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Article

On Σ-Classes in E^8. IV. The Neighbourhood of F_{22}

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Abstract: In the cone of positive quadtratic forms $\mathcal{C}^{8\times 8}$, it is shown that there exists in the neighbourhood of the quadratic form $\mathsf{Q}_{F_{22}}$ a large cluster of non-equivalent Σ_0-subcones of positive volume which are as much minimal as maximal.

Keywords: lattice of translations; parallelohedra; cone of positive quadratic forms; Σ-subcone

1. Introduction

In d-dimensional Euclidean space E^d, a bounded, convex body of finite volume congruent copies of which can be juxtaposed by translations such that they fill space without gaps, and overlap only in border points, is denoted by Fedorov [10] as *parallelohedron*. A special kind of parallelohedron, the *Dirichlet parallelohedron* is obtained by applying Dirichlet's famous construction [1] to a lattice of translations Λ^d. Voronoï [12,13] thoroughly investigated general properties of parallelohedra in spaces of arbitrary dimensions d. That is why they are often referred to as *Voronoï parallelohedra*.

Quadratic forms, translation lattices and parallelohedra play a predominant role in the geometry of numbers, but also in crystallography, where the lattice-like arrangement of atomic building blocks is a fundamental property of regular crystals. In crystallography, the discovery of quasicrystals, the structure of which can be viewed as projected from higher dimensional translation lattices, has greatly stimulated the investigation of lattices and parallelohedra in arbitrary dimensions.

In Euclidean space E^8, the collective of all lattices in the open cone $\mathcal{C}^{8\times 8}$ of positive definite quatratic forms is considered by studying its subdivision into Σ-subcones.

Voronoï proposed that each parallelohedron $\widetilde{\mathsf{P}}$ is affinely equivalent to a Dirichlet parallelohedron P, but he proved it for primitive parallelohedra only [12], §44. Primitive means that at each vertex of a parallelohedron P in its tiling exactly the minimal number $d+1$ of parallelohedra meet. These are the generic parallelohedra, for under small perturbations the combinatorial type remains fixed.

By using affinely invariant operations only, the *zone contraction* and *zone extension*, it becomes obvious that each combinatorial type of *totally contracted* Dirichlet parallelohedron is sufficient to prove that its complete *contraction family* can be represented by affine Dirichlet parallelohedra too. In dimensions $d \geq 0$ there exist totally contracted parallelohedra which are also referred to as *minimal*. But in dimensions $d \geq 6$ there exist parallelohedra which are as well minimal as *maximal* that is they allow no extension. This makes these parallelohedra particularly interesting.

In order to find Σ_0-classes in $\mathbf{C}^{8\times 8}$, the parallelohedra of the maximal finite irreducible subgroups of $GL_8(\mathbb{Z})$ given by Plesken and Pohst [11] are of importance. The five types viz.: F_5, F_8, F_{15}, F_{22}, and F_{26}, respectively, are as much minimal as maximal.

In previous papers the neighbourhood of the quadratic form Q_{F_5} [7], Q_{F_8} [8] $\mathsf{Q}_{F_{15}}$ [9] were investigated. In this note, the neighbourhood of $\mathsf{Q}_{F_{22}}$ will be looked at.

Table 1. The parallelohedra of the maximal finite irreducible subgroups of $GL_8(\mathbb{Z})$

No.	Group	Order	Comb. Type	Zones	Belts
F_1	B_8	$2^8 8!$	16.256	8(8)	4_{28}
F_2	D_8	$2^8 8!$	112.272	136(0)	6_{224}
F_3		$2 \cdot 1152^2$	48.576	24(0)	$4_{114}6_{32}$
F_4	D_8^*	$2^8 8!$	272.1120	56(0)	$4_{28}6_{512}$
F_5	E_8	$2^{14}3^5 5^2 7$	240.19449	8760(0)	6_{1120}
F_6		$3! \cdot 1152$	264.5304	36(0)	$4_{108}6_{972}$
F_7		$12^4 4!$	24.1296	12(12)	$4_{54}6_4$
F_8		$2 \cdot 6^4 4!$	348.3588	93(0)	6_{1386}
F_9		$2 \cdot 144^2$	60.10404	18(18)	$4_2 256_{48}$
F_{10}	A_8	$2 \cdot 9!$	72.510	9(9)	6_{84}
F_{11}		$2 \cdot 6^4 4!$	186.5940	174(0)	$4_{54}6_{544}$
F_{12}		$2 \cdot 6^3 3!$	402.94254	1497(0)	6_{2259}
F_{13}	A_8^*	$2 \cdot 9!$	510.362880	36(36)	6_{3025}
F_{14}		$2 \cdot 240^2$	40.900	10(10)	$4_{100}6_{20}$
F_{15}		$2^7 \cdot 15^2$	360.3840	180(0)	6_{1700}
F_{16}		$2 \cdot 240^2$	60.14400	20(20)	$4_{225}6_{50}$
F_{17}		$(2 \cdot 5!)^2$	290.43310	1195(0)	6_{1370}
F_{18}		$2 \cdot 3! \cdot 5!$	390.106200	450(0)	6_{2160}
F_{19}		$2 \cdot 3! \cdot 5!$	180.6510	15(15)	$4_{90}6_{400}$
F_{20}		1152	296.13696	580(0)	$4_{66}6_{1168}$
F_{21}		1152	200.2992	304(0)	$4_{48}6_{564}$
F_{22}		$3 \cdot 1152$	456.28632	264(0)	6_{2260}
F_{23}		672	398.90384	297(0)	6_{2192}
F_{24}		672	426.37638	21(21)	$4_{24}6_{1428}$
F_{25}		672	308.8028	1011(0)	$4_{21}6_{1386}$
F_{26}		672	398.127848	1036(0)	6_{2227}

2. The irreducible subgroup F_{22}

Notations and methods developed in previous papers [3–6] will be applied.
Referred to an optimal basis the following Gram matrix is obtained.

$$
\mathsf{Q}_{F_{22}} := \begin{pmatrix}
6 & -3 & 2 & 3 & 2 & -1 & 1 & 0 \\
-3 & 6 & -1 & -3 & 1 & 0 & -3 & -3 \\
2 & -1 & 6 & 3 & 2 & 1 & -1 & 2 \\
3 & -3 & 3 & 6 & -1 & 1 & 2 & 3 \\
2 & 1 & 2 & -1 & 6 & -3 & -3 & -2 \\
-1 & 0 & 1 & 1 & -3 & 6 & 3 & 3 \\
1 & -3 & -1 & 2 & -3 & 3 & 6 & 3 \\
0 & -3 & 2 & 3 & -2 & 3 & 3 & 6
\end{pmatrix}.
$$

The corresponding parallelohedron is computed by half-space intersections,

$$
\mathsf{P}(\mathsf{Q}_{F_{22}}) := \bigcap_{\forall \mathbf{t} \in \Lambda^d \setminus \{O\}} \mathsf{H}_{\mathbf{t}}. \tag{1}
$$

$\mathsf{P}(\mathsf{Q}_{F_{22}})$ has 456 facets, 28'632 vertices, and 264 zones all of which are open. Therefore, $\mathsf{P}(\mathsf{Q}_{F_{22}})$ is minimal. The 228 pairs of facet vectors $\pm\mathbf{f}_i$ are listed in Table 2. The set of facet vectors is denoted by $\mathcal{F}_{F_{22}}$.

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Table 2. The facet vectors $\pm\mathbf{f}_i$ of $\mathsf{P}(\mathsf{Q}_{F_{22}})$

$\mathbf{f}_{1}$	1 0 0 0 0 0 0 0	$\mathbf{f}_{3}$	0 1 0 0 0 0 0 0	$\mathbf{f}_{5}$	1 1 0 0 0 0 0 0
$\mathbf{f}_{7}$	0 0 1 0 0 0 0 0	$\mathbf{f}_{9}$	0 1 1 0 0 0 0 0	$\mathbf{f}_{11}$	1 0 -1 0 0 0 0 0
$\mathbf{f}_{13}$	1 1 -1 0 0 0 0 0	$\mathbf{f}_{15}$	0 0 0 1 0 0 0 0	$\mathbf{f}_{17}$	0 1 0 1 0 0 0 0
$\mathbf{f}_{19}$	0 1 -1 1 0 0 0 0	$\mathbf{f}_{21}$	1 1 -1 1 0 0 0 0	$\mathbf{f}_{23}$	1 0 0 -1 0 0 0 0
$\mathbf{f}_{25}$	0 0 1 -1 0 0 0 0	$\mathbf{f}_{27}$	1 0 1 -1 0 0 0 0	$\mathbf{f}_{29}$	0 0 0 0 1 0 0 0
$\mathbf{f}_{31}$	0 0 0 1 1 0 0 0	$\mathbf{f}_{33}$	0 1 -1 1 1 0 0 0	$\mathbf{f}_{35}$	1 0 0 0 -1 0 0 0
$\mathbf{f}_{37}$	0 1 0 0 -1 0 0 0	$\mathbf{f}_{39}$	1 1 0 0 -1 0 0 0	$\mathbf{f}_{41}$	0 0 1 0 -1 0 0 0
$\mathbf{f}_{43}$	0 1 1 0 -1 0 0 0	$\mathbf{f}_{45}$	1 1 1 0 -1 0 0 0	$\mathbf{f}_{47}$	1 0 0 -1 -1 0 0 0
$\mathbf{f}_{49}$	1 1 0 -1 -1 0 0 0	$\mathbf{f}_{51}$	0 0 1 -1 -1 0 0 0	$\mathbf{f}_{53}$	1 0 1 -1 -1 0 0 0
$\mathbf{f}_{55}$	1 1 1 -1 -1 0 0 0	$\mathbf{f}_{57}$	1 0 1 -2 -1 0 0 0	$\mathbf{f}_{59}$	0 0 0 0 0 1 0 0
$\mathbf{f}_{61}$	1 0 0 0 0 1 0 0	$\mathbf{f}_{63}$	1 1 0 0 0 1 0 0	$\mathbf{f}_{65}$	1 0 -1 0 0 1 0 0
$\mathbf{f}_{67}$	1 0 0 -1 0 1 0 0	$\mathbf{f}_{69}$	0 0 0 0 1 1 0 0	$\mathbf{f}_{71}$	1 0 -1 0 1 1 0 0
$\mathbf{f}_{73}$	0 1 -1 1 1 1 0 0	$\mathbf{f}_{75}$	0 0 1 0 0 -1 0 0	$\mathbf{f}_{77}$	0 0 0 1 0 -1 0 0
$\mathbf{f}_{79}$	0 1 0 1 0 -1 0 0	$\mathbf{f}_{81}$	1 0 0 0 -1 -1 0 0	$\mathbf{f}_{83}$	0 1 0 0 -1 -1 0 0
$\mathbf{f}_{85}$	1 1 0 0 -1 -1 0 0	$\mathbf{f}_{87}$	0 0 1 0 -1 -1 0 0	$\mathbf{f}_{89}$	0 1 1 0 -1 -1 0 0
$\mathbf{f}_{91}$	1 1 1 0 -1 -1 0 0	$\mathbf{f}_{93}$	0 1 0 1 -1 -1 0 0	$\mathbf{f}_{95}$	1 0 0 -1 -1 -1 0 0
$\mathbf{f}_{97}$	0 0 1 -1 -1 -1 0 0	$\mathbf{f}_{99}$	1 0 1 -1 -1 -1 0 0	$\mathbf{f}_{101}$	1 1 1 -1 -1 -1 0 0
$\mathbf{f}_{103}$	1 0 1 -1 -2 -1 0 0	$\mathbf{f}_{105}$	1 1 1 -1 -2 -1 0 0	$\mathbf{f}_{107}$	0 0 0 0 0 0 1 0
$\mathbf{f}_{109}$	0 1 0 0 0 0 1 0	$\mathbf{f}_{111}$	1 1 0 0 0 0 1 0	$\mathbf{f}_{113}$	0 0 1 0 0 0 1 0
$\mathbf{f}_{115}$	0 1 1 0 0 0 1 0	$\mathbf{f}_{117}$	0 1 0 1 0 0 1 0	$\mathbf{f}_{119}$	1 0 0 -1 0 0 1 0
$\mathbf{f}_{121}$	1 1 0 -1 0 0 1 0	$\mathbf{f}_{123}$	0 0 1 -1 0 0 1 0	$\mathbf{f}_{125}$	0 1 1 -1 0 0 1 0
$\mathbf{f}_{127}$	1 1 1 -1 0 0 1 0	$\mathbf{f}_{129}$	0 0 0 0 1 0 1 0	$\mathbf{f}_{131}$	0 1 0 0 1 0 1 0
$\mathbf{f}_{133}$	0 1 0 1 1 0 1 0	$\mathbf{f}_{135}$	0 1 -1 1 1 0 1 0	$\mathbf{f}_{137}$	1 1 1 -1 -1 0 1 0
$\mathbf{f}_{139}$	0 1 0 0 0 -1 1 0	$\mathbf{f}_{141}$	1 1 0 0 0 -1 1 0	$\mathbf{f}_{143}$	0 0 1 0 0 -1 1 0
$\mathbf{f}_{145}$	0 1 1 0 0 -1 1 0	$\mathbf{f}_{147}$	0 1 0 1 0 -1 1 0	$\mathbf{f}_{149}$	0 0 1 -1 0 -1 1 0
$\mathbf{f}_{151}$	0 1 1 -1 0 -1 1 0	$\mathbf{f}_{153}$	0 1 0 0 -1 -1 1 0	$\mathbf{f}_{155}$	1 1 0 0 -1 -1 1 0
$\mathbf{f}_{157}$	0 0 1 0 -1 -1 1 0	$\mathbf{f}_{159}$	0 1 1 0 -1 -1 1 0	$\mathbf{f}_{161}$	1 1 1 0 -1 -1 1 0
$\mathbf{f}_{163}$	0 0 1 -1 -1 -1 1 0	$\mathbf{f}_{165}$	1 0 1 -1 -1 -1 1 0	$\mathbf{f}_{167}$	0 1 1 -1 -1 -1 1 0
$\mathbf{f}_{169}$	1 1 1 -1 -1 -1 1 0	$\mathbf{f}_{171}$	0 0 2 -1 -1 -1 1 0	$\mathbf{f}_{173}$	0 1 2 -1 -1 -1 1 0
$\mathbf{f}_{175}$	0 1 1 0 -1 -2 1 0	$\mathbf{f}_{177}$	1 0 0 0 0 0 -1 0	$\mathbf{f}_{179}$	1 0 -1 0 0 0 -1 0
$\mathbf{f}_{181}$	0 0 0 1 0 0 -1 0	$\mathbf{f}_{183}$	1 0 0 0 -1 0 -1 0	$\mathbf{f}_{185}$	1 1 0 0 -1 0 -1 0
$\mathbf{f}_{187}$	0 0 1 0 -1 0 -1 0	$\mathbf{f}_{189}$	1 0 -1 0 -1 0 -1 0	$\mathbf{f}_{191}$	0 0 0 1 -1 0 -1 0
$\mathbf{f}_{193}$	1 0 0 -1 -1 0 -1 0	$\mathbf{f}_{195}$	1 -1 0 -1 -1 0 -1 0	$\mathbf{f}_{197}$	1 0 1 -1 -1 0 -1 0
$\mathbf{f}_{199}$	0 0 0 0 0 1 -1 0	$\mathbf{f}_{201}$	1 0 0 0 0 1 -1 0	$\mathbf{f}_{203}$	1 0 -1 0 0 1 -1 0
$\mathbf{f}_{205}$	1 -1 -1 0 0 1 -1 0	$\mathbf{f}_{207}$	0 0 0 1 0 1 -1 0	$\mathbf{f}_{209}$	1 0 -1 1 0 1 -1 0
$\mathbf{f}_{211}$	1 0 0 -1 0 1 -1 0	$\mathbf{f}_{213}$	1 -1 0 -1 0 1 -1 0	$\mathbf{f}_{215}$	1 0 0 0 -1 1 -1 0
$\mathbf{f}_{217}$	1 0 0 -1 -1 1 -1 0	$\mathbf{f}_{219}$	0 1 1 0 0 -1 2 0	$\mathbf{f}_{221}$	0 1 1 -1 0 -1 2 0
$\mathbf{f}_{223}$	0 0 0 0 0 0 0 1	$\mathbf{f}_{225}$	0 1 0 0 0 0 0 1	$\mathbf{f}_{227}$	1 1 0 0 0 0 0 1
$\mathbf{f}_{229}$	1 0 -1 0 0 0 0 1	$\mathbf{f}_{231}$	0 1 -1 0 0 0 0 1	$\mathbf{f}_{233}$	1 1 -1 0 0 0 0 1
$\mathbf{f}_{235}$	0 1 -1 1 0 0 0 1	$\mathbf{f}_{237}$	1 0 0 -1 0 0 0 1	$\mathbf{f}_{239}$	1 1 0 -1 0 0 0 1
$\mathbf{f}_{241}$	0 0 1 -1 0 0 0 1	$\mathbf{f}_{243}$	1 0 -1 -1 0 0 0 1	$\mathbf{f}_{245}$	0 0 0 0 1 0 0 1
$\mathbf{f}_{247}$	0 1 0 0 1 0 0 1	$\mathbf{f}_{249}$	0 1 -1 0 1 0 0 1	$\mathbf{f}_{251}$	1 1 -1 0 1 0 0 1

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Table 2. continued

$\mathbf{f}_{253}$	0 1 -1 1 1 0 0 1	$\mathbf{f}_{255}$	1 1 0 0 -1 0 0 1	$\mathbf{f}_{257}$	1 0 0 -1 -1 0 0 1
$\mathbf{f}_{259}$	1 1 0 -1 -1 0 0 1	$\mathbf{f}_{261}$	2 1 0 -1 -1 0 0 1	$\mathbf{f}_{263}$	1 1 1 -1 -1 0 0 1
$\mathbf{f}_{265}$	0 1 0 0 0 -1 0 1	$\mathbf{f}_{267}$	1 1 0 0 0 -1 0 1	$\mathbf{f}_{269}$	1 1 -1 0 0 -1 0 1
$\mathbf{f}_{271}$	0 1 0 1 0 -1 0 1	$\mathbf{f}_{273}$	0 1 -1 1 0 -1 0 1	$\mathbf{f}_{275}$	1 0 0 -1 0 -1 0 1
$\mathbf{f}_{277}$	1 1 0 -1 0 -1 0 1	$\mathbf{f}_{279}$	0 0 1 -1 0 -1 0 1	$\mathbf{f}_{281}$	0 1 0 0 -1 -1 0 1
$\mathbf{f}_{283}$	1 1 0 0 -1 -1 0 1	$\mathbf{f}_{285}$	1 2 0 0 -1 -1 0 1	$\mathbf{f}_{287}$	0 1 1 0 -1 -1 0 1
$\mathbf{f}_{289}$	1 0 0 -1 -1 -1 0 1	$\mathbf{f}_{291}$	1 1 0 -1 -1 -1 0 1	$\mathbf{f}_{293}$	2 1 0 -1 -1 -1 0 1
$\mathbf{f}_{295}$	0 0 1 -1 -1 -1 0 1	$\mathbf{f}_{297}$	1 0 1 -1 -1 -1 0 1	$\mathbf{f}_{299}$	0 1 1 -1 -1 -1 0 1
$\mathbf{f}_{301}$	1 1 1 -1 -1 -1 0 1	$\mathbf{f}_{303}$	1 0 1 -2 -1 -1 0 1	$\mathbf{f}_{305}$	1 1 1 -1 -2 -1 0 1
$\mathbf{f}_{307}$	1 1 0 -1 0 0 1 1	$\mathbf{f}_{309}$	0 1 0 0 1 0 1 1	$\mathbf{f}_{311}$	0 1 0 0 0 -1 1 1
$\mathbf{f}_{313}$	1 1 0 0 0 -1 1 1	$\mathbf{f}_{315}$	1 2 0 0 0 -1 1 1	$\mathbf{f}_{317}$	0 1 1 0 0 -1 1 1
$\mathbf{f}_{319}$	0 1 0 -1 0 -1 1 1	$\mathbf{f}_{321}$	1 1 0 -1 0 -1 1 1	$\mathbf{f}_{323}$	0 0 1 -1 0 -1 1 1
$\mathbf{f}_{325}$	0 1 1 -1 0 -1 1 1	$\mathbf{f}_{327}$	1 1 1 -1 0 -1 1 1	$\mathbf{f}_{329}$	0 1 0 0 1 -1 1 1
$\mathbf{f}_{331}$	1 1 0 -1 -1 -1 1 1	$\mathbf{f}_{333}$	0 1 1 -1 -1 -1 1 1	$\mathbf{f}_{335}$	1 1 1 -1 -1 -1 1 1
$\mathbf{f}_{337}$	1 2 1 -1 -1 -1 1 1	$\mathbf{f}_{339}$	1 1 1 -2 -1 -1 1 1	$\mathbf{f}_{341}$	0 1 1 -1 -1 -2 1 1
$\mathbf{f}_{343}$	1 1 1 -1 -1 -2 1 1	$\mathbf{f}_{345}$	1 0 0 0 0 0 -1 1	$\mathbf{f}_{347}$	1 1 0 0 0 0 -1 1
$\mathbf{f}_{349}$	1 0 -1 0 0 0 -1 1	$\mathbf{f}_{351}$	1 1 -1 0 0 0 -1 1	$\mathbf{f}_{353}$	0 1 -1 1 0 0 -1 1
$\mathbf{f}_{355}$	1 1 -1 1 0 0 -1 1	$\mathbf{f}_{357}$	1 0 0 -1 0 0 -1 1	$\mathbf{f}_{359}$	1 0 -1 -1 0 0 -1 1
$\mathbf{f}_{361}$	1 0 0 0 -1 0 -1 1	$\mathbf{f}_{363}$	1 1 0 0 -1 0 -1 1	$\mathbf{f}_{365}$	1 0 -1 0 -1 0 -1 1
$\mathbf{f}_{367}$	1 1 -1 0 -1 0 -1 1	$\mathbf{f}_{369}$	1 0 0 -1 -1 0 -1 1	$\mathbf{f}_{371}$	1 1 0 -1 -1 0 -1 1
$\mathbf{f}_{373}$	2 1 0 -1 -1 0 -1 1	$\mathbf{f}_{375}$	1 0 -1 0 0 1 -1 1	$\mathbf{f}_{377}$	1 1 -1 0 0 1 -1 1
$\mathbf{f}_{379}$	1 0 0 -1 0 1 -1 1	$\mathbf{f}_{381}$	1 0 -1 -1 0 1 -1 1	$\mathbf{f}_{383}$	1 0 -1 0 1 1 -1 1
$\mathbf{f}_{385}$	1 1 0 0 -1 -1 -1 1	$\mathbf{f}_{387}$	1 0 0 -1 -1 -1 -1 1	$\mathbf{f}_{389}$	1 0 -1 0 0 1 -2 1
$\mathbf{f}_{391}$	0 0 1 0 0 0 0 -1	$\mathbf{f}_{393}$	0 0 0 1 0 0 0 -1	$\mathbf{f}_{395}$	1 0 0 0 -1 0 0 -1
$\mathbf{f}_{397}$	0 0 1 0 -1 0 0 -1	$\mathbf{f}_{399}$	0 0 0 1 -1 0 0 -1	$\mathbf{f}_{401}$	0 0 1 -1 -1 0 0 -1
$\mathbf{f}_{403}$	1 0 1 -1 -1 0 0 -1	$\mathbf{f}_{405}$	0 0 0 0 0 1 0 -1	$\mathbf{f}_{407}$	1 0 0 0 0 1 0 -1
$\mathbf{f}_{409}$	0 0 1 0 0 1 0 -1	$\mathbf{f}_{411}$	0 0 0 1 0 1 0 -1	$\mathbf{f}_{413}$	0 0 0 0 1 1 0 -1
$\mathbf{f}_{415}$	0 0 0 1 1 1 0 -1	$\mathbf{f}_{417}$	0 0 1 0 -1 -1 0 -1	$\mathbf{f}_{419}$	0 0 0 0 0 0 1 -1
$\mathbf{f}_{421}$	0 0 1 0 0 0 1 -1	$\mathbf{f}_{423}$	0 1 1 0 0 0 1 -1	$\mathbf{f}_{425}$	0 0 0 1 0 0 1 -1
$\mathbf{f}_{427}$	0 1 0 1 0 0 1 -1	$\mathbf{f}_{429}$	0 0 1 -1 0 0 1 -1	$\mathbf{f}_{431}$	0 0 0 0 1 0 1 -1
$\mathbf{f}_{433}$	0 0 1 0 -1 0 1 -1	$\mathbf{f}_{435}$	0 0 1 -1 -1 0 1 -1	$\mathbf{f}_{437}$	0 0 0 0 1 1 1 -1
$\mathbf{f}_{439}$	0 0 1 0 0 -1 1 -1	$\mathbf{f}_{441}$	0 0 1 0 -1 -1 1 -1	$\mathbf{f}_{443}$	0 1 1 0 -1 -1 1 -1
$\mathbf{f}_{445}$	0 0 1 -1 -1 -1 1 -1	$\mathbf{f}_{447}$	0 0 2 -1 -1 -1 1 -1	$\mathbf{f}_{449}$	0 0 0 1 0 1 -1 -1
$\mathbf{f}_{451}$	1 1 0 -1 0 -1 0 2	$\mathbf{f}_{453}$	1 1 0 -1 -1 -1 0 2	$\mathbf{f}_{455}$	1 1 -1 0 0 0 -1 2

The condition for a zone vector $\mathbf{z}_k^*$ to be extendable is given by

$$L_i(\mathbf{z}_k^*) := \mathbf{f}_j \mathbf{z}_k^*, \quad i \in \{-1, 0, 1\}, \quad \forall \mathbf{f}_j \in \mathcal{F}_{F_{22}} \tag{2}$$

It is readily verified that for all zone vectors $\mathbf{z}_k^*$ with components $\mid z_l \mid \leq 4, l = 1, ..., 8$, the facet vectors $\mathbf{f}_j \in \mathcal{F}_{F_{22}}$ belong to layers $\mathsf{L}_i(\mathbf{z}_k^*)$, $-n \leq i \leq n$, where $n \geq 2$. Therefore, $\mathsf{P}(\mathsf{Q}_{F_{22}})$ is as much minimal as maximal.

The 456 facet vectors of $\mathsf{P}(\mathsf{Q}_{F_{22}})$ belong to three equivalence classes of norm 6, 8, and 10, respectively. In order to obtain the group $\mathcal{G}_{F_{22}}$, the unified polytope scheme, described in [2], was used. Each identical scheme induces a permutation S of the facet vectors under their automorphism group. There exists an isomorphism between the class of identical schemes and the group of automorphisms of $\mathsf{P}(\mathsf{Q}_{F_{22}})$.

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Below are given generating rotations of order 6 (S_1), 8 (S_2), and 12 (S_3), respectively, which generate the group $\mathcal{G}_{F_{22}}$ of order 3'456. The group contains only pure rotations.

$$\mathsf{S}_1 := \begin{pmatrix} 0 & -1 & 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathsf{S}_2 := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & -1 & 0 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \end{pmatrix},$$

$$\mathsf{S}_3 := \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & 0 \end{pmatrix}.$$

It holds:

$$\mathsf{S}_i^t \mathsf{Q}_{F_{22}} \mathsf{S}_i = \mathsf{Q}_{F_{22}}, \quad \forall \mathsf{S}_i \in \mathcal{G}_{F_{22}}. \tag{3}$$

3. The neighbourhood of $\mathsf{Q}_{F_{22}}$

In the neighourhood of $\mathsf{Q}_{F_{22}}$, a form Q_1 was found, whose parallelohedron is primitive and therefore, has the maximal number 510 of facets, as well as 362'304 vertices, and 761 zones all of which are open:

$$\mathsf{Q}_1 :=$$

$$\begin{pmatrix} 6.05228 & -3.02808 & 1.83912 & 2.92092 & 1.83868 & -0.86624 & 0.86536 & -0.10748 \\ -3.02808 & 6.05256 & -0.86504 & -2.92108 & 1.08004 & -0.10812 & -2.81348 & -2.91876 \\ 1.83912 & -0.86504 & 6.05272 & 2.81244 & 2.05364 & 0.97484 & -0.86644 & 2.05432 \\ 2.92092 & -2.92108 & 2.81244 & 6.05292 & -0.97332 & 0.86608 & 1.83976 & 2.81120 \\ 1.83868 & 1.08004 & 2.05364 & -0.97332 & 6.05352 & -3.02916 & -2.92188 & -1.94620 \\ -0.86624 & -0.10812 & 0.97484 & 0.86608 & -3.02916 & 6.05368 & 2.91984 & 2.92224 \\ 0.86536 & -2.81348 & -0.86644 & 1.83976 & -2.92188 & 2.91984 & 6.05396 & 3.02740 \\ -0.10748 & -2.91876 & 2.05432 & 2.81120 & -1.94620 & 2.92224 & 3.02740 & 6.05412 \end{pmatrix}.$$

The Gram matix Q_1 defines the Σ_0^1-subcone. For each Q in the open Σ_0^1-subcone, the set of facet vectors $\mathcal{F}_{\mathsf{Q}_1}$ is an invariant of Σ_0^1 and it is denoted by $\mathcal{F}_{\Sigma_0^1}$. The facet vectors are given in Table 3. Since $\mathsf{P}(\mathsf{Q}_1)$ is generic it has the maximal number of facets. Verifying equation (2) for all zone vectors $\mathbf{z}_k^*$ with components $\mid z_l \mid \leq 4$, it results that the facet vectors $\mathbf{f}_j \in \mathcal{F}_{\Sigma_0^1}$, lie in layers $L_i(\mathbf{z}_k^*)$, $i \in \{-n \leq i \leq n\}$, where $n \geq 2$. Thus, $\mathsf{P}(\mathsf{Q}_1)$ is as much minimal as maximal.

Table 3. The facet vectors $\pm\mathbf{f}_i$ of $\mathsf{P}(\mathsf{Q}_1)$

$\mathbf{f}_{1}$	1 0 0 0 0 0 0 0	$\mathbf{f}_{3}$	0 1 0 0 0 0 0 0	$\mathbf{f}_{5}$	1 1 0 0 0 0 0 0
$\mathbf{f}_{7}$	0 0 1 0 0 0 0 0	$\mathbf{f}_{9}$	0 1 1 0 0 0 0 0	$\mathbf{f}_{11}$	1 0 -1 0 0 0 0 0
$\mathbf{f}_{13}$	1 1 -1 0 0 0 0 0	$\mathbf{f}_{15}$	0 0 0 1 0 0 0 0	$\mathbf{f}_{17}$	0 1 0 1 0 0 0 0
$\mathbf{f}_{19}$	1 1 0 1 0 0 0 0	$\mathbf{f}_{21}$	0 1 -1 1 0 0 0 0	$\mathbf{f}_{23}$	1 1 -1 1 0 0 0 0
$\mathbf{f}_{25}$	1 0 0 -1 0 0 0 0	$\mathbf{f}_{27}$	0 0 1 -1 0 0 0 0	$\mathbf{f}_{29}$	1 0 1 -1 0 0 0 0
$\mathbf{f}_{31}$	0 0 0 0 1 0 0 0	$\mathbf{f}_{33}$	0 0 0 1 1 0 0 0	$\mathbf{f}_{35}$	0 1 -1 1 1 0 0 0
$\mathbf{f}_{37}$	1 0 0 0 -1 0 0 0	$\mathbf{f}_{39}$	0 1 0 0 -1 0 0 0	$\mathbf{f}_{41}$	1 1 0 0 -1 0 0 0
$\mathbf{f}_{43}$	0 0 1 0 -1 0 0 0	$\mathbf{f}_{45}$	0 1 1 0 -1 0 0 0	$\mathbf{f}_{47}$	1 1 1 0 -1 0 0 0
$\mathbf{f}_{49}$	0 1 0 1 -1 0 0 0	$\mathbf{f}_{51}$	1 0 0 -1 -1 0 0 0	$\mathbf{f}_{53}$	1 1 0 -1 -1 0 0 0
$\mathbf{f}_{55}$	0 0 1 -1 -1 0 0 0	$\mathbf{f}_{57}$	1 0 1 -1 -1 0 0 0	$\mathbf{f}_{59}$	1 1 1 -1 -1 0 0 0
$\mathbf{f}_{61}$	1 0 1 -2 -1 0 0 0	$\mathbf{f}_{63}$	0 0 0 0 0 1 0 0	$\mathbf{f}_{65}$	1 0 0 0 0 1 0 0
$\mathbf{f}_{67}$	0 1 0 0 0 1 0 0	$\mathbf{f}_{69}$	1 1 0 0 0 1 0 0	$\mathbf{f}_{71}$	1 0 -1 0 0 1 0 0
$\mathbf{f}_{73}$	1 1 -1 0 0 1 0 0	$\mathbf{f}_{75}$	1 0 0 -1 0 1 0 0	$\mathbf{f}_{77}$	0 0 0 0 1 1 0 0
$\mathbf{f}_{79}$	1 0 -1 0 1 1 0 0	$\mathbf{f}_{81}$	0 0 0 1 1 1 0 0	$\mathbf{f}_{83}$	0 1 -1 1 1 1 0 0
$\mathbf{f}_{85}$	0 0 1 0 0 -1 0 0	$\mathbf{f}_{87}$	0 0 0 1 0 -1 0 0	$\mathbf{f}_{89}$	0 1 0 1 0 -1 0 0
$\mathbf{f}_{91}$	1 0 0 0 -1 -1 0 0	$\mathbf{f}_{93}$	0 1 0 0 -1 -1 0 0	$\mathbf{f}_{95}$	1 1 0 0 -1 -1 0 0
$\mathbf{f}_{97}$	0 0 1 0 -1 -1 0 0	$\mathbf{f}_{99}$	0 1 1 0 -1 -1 0 0	$\mathbf{f}_{101}$	1 1 1 0 -1 -1 0 0
$\mathbf{f}_{103}$	0 1 0 1 -1 -1 0 0	$\mathbf{f}_{105}$	1 0 0 -1 -1 -1 0 0	$\mathbf{f}_{107}$	0 0 1 -1 -1 -1 0 0
$\mathbf{f}_{109}$	1 0 1 -1 -1 -1 0 0	$\mathbf{f}_{111}$	1 1 1 -1 -1 -1 0 0	$\mathbf{f}_{113}$	0 0 1 -1 -2 -1 0 0
$\mathbf{f}_{115}$	1 0 1 -1 -2 -1 0 0	$\mathbf{f}_{117}$	1 1 1 -1 -2 -1 0 0	$\mathbf{f}_{119}$	0 0 0 0 0 0 1 0
$\mathbf{f}_{121}$	0 1 0 0 0 0 1 0	$\mathbf{f}_{123}$	1 1 0 0 0 0 1 0	$\mathbf{f}_{125}$	0 0 1 0 0 0 1 0
$\mathbf{f}_{127}$	0 1 1 0 0 0 1 0	$\mathbf{f}_{129}$	0 1 0 1 0 0 1 0	$\mathbf{f}_{131}$	1 0 0 -1 0 0 1 0
$\mathbf{f}_{133}$	1 1 0 -1 0 0 1 0	$\mathbf{f}_{135}$	0 0 1 -1 0 0 1 0	$\mathbf{f}_{137}$	1 0 1 -1 0 0 1 0
$\mathbf{f}_{139}$	0 1 1 -1 0 0 1 0	$\mathbf{f}_{141}$	1 1 1 -1 0 0 1 0	$\mathbf{f}_{143}$	0 0 0 0 1 0 1 0
$\mathbf{f}_{145}$	0 1 0 0 1 0 1 0	$\mathbf{f}_{147}$	1 1 -1 0 1 0 1 0	$\mathbf{f}_{149}$	0 1 0 1 1 0 1 0
$\mathbf{f}_{151}$	0 1 -1 1 1 0 1 0	$\mathbf{f}_{153}$	1 1 1 -1 -1 0 1 0	$\mathbf{f}_{155}$	0 0 0 0 1 1 1 0
$\mathbf{f}_{157}$	0 1 0 0 0 -1 1 0	$\mathbf{f}_{159}$	1 1 0 0 0 -1 1 0	$\mathbf{f}_{161}$	0 0 1 0 0 -1 1 0
$\mathbf{f}_{163}$	0 1 1 0 0 -1 1 0	$\mathbf{f}_{165}$	0 1 0 1 0 -1 1 0	$\mathbf{f}_{167}$	0 0 1 -1 0 -1 1 0
$\mathbf{f}_{169}$	0 1 1 -1 0 -1 1 0	$\mathbf{f}_{171}$	0 1 0 0 -1 -1 1 0	$\mathbf{f}_{173}$	1 1 0 0 -1 -1 1 0
$\mathbf{f}_{175}$	0 0 1 0 -1 -1 1 0	$\mathbf{f}_{177}$	0 1 1 0 -1 -1 1 0	$\mathbf{f}_{179}$	1 1 1 0 -1 -1 1 0
$\mathbf{f}_{181}$	0 0 1 -1 -1 -1 1 0	$\mathbf{f}_{183}$	1 0 1 -1 -1 -1 1 0	$\mathbf{f}_{185}$	0 1 1 -1 -1 -1 1 0
$\mathbf{f}_{187}$	1 1 1 -1 -1 -1 1 0	$\mathbf{f}_{189}$	0 0 2 -1 -1 -1 1 0	$\mathbf{f}_{191}$	0 1 2 -1 -1 -1 1 0
$\mathbf{f}_{193}$	0 1 1 0 -1 -2 1 0	$\mathbf{f}_{195}$	1 0 0 0 0 0 -1 0	$\mathbf{f}_{197}$	1 0 -1 0 0 0 -1 0
$\mathbf{f}_{199}$	0 0 0 1 0 0 -1 0	$\mathbf{f}_{201}$	1 0 0 0 -1 0 -1 0	$\mathbf{f}_{203}$	1 1 0 0 -1 0 -1 0
$\mathbf{f}_{205}$	0 0 1 0 -1 0 -1 0	$\mathbf{f}_{207}$	1 0 -1 0 -1 0 -1 0	$\mathbf{f}_{209}$	0 0 0 1 -1 0 -1 0
$\mathbf{f}_{211}$	1 0 0 -1 -1 0 -1 0	$\mathbf{f}_{213}$	1 -1 0 -1 -1 0 -1 0	$\mathbf{f}_{215}$	0 0 1 -1 -1 0 -1 0
$\mathbf{f}_{217}$	1 0 1 -1 -1 0 -1 0	$\mathbf{f}_{219}$	0 0 0 0 0 1 -1 0	$\mathbf{f}_{221}$	1 0 0 0 0 1 -1 0
$\mathbf{f}_{223}$	1 0 -1 0 0 1 -1 0	$\mathbf{f}_{225}$	1 -1 -1 0 0 1 -1 0	$\mathbf{f}_{227}$	0 0 0 1 0 1 -1 0
$\mathbf{f}_{229}$	1 0 -1 1 0 1 -1 0	$\mathbf{f}_{231}$	1 1 -1 1 0 1 -1 0	$\mathbf{f}_{233}$	1 0 0 -1 0 1 -1 0
$\mathbf{f}_{235}$	1 -1 0 -1 0 1 -1 0	$\mathbf{f}_{237}$	1 0 -1 0 1 1 -1 0	$\mathbf{f}_{239}$	1 0 0 0 -1 1 -1 0
$\mathbf{f}_{241}$	1 0 0 -1 -1 1 -1 0	$\mathbf{f}_{243}$	0 1 1 0 0 -1 2 0	$\mathbf{f}_{245}$	0 1 1 -1 0 -1 2 0
$\mathbf{f}_{247}$	0 0 0 0 0 0 0 1	$\mathbf{f}_{249}$	1 0 0 0 0 0 0 1	$\mathbf{f}_{251}$	0 1 0 0 0 0 0 1
$\mathbf{f}_{253}$	1 1 0 0 0 0 0 1	$\mathbf{f}_{255}$	1 0 -1 0 0 0 0 1	$\mathbf{f}_{257}$	0 1 -1 0 0 0 0 1
$\mathbf{f}_{259}$	1 1 -1 0 0 0 0 1	$\mathbf{f}_{261}$	0 1 0 1 0 0 0 1	$\mathbf{f}_{263}$	0 1 -1 1 0 0 0 1
$\mathbf{f}_{265}$	1 1 -1 1 0 0 0 1	$\mathbf{f}_{267}$	1 0 0 -1 0 0 0 1	$\mathbf{f}_{269}$	1 1 0 -1 0 0 0 1
$\mathbf{f}_{271}$	0 0 1 -1 0 0 0 1	$\mathbf{f}_{273}$	1 0 -1 -1 0 0 0 1	$\mathbf{f}_{275}$	0 0 0 0 1 0 0 1
$\mathbf{f}_{277}$	0 1 0 0 1 0 0 1	$\mathbf{f}_{279}$	1 0 -1 0 1 0 0 1	$\mathbf{f}_{281}$	0 1 -1 0 1 0 0 1
$\mathbf{f}_{283}$	1 1 -1 0 1 0 0 1	$\mathbf{f}_{285}$	0 1 -1 1 1 0 0 1	$\mathbf{f}_{287}$	1 1 0 0 -1 0 0 1
$\mathbf{f}_{289}$	1 0 0 -1 -1 0 0 1	$\mathbf{f}_{291}$	1 1 0 -1 -1 0 0 1	$\mathbf{f}_{293}$	2 1 0 -1 -1 0 0 1
$\mathbf{f}_{295}$	1 1 1 -1 -1 0 0 1	$\mathbf{f}_{297}$	1 1 -1 0 1 1 0 1	$\mathbf{f}_{299}$	0 1 0 0 0 -1 0 1
$\mathbf{f}_{301}$	1 1 0 0 0 -1 0 1	$\mathbf{f}_{303}$	0 1 -1 0 0 -1 0 1	$\mathbf{f}_{305}$	1 1 -1 0 0 -1 0 1
$\mathbf{f}_{307}$	0 1 0 1 0 -1 0 1	$\mathbf{f}_{309}$	0 1 -1 1 0 -1 0 1	$\mathbf{f}_{311}$	1 0 0 -1 0 -1 0 1
$\mathbf{f}_{313}$	1 1 0 -1 0 -1 0 1	$\mathbf{f}_{315}$	0 0 1 -1 0 -1 0 1	$\mathbf{f}_{317}$	1 0 -1 -1 0 -1 0 1
$\mathbf{f}_{319}$	0 1 0 0 -1 -1 0 1	$\mathbf{f}_{321}$	1 1 0 0 -1 -1 0 1	$\mathbf{f}_{323}$	1 2 0 0 -1 -1 0 1
$\mathbf{f}_{325}$	0 1 1 0 -1 -1 0 1	$\mathbf{f}_{327}$	1 0 0 -1 -1 -1 0 1	$\mathbf{f}_{329}$	1 1 0 -1 -1 -1 0 1
$\mathbf{f}_{331}$	2 1 0 -1 -1 -1 0 1	$\mathbf{f}_{333}$	0 0 1 -1 -1 -1 0 1	$\mathbf{f}_{335}$	1 0 1 -1 -1 -1 0 1

Table 3. continued

$\mathbf{f}_{337}$	0 1 1 -1 -1 -1 0 1	$\mathbf{f}_{339}$	1 1 1 -1 -1 -1 0 1	$\mathbf{f}_{341}$	1 0 1 -2 -1 -1 0 1
$\mathbf{f}_{343}$	1 1 1 -1 -2 -1 0 1	$\mathbf{f}_{345}$	1 1 0 -1 0 0 1 1	$\mathbf{f}_{347}$	0 1 0 0 1 0 1 1
$\mathbf{f}_{349}$	0 1 -1 0 1 0 1 1	$\mathbf{f}_{351}$	0 1 0 0 0 -1 1 1	$\mathbf{f}_{353}$	1 1 0 0 0 -1 1 1
$\mathbf{f}_{355}$	1 2 0 0 0 -1 1 1	$\mathbf{f}_{357}$	0 1 1 0 0 -1 1 1	$\mathbf{f}_{359}$	0 1 0 -1 0 -1 1 1
$\mathbf{f}_{361}$	1 1 0 -1 0 -1 1 1	$\mathbf{f}_{363}$	0 0 1 -1 0 -1 1 1	$\mathbf{f}_{365}$	0 1 1 -1 0 -1 1 1
$\mathbf{f}_{367}$	1 1 1 -1 0 -1 1 1	$\mathbf{f}_{369}$	0 1 0 0 1 -1 1 1	$\mathbf{f}_{371}$	1 1 0 -1 -1 -1 1 1
$\mathbf{f}_{373}$	0 1 1 -1 -1 -1 1 1	$\mathbf{f}_{375}$	1 1 1 -1 -1 -1 1 1	$\mathbf{f}_{377}$	1 2 1 -1 -1 -1 1 1
$\mathbf{f}_{379}$	1 1 1 -2 -1 -1 1 1	$\mathbf{f}_{381}$	0 1 1 -1 -1 -2 1 1	$\mathbf{f}_{383}$	1 1 1 -1 -1 -2 1 1
$\mathbf{f}_{385}$	1 0 0 0 0 0 -1 1	$\mathbf{f}_{387}$	0 1 0 0 0 0 -1 1	$\mathbf{f}_{389}$	1 1 0 0 0 0 -1 1
$\mathbf{f}_{391}$	1 0 -1 0 0 0 -1 1	$\mathbf{f}_{393}$	1 1 -1 0 0 0 -1 1	$\mathbf{f}_{395}$	0 1 -1 1 0 0 -1 1
$\mathbf{f}_{397}$	1 1 -1 1 0 0 -1 1	$\mathbf{f}_{399}$	1 0 0 -1 0 0 -1 1	$\mathbf{f}_{401}$	1 0 -1 -1 0 0 -1 1
$\mathbf{f}_{403}$	1 0 0 0 -1 0 -1 1	$\mathbf{f}_{405}$	1 1 0 0 -1 0 -1 1	$\mathbf{f}_{407}$	1 0 -1 0 -1 0 -1 1
$\mathbf{f}_{409}$	1 1 -1 0 -1 0 -1 1	$\mathbf{f}_{411}$	1 0 0 -1 -1 0 -1 1	$\mathbf{f}_{413}$	1 1 0 -1 -1 0 -1 1
$\mathbf{f}_{415}$	2 1 0 -1 -1 0 -1 1	$\mathbf{f}_{417}$	1 0 -1 0 0 1 -1 1	$\mathbf{f}_{419}$	1 1 -1 0 0 1 -1 1
$\mathbf{f}_{421}$	1 0 0 -1 0 1 -1 1	$\mathbf{f}_{423}$	1 0 -1 -1 0 1 -1 1	$\mathbf{f}_{425}$	1 0 -1 0 1 1 -1 1
$\mathbf{f}_{427}$	1 0 0 0 -1 -1 -1 1	$\mathbf{f}_{429}$	1 1 0 0 -1 -1 -1 1	$\mathbf{f}_{431}$	1 0 0 -1 -1 -1 -1 1
$\mathbf{f}_{433}$	1 0 -1 0 0 1 -2 1	$\mathbf{f}_{435}$	0 0 1 0 0 0 0 -1	$\mathbf{f}_{437}$	0 0 0 1 0 0 0 -1
$\mathbf{f}_{439}$	1 0 0 0 -1 0 0 -1	$\mathbf{f}_{441}$	0 0 1 0 -1 0 0 -1	$\mathbf{f}_{443}$	0 0 0 1 -1 0 0 -1
$\mathbf{f}_{445}$	0 0 1 -1 -1 0 0 -1	$\mathbf{f}_{447}$	1 0 1 -1 -1 0 0 -1	$\mathbf{f}_{449}$	0 0 0 0 0 1 0 -1
$\mathbf{f}_{451}$	1 0 0 0 0 1 0 -1	$\mathbf{f}_{453}$	0 0 1 0 0 1 0 -1	$\mathbf{f}_{455}$	0 0 0 1 0 1 0 -1
$\mathbf{f}_{457}$	0 0 0 0 1 1 0 -1	$\mathbf{f}_{459}$	0 0 0 1 1 1 0 -1	$\mathbf{f}_{461}$	0 0 1 0 -1 -1 0 -1
$\mathbf{f}_{463}$	0 0 0 0 0 0 1 -1	$\mathbf{f}_{465}$	0 0 1 0 0 0 1 -1	$\mathbf{f}_{467}$	0 1 1 0 0 0 1 -1
$\mathbf{f}_{469}$	0 0 0 1 0 0 1 -1	$\mathbf{f}_{471}$	0 1 0 1 0 0 1 -1	$\mathbf{f}_{473}$	0 0 1 -1 0 0 1 -1
$\mathbf{f}_{475}$	0 0 0 0 1 0 1 -1	$\mathbf{f}_{477}$	0 0 0 1 1 0 1 -1	$\mathbf{f}_{479}$	0 0 1 0 -1 0 1 -1
$\mathbf{f}_{481}$	0 0 1 -1 -1 0 1 -1	$\mathbf{f}_{483}$	1 0 1 -1 -1 0 1 -1	$\mathbf{f}_{485}$	0 0 0 0 0 1 1 -1
$\mathbf{f}_{487}$	0 0 0 0 1 1 1 -1	$\mathbf{f}_{489}$	0 1 0 1 1 1 1 -1	$\mathbf{f}_{491}$	0 0 1 0 0 -1 1 -1
$\mathbf{f}_{493}$	0 0 1 0 -1 -1 1 -1	$\mathbf{f}_{495}$	0 1 1 0 -1 -1 1 -1	$\mathbf{f}_{497}$	0 0 1 -1 -1 -1 1 -1
$\mathbf{f}_{499}$	0 0 2 -1 -1 -1 1 -1	$\mathbf{f}_{501}$	0 0 0 1 0 1 -1 -1	$\mathbf{f}_{503}$	1 1 0 -1 0 -1 0 2
$\mathbf{f}_{505}$	1 1 0 -1 -1 -1 0 2	$\mathbf{f}_{507}$	1 1 -1 0 0 0 -1 2	$\mathbf{f}_{509}$	1 1 0 -1 -1 -1 -1 2

In order to calculate the Σ_0^1-subcone, all tripletts

$$\mathbf{f}_i + \mathbf{f}_j + \mathbf{f}_k = 0, \quad \mathbf{f}_i, \mathbf{f}_j, \mathbf{f}_k \in \mathcal{F}_{\Sigma_0^1},$$

are determined. Their number is N_b=3'793 and the number of half-spaces H_i becomes $3N_b$. The closed subcone Σ_0^1 with apex in $\widetilde{O}$ becomes

$$\Sigma_0^1 := \bigcap_{i=1}^{3N_b} \mathsf{H}_i. \tag{4}$$

Because of the heigh complexity of the Σ_0^1-subcone, the direct computation of Σ_0^1 as well as of its Φ-subcones is not practicable with small computers. Instead, existence charts may be defined which intersect the Σ_0^1-subcone.

Using condition

$$\mathbf{n}_i^t \mathbf{q} > 0, \quad i = 1, \cdots, 3N_b. \tag{5}$$

any further form $\mathsf{Q}_2 \in \Sigma_0^1$ can be found viz.:

$\mathsf{Q}_2 :=$

$$\begin{pmatrix} 6.05104 & -3.02844 & 1.83868 & 2.78892 & 1.82188 & -0.87588 & 0.86456 & -0.10868 \\ -3.02844 & 6.07416 & -0.86660 & -2.87348 & 1.07004 & -0.11692 & -2.81192 & -2.91592 \\ 1.83868 & -0.86660 & 6.14472 & 2.83496 & 2.06568 & 0.96604 & -0.86968 & 2.05112 \\ 2.78892 & -2.87348 & 2.83496 & 6.17292 & -0.96612 & 0.85528 & 1.83936 & 2.81088 \\ 1.82188 & 1.07004 & 2.06568 & -0.96612 & 6.05596 & -3.03876 & -2.92208 & -1.94636 \\ -0.87588 & -0.11692 & 0.96604 & 0.85528 & -3.03876 & 6.06728 & 2.90464 & 2.91460 \\ 0.86456 & -2.81192 & -0.86968 & 1.83936 & -2.92208 & 2.90464 & 6.06596 & 3.02140 \\ -0.10868 & -2.91592 & 2.05112 & 2.81088 & -1.94636 & 2.91460 & 3.02140 & 6.04612 \end{pmatrix}.$$

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The parallelohedron $\mathsf{P}(\mathsf{Q}_2)$ is primitive and therefore, has the maximal number 510 of facets, as well as 362'160 vertices, and 778 zones all of which are open, and it proves that $\mathsf{Q}_{F_{22}}$ lies on the boundary of Σ_0^1.

The three forms

$$\mathsf{Q}_{F_{22}},\ \mathsf{Q}_1,\ \mathsf{Q}_2 \in \Sigma_0^1$$

determine a 2-section Π^2 through the Σ_0^1-subcone. with origin in $\mathsf{Q}_{F_{22}}$ and cartesian coordinates $\overline{\mathbf{x}}_1$, and $\overline{\mathbf{x}}_2$, respectively. Each $\mathsf{Q} \in \Pi^2$ is obtained by

$$\mathsf{Q} = \mathsf{Q}_{F_{22}} + \sum_{i=1}^{2} \lambda_i \overline{\mathbf{x}}_i, \quad \lambda_i \in \mathbb{R}.$$

The condition (5) is used to decide if Q is interior to a Σ_0^k-subcone. Considering that $\mathcal{F}_{\Sigma_0^k}$ is an invariant for Σ_0^k, the parallelohedron $\mathsf{P}(\mathsf{Q})$ is easily computed.

In what follows, the surrounding of the Σ_0^1-subcone in the section Π^2 is investigated.

First, all walls W_k containing $\mathsf{Q}_{F_{22}}$ are determined along the four straight line segments

$$\begin{array}{ll} 0.1\,\overline{\mathbf{x}}_1 + \lambda_2\,\overline{\mathbf{x}}_2 & \\ -0.1\,\overline{\mathbf{x}}_1 + \lambda_2\,\overline{\mathbf{x}}_2 & \\ \lambda_1\,\overline{\mathbf{x}}_1 + 0.1\,\overline{\mathbf{x}}_2 & -0.1 \le \lambda_1,\, \lambda_2 \le 0.1, \\ \lambda_1\,\overline{\mathbf{x}}_1 - 0.1\,\overline{\mathbf{x}}_2 & \end{array}$$

forming a quadrangle arround the center $\mathsf{Q}_{F_{22}}$. Applying the method of nested intervals, the border point Q_k between two neighbouring Σ_0^k- and Σ_0^{k+1}-subcones is easily determind within an ϵ-limit. The value of ϵ is chosen to be

$$\epsilon = det(\mathsf{Q})\,10^{-10}.$$

The border point Q_k corresponds to a limiting parallelohedron $\mathsf{P}(\mathsf{Q}_k)$ which is characterized by haveing some vertex $\mathbf{v} \in \mathsf{P}(\mathsf{Q}_k)$ where at least d+1 facets meet. The wall W_k contains Q_k and the origin $\widetilde{O}$. Thus, the wall normal $\mathbf{n}_k$ is easily calculated.

It proves that there exists a highly complex cluster $\mathsf{C}_{F_{22}}$ of Σ_0^k-subcones. In Figure 1 is shown the section through the $\mathsf{C}_{F_{22}}$-cluster. Only the border lines of the Σ_0^k-subcones are drawn. Each border line corresponds to a wall W_h having wall normal $\mathbf{n}_h$ of some Σ_0^k-subcone, and therefore, has a representation as a tensor product

$$\mathbf{n}_h = \mathbf{f}_a \otimes \mathbf{f}_b, \quad \mathbf{f}_a, \mathbf{f}_b \in \mathcal{F}_{\Sigma_0^k}.$$

There exist two kinds of walls. Walls $\mathsf{W}^{(i)}$ that separate two adjacent Σ_0-subcones, and walls $\mathsf{W}^{(a)}$ that separate a Σ_0-subcone from a adjacent Σ_1-subcone. In order to determine all walls, it would require to compute a huge number of Φ-subcones which at present is far out of reach.

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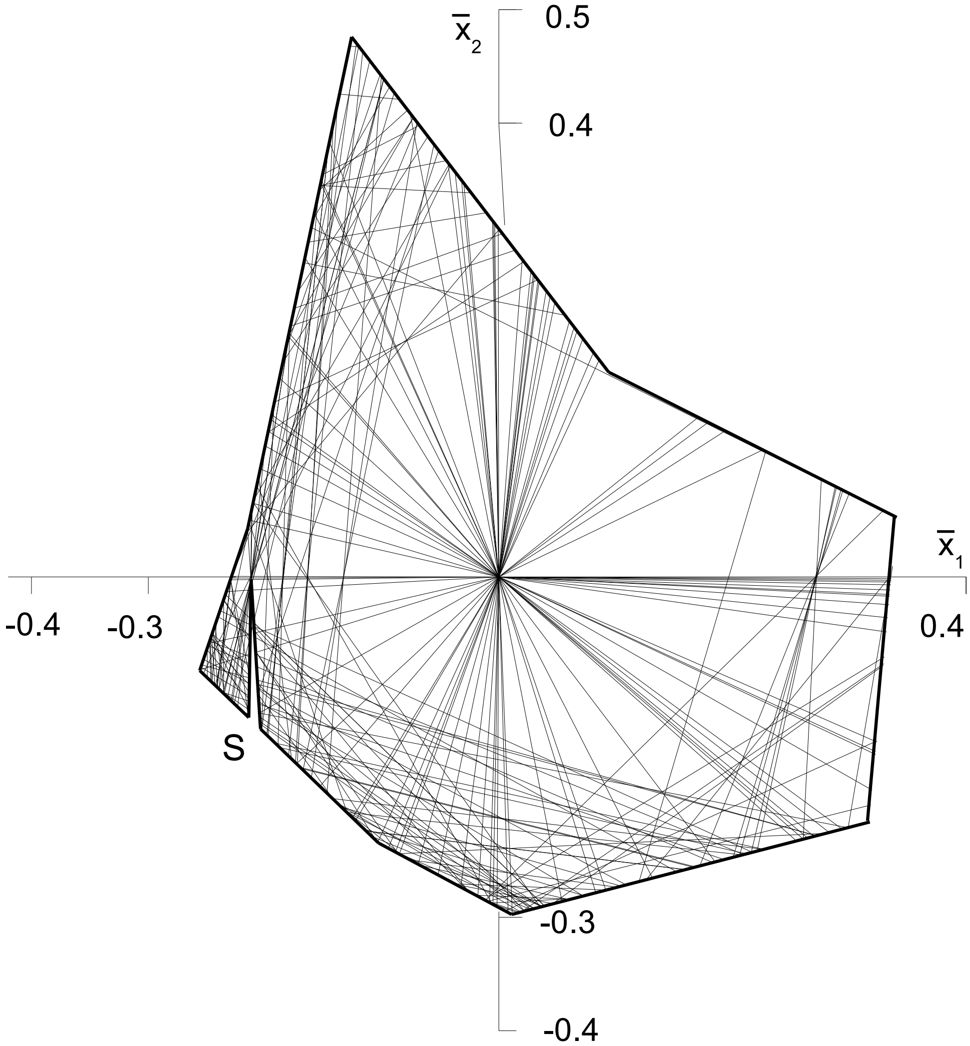

Figure 1. Existence chart through the $\mathsf{C}_{F_{22}}$-cluster in E^8

The walls containing the center $\mathsf{Q}_{F_{22}}$ belong to three non-equivalent wall classes $\mathcal{W}_h$, $h = 1, 2, 3$, under the group $\mathcal{G}_{F_{22}}$ which are shown in Table 4.

Table 4. The wall classes $\mathcal{W}_h$, $h = 1, 2, 3$, of walls containing $\mathsf{Q}_{F_{22}}$

Class	Representative wall normal $\mathbf{n}_j$	Order	Intersecting Π^2
$\mathcal{W}_1$	(1 0 0 0 0 0 0 0) $\otimes$ (0 0 0 0 0 0 0 1)	288	84
$\mathcal{W}_2$	(0 1 0 -1 1 -1 1 1) $\otimes$ (0 0 0 0 1 0 0 0)	288	13
$\mathcal{W}_3$	(1 0 -1 0 0 0 0 0) $\otimes$ (0 0 0 1 0 1 0 -1)	108	5
Total		684	102

A representative wall normal $\mathbf{n}_j$ of $\mathsf{W}_j \in \mathcal{W}_h$ is given by a tensor product

$$\mathbf{n}_j := (\mathbf{f}_a \otimes \mathbf{f}_b), \quad \mathbf{f}_a, \mathbf{f}_b \in \mathcal{F}_{\Sigma_0^k}.$$

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Although the walls $W_i \in \mathcal{W}_h$ are equivalent, the Σ_0^i-subcones in the $C_{F_{22}}$-cluster are not equivalent because the section Π^2 generally contains only one element of the orbit of some $Q \in \Sigma_0^1$ under the group $\mathcal{G}_{F_{22}}$. Altogether there exist 684 walls containing $Q_{F_{22}}$, whereof 102 intersect the $C_{F_{22}}$-cluster.

For fixing the outer border of the $C_{F_{22}}$-cluster further walls along suitable straight line segments were determined in order to find the walls which separate a Σ_0^i-subkone from a Σ_1^j-subcone. Thereby, the border lines avoiding the center $Q_{F_{22}}$ were found which are shown in Figure 1. Remarkable in the Figure is the spike S of Σ_1^j-subcones entering into the $C_{F_{22}}$-cluster. In the Π^2-section, the border lines do not necessarily extend over the whole cluster. When several border lines intersect then some border lines may end in the point of intersection. In Figure 1 it is seen that many of the walls $W_i \in \mathcal{W}^{(i)}$ end in the center $Q_{F_{22}}$. We don't have further investigated this behaviour.

A rough estimate for the number of non-equivalent Σ_0^k-subcones within the $C_{F_{22}}$-cluster in E^8 could amount up to 10^{10}.

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