Article

Not peer-reviewed version

Alternative Proof of the Ribbonness on Classical Link

Akio Kawauchi *

Posted Date: 15 January 2025

doi: 10.20944/preprints202501.1084.v1

Keywords: Ribbon surface; Slice link; Ribbon link

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 January 2025 doi:10.20944/preprints202501.1084.v1

Article

Alternative Proof of the Ribbonness on Classical Link

Akio Kawauchi

Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University, Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan; kawauchi@omu.ac.jp

Abstract: Alternative proof is given for an earlier presented result that if a link in 3-space bounds a proper oriented surface (without closed component) in the upper half 4-space, then the link bounds a proper oriented ribbon surface in the upper half 4-space which is a renewal embedding of the original surface

Keywords: Ribbon surface; Slice link; Ribbon link

MSC: Primary 57K45; Secondary 57K40

---

1. Introduction

For a set A in the 3-space $\mathbf{R}^3 = \{(x,y,z) | -\infty < x,y,z < +\infty\}$ and an interval $J \subset \mathbf{R}$, let

$$AJ = \{(x,y,z,t) | (x,y,z) \in A, \ t \in J\}.$$

The *upper-half 4-space* $\mathbf{R}^4_+$ is denoted by $\mathbf{R}^3[0,+\infty)$. Let k be a link in the 3-space $\mathbf{R}^3$, which always bounds a proper oriented surface F embedded smoothly in the upper-half 4-space $\mathbf{R}^4_+$, where $\mathbf{R}^3[0]$ is canonically identified with $\mathbf{R}^3$. Two proper oriented surfaces F and F' in $\mathbf{R}^4_+$ are *equivalent* if there is an orientation-preserving diffeomorphism f of $\mathbf{R}^4_+$ sending F to F', where f is called an *equivalence*. Let $\mathbf{b}$ be a *band system* spanning the link k in $\mathbf{R}^3$, namely a system of finitely many disjoint oriented bands b_i, $(i = 1,2\ldots,m)$ spanning the link k in $\mathbf{R}^3$. Let k' be a link in $\mathbf{R}^3$ obtained from k by surgery along this band system $\mathbf{b}$. This band surgery operation is denoted by $k \to k'$. If the link k' has $r-m$ or $r+m$ knot components for a link k of r knot components, then the band surgery operation $k \to k'$ is called a *fusion* or *fission*, respectively. These terminologies are used in [6]. Assume that a band surgery operation $k \to k'$ consists of a band surgery operation $k_i \to k'_i$ for the knot components $k_i \, (i = 1,2,\ldots,n)$ of k. Then if the link k'_i is a knot for every i, then the band surgery operation $k \to k'$ is called a *genus addition*. Every band surgery operation $k \to k'$ along a band system $\mathbf{b}$ is realized as a proper surface F^u_s in $\mathbf{R}^3[s,u]$ for any interval $[s,u]$ as follows.

$$F^u_s \cap \mathbf{R}^3[t] = \begin{cases} k'[t], & \text{for } \frac{s+u}{2} < t \leq u, \\ (k \cup \mathbf{b})[t], & \text{for } t = \frac{s+u}{2}, \\ k[t], & \text{for } s \leq t < \frac{s+u}{2}. \end{cases}$$

For every band surgery sequence $k_0 \to k_1 \to k_2 \to \cdots \to k_n$, the *realizing surface* F^u_s in $\mathbf{R}^3[s,u]$ is given by the union

$$F^{s_1}_{s_0} \cup F^{s_2}_{s_1} \cup \cdots \cup F^{s_n}_{s_{n-1}}$$

for any division

$$s = s_0 < s_1 < s_2 < \cdots < s_n = u$$

of the interval $[s,u]$. For a band surgery sequence $k_0 \to k_1 \to k_2 \to \cdots \to o_n$ with $\mathbf{o}_n$ a trivial link, the *upper-closed realizing surface* in $\mathbf{R}^3[s,t]$ is the surface

$$\mathrm{ucl}(F^u_s) = F^u_s \cup \delta[u]$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 January 2025 doi:10.20944/preprints202501.1084.v1

in $\mathbf{R}^3[s,t]$ with boundary $\partial \mathrm{ucl}(F_s^u) = k_0[s]$ in $\mathbf{R}^3[s]$, where δ denotes a disk system in $\mathbf{R}^3$ of mutually disjoint disks with $\partial\delta = \mathbf{o}_n$. Further, if the link k_0 is the split sum $k+\mathbf{o}$ of a link k and a trivial link $\mathbf{o}$ in $\mathbf{R}^3$, then a *bounded realizing surface for* the link k $\mathrm{p}(F_s^u)$ in $\mathbf{R}^3[s,u]$ is defined to be the surface

$$\mathrm{p}(F_s^u) = F_s^u \cup \mathbf{d}[s] \cup \delta[u]$$

in $\mathbf{R}^3[s,u]$ with $\partial \mathrm{p}(F_s^u) = k[s]$ in $\mathbf{R}^3[s]$, where $\mathbf{d}$ is a disk system in $\mathbf{R}^3$ with $\partial\mathbf{d} = \mathbf{o}$ and $\mathbf{d}\cap k = \emptyset$. A *proper realizing surface* for the link k is a *proper* surface $\mathrm{p}(F_s^u)^+$ in $\mathbf{R}^3[s,+\infty)$ with $\partial \mathrm{p}(F_s^u)^+ = k[s]$ in $\mathbf{R}^3[s]$ which is obtained from $\mathrm{p}(F_s^u)$ by raising the level s of the disk system d into the level $s+\varepsilon$ for a sufficiently small $\varepsilon > 0$. The upper-closed proper surface $\mathrm{ucl}(F_s^u)$ for k_0 in $\mathbf{R}^4_+$ does not depend on choices of δ and is determined up to equivalences only by the band surgery sequence $k_0 \to k_1 \to k_2 \to \cdots \to \mathbf{o}_n$ with $\mathbf{o}_n$ a trivial link by Horibe-Yanagawa's lemma, [6]. Also, the proper realizing surface $\mathrm{p}(F_s^u)^+$ for k in $\mathbf{R}^4_+$ does not depend on choices of δ, $\mathbf{d}$, ε and is determined up to equivalences only by the band surgery sequence $k+\mathbf{o} \to k_1 \to k_2 \to \cdots \to \mathbf{o}_n$ with $\mathbf{o}, \mathbf{o}_n$ trivial links. A *stable-exact band surgery sequence* for a link k in $\mathbf{R}^3$ is a band surgery sequence

$$(\#) \qquad k+\mathbf{o} \to k_1 \to k_2 \to k_3 \to \mathbf{o}_4$$

for trivial links $\mathbf{o}, \mathbf{o}_4$ in $\mathbf{R}^3$ such that

(0) the operation $k+\mathbf{o} \to k_1$ is a fusion along a band system $\mathbf{b}_1$ connecting every component of $\mathbf{o}$ to k with just one band,

(1) the operation $k_1 \to k_2$ is a fusion along a band system $\mathbf{b}_2$,

(2) the operation $k_2 \to k_3$ is a genus addition along a band system $\mathbf{b}_3$, and

(2) the operation $k_3 \to \mathbf{o}_4$ is a fission along a band system $\mathbf{b}_4$.

In (0), the link k_1 is called a *band sum* of the link k and the trivial link $\mathbf{o}$. By band slides, assume that the band systems $\mathbf{b}_i$ $(i = 2,3,4)$ do not meet with the trivial link $\mathbf{o}$. For every stable-exact band surgery sequence (#) for a link k in $\mathbf{R}^3$, a proper realizing surface $\mathrm{p}(F_0^1)$ for k in $\mathbf{R}^4_+$ with $\partial\mathrm{p}(F_0^1) = k$ is constructed for any division $0 = s_0 < s_1 < s_2 < s_3 < s_4 = 1$ of the interval $[0,1]$. The following theorem is known, [6].

Normal form theorem. For every proper oriented surface F without closed component in the upper-half 4-space $\mathbf{R}^4_+$, there is a stable-exact band surgery sequence (#) for the link $k = \partial F$ in $\mathbf{R}^3$ such that the proper realizing surface $\mathrm{p}(F_0^1)^+$ is equivalent to F in $\mathbf{R}^4_+$.

The surface $\mathrm{p}(F_0^1)^+$ in $\mathbf{R}^4_+$ is called a *normal form* of the proper surface F in $\mathbf{R}^4_+$. In the stable-exact band surgery sequence (#), if the trivial link o is taken the empty set $\emptyset$, then the item (0) is omitted and the stable-exact band surgery sequence (#) is reduced to the band surgery sequence

$$(\#\#) \qquad k = k_1 \to k_2 \to k_3 \to \mathbf{o}_4$$

This band surgery sequence (##) is called an *exact band surgery sequence* for the link k. In classical knot theory, a proper surface F in $\mathbf{R}^4_+$ is a *ribbon surface* in $\mathbf{R}^4_+$ for the link $k = \partial F$ in $\mathbf{R}^3$ if it is equivalent to the upper-closed realizing surface $\mathrm{ucl}(F_0^1)$ of an exact band surgery sequence (##) for the link k. In the following example, it is observed that there are lots of proper oriented surfaces without closed component in $\mathbf{R}^4_+$ which is not equivalent to any ribbon surface.

Example 1. *For every link k, let F' be any ribbon surface in $\mathbf{R}^4_+$ with $k = \partial F'$. For example, let F' be a proper surface in $\mathbf{R}^4_+$ obtained from a Seifert surface for k in $\mathbf{R}^3$ by an interior push into $\mathbf{R}^4_+$. Take a connected sum $F = F'\#K$ of F' and a non-trivial S^2-knot K in $\mathbf{R}^4$ with non-abelian fundamental group. The ribbon surface F'*

 doi:10.20944/preprints202501.1084.v1

is a renewal embedding of F into $\mathbf{R}^4_+$ with $k = \partial F' = \partial F$. The fundamental groups of k, F', F, K are denoted as follows.

$$\pi(k) = \pi_1(\mathbf{R}^3 \setminus k, x_0), \quad \pi(F') = \pi_1(\mathbf{R}^4 \setminus F', x_0),$$

$$\pi(F) = \pi_1(\mathbf{R}^4 \setminus F, x_0), \quad \pi(K) = \pi_1(S^4 \setminus K, x_0).$$

Let $\pi(k)^*, \pi(F')^* \pi(F)^*, \pi(K)^*$ be the kernels of the canonical epimorphisms from the groups $\pi(k), \pi(F'), \pi(F)$, $\pi(K)$ to the infinite cyclic group sending every meridian element to the generator, respectively. It is a special feature of a ribbon surface F' that the canonical homomorphism $\pi(k) \to \pi(F')$ is an epimorphism, so that the induced homomorphism $\pi(k)^* \to \pi(F')^*$ is onto. On the other hand, the canonical homomorphism $\pi(k) \to \pi(F)$ is not onto, because the group $\pi(F)^*$ is the free product $\pi(F')^* * \pi(K)^*$ and $\pi(K)^* \neq 0$ and the image of the induced homomorphism $\pi(k)^* \to \pi(F)^*$ is just the free product summand $\pi(F')^*$. Thus, the proper surface F in $\mathbf{R}^4_+$ is not equivalent to any ribbon surface, in particular to F'.

A proper surface F' in $\mathbf{R}^4_+$ is a *renewal embedding* of a proper surface F into $\mathbf{R}^4_+$ if $\partial F' = \partial F$ in $\mathbf{R}^3$ and there is an orientation-preserving surface-diffeomorphism $F' \to F$ keeping the boundary fixed. The proof of the following theorem is given, [4]. In this paper, an alternative proof of this theorem is given from a viewpoint of deformations of a ribbon surface-link in $\mathbf{R}^4$.

Classical ribbon theorem. Assume that a link k in the 3-space $\mathbf{R}^3$ bounds a proper oriented surface F without closed component in the upper-half 4-space $\mathbf{R}^4_+$. Then the link k in $\mathbf{R}^3$ bounds a ribbon surface F' in $\mathbf{R}^4_+$ which is a renewal embedding of F.

A link k in $\mathbf{R}^3$ is a *ribbon link* if there is a fission $k \to \mathbf{o}_1$ for a trivial link $\mathbf{o}_1$. A link k in $\mathbf{R}^3$ is a *slice link in the strong sense* if k bounds a proper disk system embedded smoothly in $\mathbf{R}^4_+$. Then there is a stable-exact band surgery sequence (#) with $k_1 = k_2 = k_3$ for k. The following corollary is a spacial case of Classical ribbon theorem.

Corollary 1. *Every slice link in the strong sense in $\mathbf{R}^3$ is a ribbon link.*

Thus, Classical ribbon theorem solves *Slice-Ribbon Problem*, [1], [2]. If there is a fusion $k + \mathbf{o} \to k_1$ for a trivial link $\mathbf{o}$ and a ribbon link k_1, then k is a slice link in the strong sense, for k bounds a proper disk system as a proper realizing surface $\mathrm{p}(F_0^1)^+$ of a band surgery sequence $k + \mathbf{o} \to k_1 \to \mathbf{o}_2$ with a fission $k_1 \to \mathbf{o}_2$ for a trivial link $\mathbf{o}_2$. Hence, the following corollary is obtained from Corollary 1.

Corollary 2. *A link k in $\mathbf{R}^3$ is a ribbon link if if there is a fusion $k + \mathbf{o} \to k_1$ for a trivial link $\mathbf{o}$ and a ribbon link k_1.*

An idea of the present proof of this theorem is to consider the ribbon surface-link $\mathrm{cl}(F^1_{-1})$ in the 4-space $\mathbf{R}^4$ obtained by doubling from the upper-closed realizing surface $\mathrm{ucl}(F_0^1)$ in $\mathbf{R}^4_+$ of a stable-exact band surgery sequence (#) for a link k in $\mathbf{R}^3$ obtained from F by the normal form theorem. The effort is to remove the interior intersection between the 2-handle systems on $\mathrm{cl}(F^1_{-1})$ arising from the band systems of a stable-exact band surgery sequence (#) and the 2-handle system on $\mathrm{cl}(F^1_{-1})$ arising from the disk system $\mathbf{d}$ at the expense of type of the ribbon surface-link $\mathrm{cl}(F^1_{-1})$.

2. Proof of Classical Ribbon Theorem

Throughout this section, the proof of the classical ribbon theorem is done. Let F be a proper oriented surface without closed component in $\mathbf{R}^4_+$, and $\partial F = k$ a link in $\mathbf{R}^3$. By the normal form theorem, consider a stable-exact band surgery sequence (#) for k such that $\mathrm{p}(F_0^1)^+$ is equivalent to F in $\mathbf{R}^4_+$. Also, consider the ribbon surface-link $\mathrm{cl}(F^1_{-1})$ in the 4-space $\mathbf{R}^4$ constructed by doubling from the upper-closed realizing surface $\mathrm{ucl}(F_0^1)$ in $\mathbf{R}^4_+$ of the stable-exact band surgery sequence (#) , [6]. Let $\mathbf{b}_i$, $(i = 1,2,3,4)$ be the band system used for the operations $k + o \to k_1, k_1 \to k_2, k_2 \to k_3$ and $k_3 \to o_4$, respectively in (#), which are taken disjoint. Further, the band systems $\mathbf{b}_i$, $(i = 1, 2, 3, 4)$ are taken to be attached to k. Then the disjoint 2-handle systems $\mathbf{b}_i[-t_i, t_i]$, $(i = 1, 2, 3, 4)$ on the ribbon surface-link $\mathrm{cl}(F^1_{-1})$ are obtained, where $t_i = (s_{i-1} + s_i)/2\,(i = 1, 2, 3, 4)$. The disk system $\mathbf{d}$ with $\partial\mathbf{d} = \mathbf{o}$ and $\mathbf{d} \cap k = \emptyset$ for the split sum $k + \mathbf{o}$ also constructs the 2-handle system $\mathbf{d}[-\varepsilon, \varepsilon]$ on the ribbon surface-link $\mathrm{cl}(F^1_{-1})$ for a sufficiently small $\varepsilon > 0$. Let ι be the reflection of $\mathbf{R}^4$ sending every point (x, y, z, t) to the point $(x, y, z, -t)$. The ribbon surface-link $\mathrm{cl}(F^1_{-1})$ and the 2-handles $\mathbf{b}_i[-t_i, t_i]$, $(i = 1, 2, 3, 4)$, $\mathbf{d}[-\varepsilon, \varepsilon]$ are ι-invariant. Let δ be a disk system in $\mathbf{R}^3$ bounded by the trivial link $\boldsymbol{\lambda} = \mathbf{o}_4$. The band systems $\boldsymbol{\alpha}_i$, $(i = 1, 2, 3, 4)$ in $\mathbf{R}^3$ spanning the trivial link $\boldsymbol{\lambda}$ are obtained, as a dual viewpoint, from the band systems $\mathbf{b}_i$, $(i = 1, 2, 3, 4)$ spanning the link $k + \mathbf{o}$ in $\mathbf{R}^3$, respectively. It is considered that the ribbon surface-link $\mathrm{cl}(F^1_{-1})$ is obtained from the trivial S^2-link $O = \partial(\delta[-1, 1])$ by surgery along the ι-invariant 1-handle systems $\boldsymbol{\alpha}_i[-t_i, t_i]$, $(i = 1, 2, 3, 4)$ on O which are the duals of the 2-handles $\mathbf{b}_i[-t_i, t_i]$, $(i = 1, 2, 3, 4)$ on $\mathrm{cl}(F^1_{-1})$, respectively, [6]. In recent terms, the ribbon surface-link $\mathrm{cl}(F^1_{-1})$ is presented by the pair $(\boldsymbol{\lambda}, \boldsymbol{\alpha})$ of a *based loop system* $\boldsymbol{\lambda}$ and a *chord system* $\boldsymbol{\alpha}$ consisting of the band systems $\boldsymbol{\alpha}_i$ $(i = 1, 2, 3, 4)$ spanning $\boldsymbol{\lambda}$ in $\mathbf{R}^3$, [3]. The chord system α is generally understood as a system of spanning strings, but here it is a system of spanning bands. Since every component o of $\mathbf{o}$ meets with a based loop λ in $\boldsymbol{\lambda}$ in an arc I not meeting $\boldsymbol{\alpha}_1$, let $\mathbf{I}$ be an arc system obtained by choosing one such arc I for every component o of $\mathbf{o}$. Then there is a disk system $\mathbf{d}_0$ in $\mathbf{d}$ not meeting the band system $\boldsymbol{\alpha}$ such that $\mathbf{I} \subset \partial\mathbf{d}_0$ and the complement $\mathbf{d}' = \mathrm{cl}(\mathbf{d} \setminus \mathbf{d}_0)$ is a disk system which is a strong deformation retract of $\mathbf{d}$. Note that every band α in the band system $\boldsymbol{\alpha}$ meets the interior of the disk system $\mathbf{d}$ in an arc system consisting of proper arcs parallel to the centerline of α. The following claim (2.1) is obtained.

(2.1) There is a band system $\boldsymbol{\alpha}'$ spanning $\boldsymbol{\lambda}$ isotopic to the band system $\boldsymbol{\alpha}$ by band slide moves on $\mathbf{d}$ keeping $\boldsymbol{\lambda}$ fixed and keeping $\mathbf{d}$ setwise fixed such that every band α' of $\boldsymbol{\alpha}'$ meets $\mathbf{d}$ only in the disk system $\mathbf{d}_0$.

After the claim (2.1), let

$$\boldsymbol{\lambda}' = \mathrm{cl}(\boldsymbol{\lambda} \setminus \mathbf{I}) \cup \mathrm{cl}(\partial\mathbf{d}_0 \setminus \mathbf{I})$$

be a trivial link spanned by the band system $\boldsymbol{\alpha}$, so that the pair $(\boldsymbol{\lambda}', \boldsymbol{\alpha}')$ is a chord system in $\mathbf{R}^3$. Let $\mathrm{cl}(F^1_{-1})'$ be the ι-invariant surface-link obtained from the chord system $(\boldsymbol{\lambda}', \boldsymbol{\alpha}')$ in $\mathbf{R}^3$. Then the middle cross-sectional link $\mathrm{cl}(F^1_{-1})' \cap \mathbf{R}^3[0]$ is the split sum $k + \mathbf{o}'$ for the trivial link $\mathbf{o}' = \partial\mathbf{d}'$. In fact, the link obtained from $\boldsymbol{\lambda}$ by surgery along $\boldsymbol{\alpha}$ is the split sum $k + \mathbf{o}$ and the link obtained from $\boldsymbol{\lambda}$ by surgery along $\boldsymbol{\alpha}_1$ is a link $k' \cup \mathbf{o}$ for a link k', so that k is obtaine from k' by surgery along $\boldsymbol{\alpha}_i$ $(i = 2, 3, 4)$. On the other hand, the link obtained from $\boldsymbol{\lambda}'$ by surgery along $\boldsymbol{\alpha}'_1$ is the split sum $k' + \mathbf{o}'$, so that the link obtained from $\boldsymbol{\lambda}'$ by surgery along $\boldsymbol{\alpha}'$ is the split sum $k + \mathbf{o}'$. Note that the surface-link $\mathrm{cl}(F^1_{-1})'$ is obtained by sacrificing an equivalence to the surface-link $\mathrm{cl}(F^1_{-1})$ although they are the same surface. By replacing $\mathbf{d}'$, $\boldsymbol{\lambda}'$ and $\mathbf{o}'$ with $\mathbf{d}$, $\boldsymbol{\lambda}$ and $\mathbf{o}$, respectively, the following claim (2.2) is obtained.

(2.2) There is a stable-exact band surgery sequence (#) for the link k with $\mathrm{p}(F^1_0)^+$ a renewal embedding of F in $\mathbf{R}^4_+$ such that the band system $\boldsymbol{\alpha}$ does not meet the interior of the disk system $\mathbf{d}$.

Take a stable-exact band surgery sequence (#) for the link k of (2.2). Then the link k_1 is isotopic to the link k in $\mathbf{R}^3$. Thus, there is an exact band surgery sequence (##) for the knot k such that the upper-closed realizing surface $\mathrm{ucl}(F^1_0)$ is a renewal embedding of F. This completes the proof of the classical ribbon theorem.

In (2.2), the ribbon surface-link $\mathrm{cl}(F^1_{-1})$ in the 4-space $\mathbf{R}^4$ constructed by doubling from the upper-closed realizing surface $\mathrm{ucl}(F^1_0)$ in $\mathbf{R}^4_+$ of the stable-exact band surgery sequence (#) admits the O2-handle pair system $(\boldsymbol{\alpha}_1[-t_1, t_1], \mathbf{d}[-\varepsilon, \varepsilon])$, [5]. The last explanation above is related to the surgery of $\mathrm{cl}(F^1_{-1})$ along the O2-handle pair system.

Acknowledgments: This paper is motivated by T. Shibuya's comments pointing out insufficient explanation on Lemma 2.3 in [4] (Corollary 2 in this paper). This work was partly supported by JSPS KAKENHI Grant Number JP21H00978 and MEXT Promotion of Distinctive Joint Research Center Program JPMXP0723833165.

 doi:10.20944/preprints202501.1084.v1

References

1. R. H. Fox, Some problems in knot theory, Topology of 3-manifolds and related topics, (1962), 168-176, Prentice-hall, Inc., Engelwood Cliffs, N. J. USA.
2. R. H. Fox, Characterization of slices and ribbons, Osaka J. Math 10. (1973), 69-76.
3. A. Kawauchi, A chord diagram of a ribbon surface-link, Journal of Knot Theory and Its Ramifications 24 (2015), 1540002 (24 pages).
4. A. Kawauchi, Ribbonness on classical link. Journal of Mathematical Techniques and Computational Mathematics, 2 (8) (2023), 375-377, DOI:10.33140/JMTCM.
5. A. Kawauchi, Ribbonness of a stable-ribbon surface-link, I. A stably trivial surface-link, Topology and its Applications 301(2021), 107522 (16pages).
6. A. Kawauchi, T. Shibuya, S. Suzuki, Descriptions on surfaces in four-space I : Normal forms, Mathematics Seminar Notes, Kobe University, 10 (1982), 75-125; II: Singularities and cross-sectional links, Mathematics Seminar Notes, Kobe University 11 (1983), 31-69. Available from: https://sites.google.com/view/kawauchiwriting