\mathcal{R}$

On a local spinorial trivialization, the self-dual bundle is identified with  $\text{Sym}^2 S$ , where  $S$  is the rank-two complex Weyl spinor bundle [13,14]. A Hermitian endomorphism of  $\Lambda_+^2(\eta) \cong \text{Sym}^2 S$  is therefore represented by a Hermitian spinor endomorphism

$$Q_{AB}{}^{CD}. \quad (\text{A107})$$

Under the standard identification between Hermitian endomorphisms on  $\text{Sym}^2 S$  and real traceless symmetric rank-two tensors, the representation map  $\mathcal{R}$  is obtained. Equation (A109) is taken as its local definition in a self-dual frame, and the compatibility under change of self-dual basis shows that it defines a global real-linear map

$$\mathcal{R} : \{A \in \text{End}(\Lambda_+^2(\eta)|_U) : A^\dagger = A\} \rightarrow \text{Sym}_0^2 T^*U. \quad (\text{A108})$$

In a local self-dual basis  $(\Sigma_1, \Sigma_2, \Sigma_3)$ , if  $Q = (Q_{ij})$ , then

$$\mathcal{R}(Q)_{\mu\nu} = 2 \sum_{i,j=1}^3 Q_{ij} (\Sigma_i)_{\mu}{}^{\alpha} \overline{(\Sigma_j)_{\nu\alpha}}. \quad (\text{A109})$$

Under a change of local self-dual basis, the matrix  $Q = (Q_{ij})$  and the basis forms  $(\Sigma_i)$  transform compatibly, so the tensor (A109) is unchanged. This formula is therefore independent of the chosen local self-dual basis, defines a global real-linear map, and matches the normalization used in (7).

### Appendix A.2. Real and Rotating Sectors of the Visible Block

The explicit local formula for  $\mathcal{R}$  already shows that the Hermitian visible block splits kinematically into a real sector and a rotating sector. In the standard local self-dual frame of Appendix A.1, the diagonal part of  $Q$  controls the diagonal part of  $Y$ , the real off-diagonal part controls the spatial off-diagonal sector, and the imaginary off-diagonal part controls the mixed time-space sector. The rotating examples of Section 2.3 are based on this observation.

**Proposition A13.** *Work in the standard local self-dual frame of Appendix A.1, and write  $Q$  as in (52). Then the following are equivalent:*

- (i)  $Q$  is real symmetric;
- (ii)  $Y_{0i} = 0$  for  $i = 1, 2, 3$ .

*Equivalently, the visible tensor has no mixed time-space sector if and only if the Hermitian visible block lies in the real-symmetric subcone of  $\{A \in \text{End}(\Lambda_+^2(\eta)|_U) : A^\dagger = A\}$ .*

**Proof.** By (53), the mixed time-space components are

$$Y_{01} = -2b_{23}, \quad Y_{02} = 2b_{13}, \quad Y_{03} = -2b_{12}. \quad (\text{A110})$$

They vanish identically if and only if

$$b_{12} = b_{13} = b_{23} = 0. \quad (\text{A111})$$

Since  $Q$  is Hermitian, condition (A111) is equivalent to the statement that all off-diagonal entries of  $Q$  are real. Hence  $Q$  is real symmetric if and only if  $Y_{0i} = 0$  for  $i = 1, 2, 3$ .  $\square$

**Remark A34.** In the same frame, (53) gives a direct kinematical splitting of the visible sector. The diagonal entries of  $Q$  determine the diagonal part of  $Y$ . The real off-diagonal entries determine the spatial off-diagonal components  $Y_{ij}$  with  $i, j = 1, 2, 3$ . The imaginary off-diagonal entries determine the mixed time-space components  $Y_{0i}$ . No further change in the definition of regular Alena-Urbantke geometry is required in order to pass from the non-rotating sector to a rotating one.

**Remark A35.** The constant model (45) and the stationary-axisymmetric ansatz (50) show that the minimal route from the diagonal model (38) to a rotating visible sector is obtained by activating the imaginary off-diagonal Hermitian part of  $Q$ . The ordered spectrum may remain regular throughout.

**Remark A36.** This decomposition is frame-dependent. A physically suggestive reading of a given mixed component is obtained only after an adapted stationary-axisymmetric splitting has been chosen. In particular, the ansatz (50) is distinguished because it isolates a single mixed component, namely  $Y_{03}$  in (54). The resulting sector is of the same kinematical type as a  $t$ - $\phi$  term in a stationary axisymmetric metric, but no identification with a specific spacetime solution is implied by this observation alone.

### Appendix A.3. Spectral Perturbation Formulas

Let  $Q(t)$  be a smooth family of regular Hermitian endomorphisms. On a sufficiently small interval, the eigenvalues and spectral projectors may be chosen smoothly. Standard perturbation theory for smooth families of Hermitian operators then gives [48]

$$\dot{Q} = \sum_{i=1}^3 \dot{\lambda}_i P_i + [A, Q], \quad A^\dagger = -A. \quad (\text{A112})$$

The first term is diagonal in the instantaneous spectral basis and changes the ordered spectrum. The second term is off-diagonal and moves the spectral flag.

### Appendix A.4. Carrier-Space Automorphism Types and Reduced Sectors

Let  $Q$  be regular, with ordered spectral decomposition as in (24). The corresponding canonical flag (26) determines the carrier spaces

$$L = \text{Im } P_1, \quad E = \text{Im}(P_1 + P_2), \quad V = \Lambda_+^2(\eta). \quad (\text{A113})$$

These spaces have complex dimensions 1, 2, and 3. After the induced Hermitian volume data are fixed on  $E$  and  $V$ , the associated standard compact automorphism types are

$$L \rightsquigarrow U(1), \quad E \rightsquigarrow SU(2), \quad V \rightsquigarrow SU(3). \quad (\text{A114})$$

They are attached to the Hermitian carrier spaces determined by the ordered flag. They are not stabilizer groups of an individual regular endomorphism  $Q$ , and no independent gauge dynamics is inferred from them.

On loci where  $\det Q = 0$ , the full rank-three carrier is no longer realized by the visible block. On loci where  $\Delta(Q) = 0$ , the ordered spectral splitting ceases to be simple. Reduced sectors are therefore most naturally read as degenerations of the regular flag geometry.

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