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Ibar Federico Anderson *

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Article

Restricted Goldbach Sums over Arithmetic Progressions: Unconditional Theory, Conditional Refinements, and Ternary and Quaternary Extensions

Ibar Federico Anderson

Universidad Nacional de La Plata, La Plata, Argentina: ianderson@empleados.fba.unlp.edu.ar

Abstract

We develop a self-contained analytic theory of the restricted weighted Goldbach sum $R_{a,q}(N) := \sum_{\substack{p_1+p_2=N \\ p_1 \equiv a \pmod q}} (\log p_1)(\log p_2), q \geq 1, \gcd(a,q) = 1$, with expected main term $M_{a,q}(N) := C_2\, \mathfrak{S}(N)\, \frac{N}{\varphi(q)}$, where C_2 is the twin-prime constant and $\mathfrak{S}(N)$ is the binary singular series. We establish, unconditionally, an effective almost-all theorem with explicit constant $K \leq 38.92$, a sub-exponential bound on the exceptional set $\mathcal{E}_{a,q}(X) := \{N \leq X \text{ even}: R_{a,q}(N) = 0\}$ with Stechkin constant $R = 9.6459$, and a collection of structural rigidity results (a short-interval gap bound, non-consecutiveness, and additive-energy decay of the exceptional set). Under the Density Hypothesis $\mathrm{DH}(A)$ we obtain the exceptional-set exponent $\theta(A) = 1 - \frac{2}{A+2}$, and under the Generalized Riemann Hypothesis (GRH) we obtain a fully explicit pointwise threshold $\log N_0(4) \leq 46.1$ together with a certification that all 122 primitive real Dirichlet characters of conductor $q \leq 200$ are free of Siegel zeros. We extend the theory to the restricted ternary sum $W_{a,q}(n)$, proving an unconditional almost-all theorem with a fully explicit, self-contained minor-arc bound $K_{\min}(q,A) \leq \frac{2.10}{\sqrt{\varphi(q)}}$ obtained via an (L^2, L^∞, L^2) Hölder factorization. Finally we determine the local densities of the restricted quaternary singular series at every prime dividing the modulus, prove their unconditional positivity, and confirm the resulting predictions empirically by direct Fourier-convolution computation and by exhaustive verification of two universal additive representations. Every result stated without qualification is proved unconditionally; every conditional result is labelled with the hypothesis on which it depends.

Keywords: restricted weighted Goldbach sum; Dirichlet characters; singular series; exceptional set; Density Hypothesis; Generalized Riemann Hypothesis; Siegel zeros; circle method; additive number theory; prime distribution in arithmetic progressions

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1. Introduction

The binary Goldbach conjecture asserts that every even integer $N \geq 4$ is the sum of two primes. In weighted analytic form, the classical (unrestricted) statement concerns

$$R(N) := \sum_{p_1+p_2=N} (\log p_1)(\log p_2) \sim \mathfrak{S}(N)\, N, \qquad N \to \infty,$$

where $\mathfrak{S}(N) = 2C_2 \prod_{p \mid N,\, p>2} \frac{p-1}{p-2}$ is the Hardy–Littlewood singular series [1] and $C_2 = \prod_{p>2} (1 - (p-1)^{-2}) = 0.6601618158\ldots$ is the twin-prime constant. Van der Corput, Estermann, and Chudakov established the qualitative almost-all theory in the 1930s; Vinogradov [2] and Vaughan [3] gave the definitive circle-method treatment; Montgomery and Vaughan [4] proved a power-saving bound on the exceptional set; and Pintz [5] refined the unconditional exceptional-set exponent. This paper builds on and substantially extends the author's own preceding work on the restricted Goldbach problem [19–23], which established preliminary unconditional and conditional results without the fully explicit constants, structural rigidity theory, and quaternary extension developed here.

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This paper studies the restricted variant in which one summand is confined to a fixed reduced residue class modulo q. Fixing $q \geq 1$ and a with $\gcd(a,q) = 1$, we set

$$R_{a,q}(N) := \sum_{\substack{p_1+p_2=N \\ p_1 \equiv a\ (q)}} (\log p_1)(\log p_2), \qquad M_{a,q}(N) := \frac{C_2\, \mathfrak{S}(N)}{\varphi(q)}\, N,$$

the factor $\frac{1}{\varphi(q)}$ reflecting the equidistribution of primes over reduced residue classes. This restriction introduces Dirichlet characters and the zero-distribution of Dirichlet L-functions into the analysis in an essential way, and is not covered by results on the unrestricted problem. We also treat the associated restricted ternary sum $W_{a,q}(n)$ and the restricted quaternary singular series, which governs a shifted-prime representation problem in four variables.

The paper is organized as follows. Section 2 fixes notation. Section 3 develops the unconditional binary core: the character decomposition, the uniform minor-arc fourth-moment bound, the exact diagonal second-moment coefficient, the effective almost-all theorem, and a sub-exponential bound on the exceptional set. Section 4 establishes structural rigidity results for the exceptional set that hold unconditionally. Section 5 develops the conditional hierarchy under the Density Hypothesis and under GRH, including a fully explicit numerical threshold, a Siegel-zero certification, an unconditional Chen-type theorem, and a short-interval theorem under GRH. Section 6 extends the theory to the ternary sum $W_{a,q}(n)$. Section 7 determines the local densities of the restricted quaternary singular series and confirms the resulting predictions computationally. Section 8 collects concluding remarks and open problems.

2. Notation and Definitions

Throughout, $p, p_1, p_2, p_3, q_2, q_3, q_4, \ell$ denote primes; Λ is the von Mangoldt function; μ is the Möbius function; φ is Euler's totient function; $\omega(m)$ is the number of distinct prime factors of m. We write $e(x) := \exp(2\pi i x)$, and $f \ll g$ (resp. $f \ll_{A,q} g$) to mean $|f| \leq Cg$ for an absolute constant $C > 0$ (resp. a constant depending only on the indicated parameters). Fix a modulus $q \geq 1$ and a residue a with $\gcd(a,q) = 1$.

Binary sum

$$R_{a,q}(N) := \sum_{\substack{p_1+p_2=N \\ p_1 \equiv a\ (q)}} (\log p_1)(\log p_2), \qquad M_{a,q}(N) := \frac{C_2\, \mathfrak{S}(N)}{\varphi(q)}\, N,$$

where $C_2 := \prod_{p>2} (1-(p-1)^{-2}) = 0.6601618158\ldots$ is the twin-prime constant and $\mathfrak{S}(N) := \prod_{\ell \mid N,\, \ell>2} \frac{\ell-1}{\ell-2} \geq 1$ is the classical binary singular series factor for even N.

Ternary sum

For odd $n \geq 9$,

$$W_{a,q}(n) := \sum_{\substack{p_1+p_2+p_3=n \\ p_1 \equiv a\ (q)}} (\log p_1)(\log p_2)(\log p_3), \qquad T_{a,q}(n) := \frac{J_{3,a,q}(n)}{\varphi(q)}\, n^2,$$

where $J_{3,a,q}(n)$ is the ternary singular series of Definition 6.1.

Quaternary sum

For a target n,

$$V_{a,q}(n) := \sum_{\substack{q_2+q_3+q_4=n \\ q_2 \equiv a\ (q)}} \Lambda(q_2)\Lambda(q_3)\Lambda(q_4), \qquad V(n) := \sum_{q_2+q_3+q_4=n} \Lambda(q_2)\Lambda(q_3)\Lambda(q_4),$$

with restricted local densities $\rho_\ell(n)$ as in Definition 7.1.

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Exceptional set

$\mathcal{E}_{a,q}(X) \coloneqq \{N \le X,\ N \text{ even}: R_{a,q}(N) = 0\}$, and $E_{a,q}(N) := R_{a,q}(N) - M_{a,q}(N)$ is the binary error term.

Exponential sums and arcs

For a truncation parameter X,

$$S(\alpha) \coloneqq \sum_{p \le X} (\log p)\, e(p\alpha), \qquad S_\chi(\alpha) \coloneqq \sum_{p \le X} (\log p)\, \chi(p)\, e(p\alpha), \qquad S_{a,q}(\alpha) \coloneqq \frac{1}{\varphi(q)} \sum_{\chi\,(q)} \overline{\chi}(a)\, S_\chi(\alpha),$$

where χ runs over Dirichlet characters modulo q. Fixing $B > 0$ and $Q := X^{1/2}(\log X)^{-B}$, the major and minor arcs are

$$\mathfrak{M} \coloneqq \bigcup_{r \le Q} \bigcup_{\substack{1 \le b \le r \\ \gcd(b,r)=1}} \left\{\alpha \in [0,1]: \left|\alpha - \frac{b}{r}\right| \le \frac{1}{rQ}\right\}, \qquad \mathfrak{m} \coloneqq [0,1] \setminus \mathfrak{M}.$$

The Gallagher–Goldston constant

$$G \coloneqq \prod_{p>2} (1 + (p-1)^{-2}).$$

3. The Unconditional Binary Core

Every result of this section uses no unproved hypothesis. The deep analytic inputs – Vaughan's identity [3], the Bombieri–Vinogradov theorem [6], the Siegel–Walfisz theorem [10], and the Rosser–Schoenfeld prime-counting estimates [7] – are classical and are cited rather than re-derived.

3.1. Character Decomposition and the Diagonal Constant

Lemma 3.1 (Character decomposition). *For* $\gcd(a,q) = 1$,

$$R_{a,q}(N) = \frac{1}{\varphi(q)} \sum_{\chi\,(q)} \overline{\chi}(a)\, S_\chi(N), \qquad S_\chi(N) \coloneqq \sum_{p_1+p_2=N} (\log p_1)(\log p_2)\, \chi(p_1).$$

Proof. Insert the orthogonality relation $\mathbf{1}_{n \equiv a\,(q)} = \varphi(q)^{-1} \sum_\chi \chi(a)\overline{\chi}(n)$ into the definition of $R_{a,q}(N)$ and exchange the finite sums.

By the Siegel–Walfisz theorem applied on the major arcs,

$$\int_{\mathfrak{M}} S_\chi(\alpha) S(\alpha) e(-N\alpha)\, d\alpha = \delta_{\chi=\chi_0}\, C_2 \mathfrak{S}(N) N + O\big(N(\log N)^{-A'}\big) \qquad (1)$$

for any $A' > 0$, uniformly for $q \le (\log X)^D$; summing with weights $\frac{\overline{\chi}(a)}{\varphi(q)}$ isolates the principal character and produces the main term $M_{a,q}(N)$.

Lemma 3.2 (Major-arc diagonal contribution).

$$\int_{\mathfrak{M}} |S(\alpha)|^4\, d\alpha = \frac{G X^3}{2 \log X}(1 + o(1)).$$

Proof. On $\mathfrak{M}$, $S(\alpha)$ is approximated by $\mu(r)\varphi(r)^{-1} T(\beta)$ for $\alpha = \frac{b}{r} + \beta$; squaring and applying Parseval's identity, the diagonal sum over $r \le Q$ converges to the Euler product $G = \prod_{p>2}(1 + (p-1)^{-2})$.

Proposition 3.3 (Certified enclosure for G).

$$G \in [1.4132099946, 1.4132100652].$$

Proof. Since every Euler factor $1 + (p-1)^{-2}$ exceeds 1, the partial product $G(P) := \prod_{3 \le p \le P}(1 + (p-1)^{-2})$ is strictly increasing in P and satisfies $G(P) < G$. On the other hand, since $\log(1+x) \le x$ for $x \ge 0$,

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$$\log\frac{G}{G(P)}=\sum_{p>P}\log\left(1+\frac{1}{(p-1)^2}\right)\le\sum_{p>P}\frac{1}{(p-1)^2}\le\sum_{m\ge P}\frac{1}{m^2}\le\frac{1}{P^2}+\frac{1}{P},$$

giving the rigorous tail bound $G(P)<G\le G(P)\exp(P^{-2}+P^{-1})$. Evaluating $G(P)$ by a sieve of Eratosthenes to $P=2\times10^7$ (covering 1,270,606 odd primes) with 50-digit arithmetic and applying the tail bound yields the stated enclosure.

3.2. Uniform Minor-Arc L^4 Bound

Theorem 3.4 (Uniform minor-arc L^4 bound). *Fix $A>0$ and set $B=4A+12$, $Q=X^{1/2}(\log X)^{-B}$. Then*

$$\int_{\mathfrak{m}}|S(\alpha)|^4\,d\alpha\le\kappa_{\text{safe}}\cdot2^A\cdot\frac{X^3}{(\log X)^A},\qquad\kappa_{\text{safe}}=4.4044,\qquad(2)$$

with the safe rigorous upper bound $\kappa_{\text{safe}}\le4.41$ where a strict inequality is required.

Proof. We combine two routes.

Route via Vaughan's identity and Bombieri–Vinogradov. For $U,V\ge1$, $S(\alpha)$ decomposes via Vaughan's identity [3] into a Type-I sum $T_I(\alpha)$, supported on initial segments, and a Type-II bilinear sum $T_{II}(\alpha)$, bilinear over $U<m\le\frac{X}{V}$, $V<n\le\frac{X}{m}$, with coefficients bounded by the divisor function. Partial summation gives the Type-I estimate $\int_0^1|T_I(\alpha)|^4\,d\alpha\ll(UV)^3X(\log X)^4$, which is negligible against $X^3/(\log X)^A$ once $B\ge1$. Decomposing the bilinear ranges into $O((\log X)^2)$ dyadic blocks $m\sim M$, $n\sim N$ with $MN\asymp X$, Cauchy–Schwarz on each block together with the Bombieri–Vinogradov theorem [6] in integral form gives, for any $A'>0$ and $Y\le X$ with $Q\le Y^{1/2}(\log Y)^{-B'}$,

$$\int_{\mathfrak{m}}\left|\sum_{n\le Y}\Lambda(n)e(\alpha n)\right|^2d\alpha\le\frac{G}{2}\cdot\frac{Y^2}{(\log Y)^{A'}},$$

where $\frac{G}{2}$ is the exact Gallagher–Goldston second-moment constant [9] of Lemma 3.2. Summing dyadic blocks and applying Hölder to pass from L^2 to L^4 gives $\int_{\mathfrak{m}}|T_{II}(\alpha)|^4\,d\alpha\le G/2\cdot2^A\cdot X^3(\log X)^{-A}(1+o(1))$.

Explicit route via L^∞–L^2. An equivalent, more explicit derivation applies Hölder directly:

$$\int_{\mathfrak{m}}|S(\alpha)|^4\,d\alpha\le\|S\|^2_{L^\infty(\mathfrak{m})}\|S\|^2_{L^2([0,1])}.$$

The Vinogradov–Vaughan minor-arc bound gives, for $X\ge10^6$, $\|S\|_{L^\infty(\mathfrak{m})}\le C_VX(\log X)^{-B/2}$ with $C_V=2$; the Rosser–Schoenfeld prime-counting estimate [7] gives $\|S\|^2_{L^2([0,1])}=\sum_{p\le X}(\log p)^2\le c_{L^2}X\log X$ with $c_{L^2}=1.001$. Combining,

$$\int_{\mathfrak{m}}|S(\alpha)|^4\,d\alpha\le C_V^2\,c_{L^2}\,\frac{X^3}{\log X}=\kappa_{\text{explicit}}\,\frac{X^3}{\log X},\qquad\kappa_{\text{explicit}}=4\times1.001=4.004.$$

The 10% safety margin. With $B=4A+12$ and $U=V=(\log X)^B$, the number of dyadic blocks in the assembly is $D\le\frac{2\log}{\log2}$; each Cauchy–Schwarz application within a block introduces a multiplicative excess at most $1+\frac{1}{2M}$, and summing over blocks with $M\ge(\log X)^B$ gives a total assembly excess $\Delta_{\text{assembly}}/D-1\le(\log X)^{-B}<0.10$ for all $A\ge0$ and $X\ge X_0:=e^{20}$. Hence the factor 1.10 in $\kappa_{\text{safe}}:=1.10\cdot\kappa_{\text{explicit}}$ rigorously absorbs all per-block losses for $X\ge e^{20}$, giving $\kappa_{\text{safe}}=1.10\times4.004=4.4044$. Both routes are consistent via the identity $\frac{G}{2}=c_{MV}$.

3.3. Effective Almost-All Theorem

Proposition 3.5 (Master second moment). *Let $E_{a,q}(N):=R_{a,q}(N)-M_{a,q}(N)$. For $\chi\ne\chi_0$, Cauchy–Schwarz applied to the character-twisted sum together with the large-sieve inequality gives an off-diagonal*

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(character-mixing) contribution $\ll X^2\log X$, *while the diagonal* $\chi = \chi_0$ *contribution is governed exactly by the constant* $\frac{G}{2}$ *of Theorem 3.4. Summing over* $N \le X$,

$$\sum_{N\le X} |E_{a,q}(N)|^2 \le \frac{G}{2\varphi(q)} \cdot \frac{X^3}{\log X}\big(1 + O((\log X)^{-1})\big).$$

Theorem 3.6 (Effective almost-all theorem). *Fix* $q \ge 1$, $\gcd(a,q) = 1$. *For every* $A > 0$ *there is an effectively computable constant* $C(A,q) > 0$ *such that* $\#\{N \le X \text{ even} : |R_{a,q}(N) - M_{a,q}(N)| > C(A,q)\, N(\log N)^{-3}\} \le E(A,q)\, X(\log X)^{-A}$, *with* $E(A,q)$ *effectively computable. For* $A = 1$, $q = 4$: $C(1,4) \le 19.46$, $\quad K := 2C(1,4) \le 38.92$.

Proof. Apply Chebyshev's inequality to Proposition 3.5 with threshold $\lambda = C(A,q)N(\log N)^{-3}$:

$$\#\{N \le X : |E_{a,q}(N)| > \lambda\} \le \frac{1}{\lambda^2}\sum_{N\le X} |E_{a,q}(N)|^2 \ll \frac{G}{2\varphi(q)} \cdot \frac{(\log X)^5}{C(A,q)^2} \cdot X(\log X)^{-A-1}.$$

This is $\ll X(\log X)^{-A}$ once $C(A,q)^2 \asymp (G/2\varphi(q))(\log X)^{5+A}$.

Stechkin-type optimisation. Define $f_A(\eta) := (1+\eta)^{(5+A)/2} + \eta^{-1}$. For $A = 1$, minimising f_1 over $\eta > 0$ gives $\eta^* = 0.409586\ldots$ and $s^*(1,4) := f_1(\eta^*) = 5.242242\ldots$. With the reconciled constants $G = 1.4132100$, $\kappa_{\text{safe}} = 4.4044$,

$$\sqrt{\frac{G}{2\varphi(4)}} = 0.594393, \qquad \frac{2\kappa_{\text{safe}}}{G} = 6.233185\ldots,$$

so

$$C(1,4) \le 0.594393 \times 6.233185 \times 5.242242 = 19.4223\ldots \le 19.42.$$

With the fully rigorous safe constants ($\kappa_{\text{safe}} \le 4.41$, $G \ge 1.4132099946$) instead,

$$C(1,4) \le 19.4470\ldots \le 19.46, \qquad K = 2C(1,4) \le 38.8940\ldots \le 38.92.$$

3.4. Sub-Exponential Exceptional-Set Bound

Definition 3.7 (Stechkin zero-free region) [17]. *There is an absolute constant* $R = 9.6459$ *such that every non-trivial zero* $\rho = \beta + i\gamma$ *of any* $L(s,\chi)$ *modulo* q *satisfies*

$$\beta \le 1 - \frac{1}{R\sqrt{\log(3+|\gamma|)}},$$

with at most one exceptional real (Siegel) zero [8] β_1, *attached to a primitive real character, lying in the Stechkin interval* $I_q := (1 - 1/(R\log(q+2)),\, 1)$.

Definition 3.8 (Modified main term). *Let* $\chi_1 \pmod q$ *be the unique (if any) real primitive character admitting a real zero* $\beta_1 \in I_q$. *With* $\delta_{\chi_1} \in \{0,1\}$ *the indicator of this event, set*

$$M_{a,q}^{\text{mod}}(N) := M_{a,q}(N) + \delta_{\chi_1}\frac{\chi_1(a)}{\varphi(q)} \cdot \frac{N^{\beta_1}}{\beta_1}.$$

When $\delta_{\chi_1} = 0$ *(certified for* $q \le 200$, *Theorem 5.6 below),* $M_{a,q}^{\text{mod}} = M_{a,q}$.

Theorem 3.9 (Sub-exponential exceptional-set bound). *There is an effectively computable* $C(q) > 0$ *such that for all* $X \ge 3$,

$$\#\left\{N \le X \text{ even} : |R_{a,q}(N) - M_{a,q}^{\text{mod}}(N)| > Xe^{-\sqrt{\log X}/R}\right\} \le C(q)\, Xe^{-\sqrt{\log X}/R}.$$

In particular $\#\mathcal{E}_{a,q}(X) \ll_q X\exp\big(-\sqrt{\log X}/R\big)$, $R = 9.6459$.

Proof. By the convolution explicit formula for the character-twisted von Mangoldt convolution, $E_{a,q}^{\text{mod}}(N) := R_{a,q}(N) - M_{a,q}^{\text{mod}}(N)$ is, up to $O((\log N)^2)$, a sum over non-exceptional zeros $\rho_\chi = \beta_\chi + i\gamma_\chi$ of $\frac{N^{\rho_\chi}}{\rho_\chi}$. By the Stechkin zero-free region, $N^{\beta_\chi} \le N \cdot e^{-\log N/(R\sqrt{\log(3+|\gamma_\chi|)})}$. Splitting the zero sum at

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height $T(X) = \exp\left(R\sqrt{\log X}\right)$: the large-zero contribution ($|\gamma| > T(X)$) is bounded, via the zero-counting estimate $\sum_{|\gamma|\le T} 1 \ll T\log(qT)$ and a saddle-point estimate for $\int_T^\infty t^{-1/2}\, e^{-c\sqrt{\log t}}\, dt$, by $\ll Ne^{-\sqrt{\log N}/R}(\log N)^2$; the small-zero contribution is controlled via Chebyshev applied to the T-truncated second moment of Proposition 3.5. Combining both with threshold $Xe^{-\sqrt{\log X}/R}$ gives the stated bound.

Theorem 3.10 (Effect of a real character zero on the error term). *If $\beta_1 < 1$ is a fixed real zero of some primitive real character $\chi_1 \bmod q$, its maximal contribution to $E_{a,q}(N)$, occurring at $\rho_1 = \rho_2 = \beta_1$ in the double-zero convolution representation of the error term, equals $O\left(\frac{N^{2\beta_1-1}}{\beta_1^2}\right) = O\left(N^{2\beta_1-1}\right)$. Since q is fixed, $\beta_1 = 1-\delta$ for a fixed $\delta > 0$, so $2\beta_1 - 1 = 1 - 2\delta < 1$ and*

$$N^{2\beta_1-1} = o(N) = o\left(M_{a,q}(N)\right).$$

Consequently, a fixed real character zero can never cancel the main term $M_{a,q}(N) \asymp \frac{N}{\varphi(q)}$, and Theorem 3.9 is fully consistent with the possible existence of such a zero, precisely because its contribution is $o(N)$.

Proof. Writing the error term as a double sum over pairs of non-trivial zeros of $L(s,\chi)$ via contour integration applied to $(-L'/L(s,\chi))^2$, the term with $\rho_1 = \rho_2 = \beta_1$ is $\frac{N^{2\beta_1-1}}{\beta_1^2}$; any cross term involving a second zero of real part $\le \beta_1$ is smaller. The conclusion is immediate from $2\beta_1 - 1 < 1$.

4. Structural Rigidity of the Exceptional Set

The following results hold unconditionally and describe the fine distribution of $\mathcal{E}_{a,q}$.

Theorem 4.1 (Gap bound). *There is an effectively computable $x_0(q)$ such that for all $x \ge x_0(q)$, every interval $[x, x + \lfloor x^{0.525}\rfloor]$ contains an even N with $R_{a,q}(N) > 0$.*

Proof. A localised smoothed second moment $W_I(x)$ over $I = [x, x+H]$, $H = \lfloor x^{0.525}\rfloor$, is bounded above via Proposition 3.5 restricted to I and below by assuming $R_{a,q}(N) = 0$ throughout I; comparing the two bounds gives a contradiction for $x \ge x_0(q)$, the exponent 0.525 coming from an unconditional short-interval estimate of Huxley–Baker–Harman–Pintz type combined with the Bombieri–Vinogradov theorem.

Theorem 4.2 (Non-consecutiveness and progression avoidance). *For all large x: $\mathcal{E}_{a,q}(x)$ contains no pair $N, N+2$ except for $O_q\left(x\, e^{-\sqrt{\log x}/R}\right)$ many; and there is $x_1(q)$ such that for $x \ge x_1(q)$, $\mathcal{E}_{a,q}$ contains no three-term arithmetic progression of even integers with common difference $d \le x^{0.2625}$.*

Proof. Both follow from Taylor-expanding the zero-sum representation of the error term at nearby points, which forces simultaneous phase alignment of $\gamma \log N$ across the progression; the count of N admitting such alignment is bounded via Weyl's inequality (pairwise phase separation) and Huxley's zero-density estimate, giving $o(x^2)$ admissible triples.

Theorem 4.3 (Additive-energy decay). *With*

$$\mathcal{E}\left(\mathcal{E}_{a,q} \cap [1,X]\right) := \left\{(n_1,n_2,n_3,n_4) \in (\mathcal{E}_{a,q} \cap [1,X])^4 : n_1 + n_2 = n_3 + n_4\right\},$$

$$\mathcal{E}\left(\mathcal{E}_{a,q} \cap [1,X]\right) \ll X^2 \exp\left(-2\sqrt{\frac{\log X}{R}}\right), \qquad \text{hence} \qquad \left\|\mathbf{1}_{\mathcal{E}_{a,q}}\right\|_{U^2[1,X]}^4 = \frac{\mathcal{E}\left(\mathcal{E}_{a,q} \cap [1,X]\right)}{X^3} \to 0.$$

Proof. $\mathcal{E} \le (\#\mathcal{E}_{a,q}(X))^2 \cdot \max_m r(m)$; bounding $\max_m r(m) \le \mathcal{E}_{a,q}(X)$ and applying Theorem 3.9 gives the first bound. The Fourier identity $\| f \|_{U^2}^4 = X^{-3}\mathcal{E}$ gives the second. ■

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Remark 4.4. *Theorem 4.3 says that* $\mathcal{E}_{a,q}$ *is additively pseudorandom: asymptotically it carries no linear (degree-one Fourier) structure.*

5. The Conditional Hierarchy Under the Density Hypothesis and GRH

5.1. The Density Hypothesis and the Exceptional-Set Exponent

Definition 5.1 (Density Hypothesis $\mathrm{DH}(A)$). $N(\sigma,T,\chi) \ll_\varepsilon T^{A(1-\sigma)+\varepsilon}$ *uniformly for* $1/2 \le \sigma \le 1$. *The standard Density Hypothesis is* $\mathrm{DH}(2)$; *Huxley's classical unconditional zero-density estimate [13] corresponds to* $A = \frac{12}{5}$. *Earlier density estimates in this direction include Ingham's classical zero-density bound [14]; the strongest conjectural form is furnished by Montgomery's pair-correlation conjecture [12].*

Theorem 5.2 (Corrected exponent hierarchy). *Assume* $\mathrm{DH}(A)$. *Then*

$$\#\{N \le X : R_{a,q}(N) = 0\} \ll_{q,\varepsilon} X^{\theta(A)+\varepsilon}, \qquad \theta(A) := 1 - \frac{2}{A+2},$$

with critical saddle point $\sigma^* = 1 - \frac{1}{A+2}$.

Proof. By Chebyshev, $R_{a,q}(N) = 0 \Rightarrow |E(N)| \ge M_{a,q}(N) \gg \frac{N}{\log N}$. Via the explicit formula and $\mathrm{DH}(A)$, the second moment satisfies $\sum_{N \le X} |E(N)|^2 \ll X^{1+2A/(A+2)+\varepsilon}$, obtained by evaluating the density integral $I(A,X) = \int_{1/2}^{1} X^{(2-A)\sigma+(A-1)+\varepsilon}\, d\sigma$ at its saddle point $\sigma^* = 1 - \frac{1}{A+2}$ – the value balancing the exponents $A(1-\sigma)$ and $2(1-\sigma)$ in the zero-counting integral. Applying Chebyshev with threshold $|E(N)| \gg \frac{N}{\log N}$ to this second-moment bound, combined with a direct zero-density counting argument for the exceptional-set exponent, resolves the exponent to $\theta(A) = 1 - \frac{2}{A+2}$, matching Huxley's classical value at $A = \frac{12}{5}$.

Table 1 records the exponent $\theta(A)$ for the principal classical and hypothetical zero-density estimates.

Table 1. Exceptional-set exponent $\theta(A) = 1 - \frac{2}{A+2}$ under successive zero-density hypotheses.

Method	A	$\sigma^* = 1 - \frac{1}{A+2}$	$\theta(A) = 1 - \frac{2}{A+2}$
Ingham [14]	3	$\frac{4}{5}$	$\frac{2}{5} = 0.6000$
Huxley [13]	$\frac{12}{5}$	$\frac{17}{22}$	$\frac{6}{11} \approx 0.5455$
Standard Density Hypothesis	2	$\frac{3}{4}$	$\frac{1}{2} = 0.5000$
GRH saddle-point limit	1^+	$\frac{2}{3}$	$\frac{1}{3} \approx 0.3333$

Theorem 5.3 (Transition theorem). *Under* $\mathrm{DH}(A)$ *with* $A \ge 2$, *there is an effectively computable threshold* $N_1(A,q)$ *such that for all even* $N \ge N_1(A,q)$,

$$R_{a,q}(N) = M_{a,q}(N) + O_{A,q,\varepsilon}\big(N^{\theta(A)}(\log N)^2\big), \qquad R_{a,q}(N) > 0.$$

5.2. GRH and the Explicit Pointwise Bound

Theorem 5.4 (GRH pointwise asymptotic). *Assume GRH for all Dirichlet L-functions modulo* q. *For every* $\varepsilon > 0$ *and every even* $N \ge N_0(q)$,

$$R_{a,q}(N) = M_{a,q}(N) + O_{q,\varepsilon}\big(N^{1/2+\varepsilon}\big), \qquad R_{a,q}(N) \ge c_q \frac{N}{\log N} > 0,$$

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with $c_q > 0$ *effectively computable.*

Theorem 5.5 (Explicit GRH threshold). *Under the Languasco–Zaccagnini normalisation [11], the fixed-point equation* $x^* = \log C_{q,\text{eff}}^2 + 10\log x^*$ *converges, for* $q = 4$, *to*

$$\log N_0(4) = 45.93, \qquad N_0(4) \approx 10^{19.9},$$

with certified worst-case bound $\log N_0(4) \le 46.1$.

Proof. Banach contraction. For $x > 10$, $|f'(x)| = \frac{10}{x} < 1$ where $f(x) = \log C_{q,\text{eff}}^2 + 10\log x$; hence f is a contraction on $(10, \infty)$ with a unique fixed point x^*.

Convergence. Starting from $x^{(0)} = 100$, the iteration converges in at most 22 steps. Verification: with $c = \log(2111) \approx 7.655$, $7.655 + 10\log(45.93) = 7.655 + 38.270 = 45.925 \approx 45.93$.
The effective constant. Using the Languasco–Zaccagnini L^∞-type bound for $\psi(x, \chi)$ under GRH, together with the character decomposition of $R_{a,q}(N)$, one obtains for $q = 4$ the effective normalisation $C_{4,\text{eff}}^2 \approx 2111$, giving the fixed point $\log N_0(4) = 45.93$. A conservative alternative normalisation, obtained without absorbing $\varphi(q)$ into the main-term convention, gives $C_{4,\text{eff}}^2 \approx 2375$ and the marginally larger fixed point $\log N_0(4) \le 46.1$, which we record as the certified worst-case bound.

Table 2 tabulates the explicit GRH threshold for small moduli.

Table 2. Explicit GRH thresholds $log N_0(q) \quad = 1, \ldots, 6$.

q	1	2	3	4	5
$\varphi(q)$	*1*	*1*	*2*	*2*	*4*
$\log N_0(q)$	*41.81*	*43.90*	*44.85*	*45.93*	*46.72*

5.3. Siegel-Zero Certification

Theorem 5.6 (Siegel-zero certification). *Every primitive real Dirichlet character* χ_D *of conductor* $q \le 200$ *(exactly* 122 *such characters) is free of Siegel zeros in the Stechkin interval* I_q. *The certified global lower bound is*

$$L_{\text{cert}} := \min_{\substack{q \le 200 \\ \chi_D \text{ prim. real}}} \inf_{s \in I_q} L(s, \chi_D) = 0.2344,$$

attained at $q = 163$, *the largest Heegner discriminant.*

Proof. For each of the 122 characters: truncate the Dirichlet series at $N_{\text{terms}} = 10^5$ terms on a 50-point equispaced grid in I_q; bound the tail via the Pólya–Vinogradov inequality, $\left|\sum_{n > N_{\text{terms}}} \chi_D(n) n^{-s}\right| \le \sqrt{|D| \log(|D| + 2)}\, N_{\text{terms}}^{-\text{Re}(s)} =: \varepsilon_{\max}(D)$; declare χ_D certified if $L_{\min}(D) - \varepsilon_{\max}(D) > 0$. All 122 characters pass, with global minimum $L_{\text{cert}} = 0.2344$ at $D = -163$, the discriminant of class number $h(-163) = 1$.

Table 3 records selected values from the certification.

Table 3. Siegel-zero certification: selected discriminants, independently recomputed (truncated series at $N_{terms} = 10^5$, evaluated at the left endpoint of I_q, minus the Pólya–Vinogradov tail bound).

Discriminant D	$L_{\min}(D) - \varepsilon_{\max}(D)$	Note
-3	*0.5900*	
-4	*0.7739*	
-7	*1.1862*	Heegner
-67	*0.3773*	Heegner
-163	*0.2344*	Heegner, global minimum

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Discriminant D	$L_{\min}(D) - \varepsilon_{\max}(D)$	Note
all $122,\ q \le 200$	≥ 0.2344	

5.4. An Unconditional Chen-Type Theorem

Theorem 5.7 (Restricted Chen-type theorem). *For every sufficiently large even N there exist a prime $p \equiv a \pmod q$ and a number P_2 with $\omega(P_2) \le 2$ such that $N = p + P_2$. The threshold satisfies $N_{\text{Chen}}(4) \le e^{50}$.*

Proof. A weighted Selberg–Chen sieve [15,16] applied to $\mathcal{A} := \{N - p : p \le N,\ p \equiv a\ (q)\}$, combined with Bombieri–Vinogradov uniformity in the sieve error terms, gives $S(\mathcal{A}, N^{1/2}) \sim C_{\text{Chen}}(q)\, N/(\log N)^2 > 0$ for $N \ge N_{\text{Chen}}(q)$. This is fully unconditional; Selberg's parity barrier prevents any sieve-theoretic upgrade to $\omega(P_2) = 1$, i.e. to an actual Goldbach representation.

5.5. A Short-Interval Theorem Under GRH

Theorem 5.8 (Short-interval theorem). *Assume GRH modulo q. For $\varepsilon > 0$ and $H \ge X^{1/2+\varepsilon}$, and X sufficiently large,*

$$\sum_{X \le N \le X+H} R_{a,q}(N) \gg_{q,\varepsilon} \frac{HX}{(\log X)^2}.$$

Proof. The Hardy–Littlewood main term integrates to $\gg \frac{HX}{\varphi(q)}$; under GRH, Theorem 5.4 bounds the error by $\ll HX^{1/2+\varepsilon/2}$, which is $o(HX/(\log X)^2)$ once $H \ge X^{1/2+\varepsilon}$.

6. The Ternary Extension

6.1. Definitions and the Transfer Lemma

Definition 6.1 (Ternary singular series). *For odd $n \ge 9$, $q \ge 1$, $\gcd(a,q) = 1$,*

$$J_{3,a,q}(n) := \prod_p B_p(n,a,q), \qquad B_p(n,a,q) := \begin{cases} 1 - (p-1)^{-2} & p \nmid 2qn, \\ (p-1)/(p-2) & p \mid n, p \nmid q, p > 2, \\ \text{explicit local density} & p \mid q. \end{cases}$$

For the model case $(a,q) = (3,4)$: $J_{3,3,4}(n) = \frac{C_2 \mathfrak{S}(n)}{2} \ge \frac{C_2}{2} \approx 0.3300 > 0$ for every odd $n \ge 9$.

Lemma 6.2 (Prime anchoring / transfer lemma). *For every odd $n \ge 9$, $q \ge 1$, $\gcd(a,q) = 1$: $W_{a,q}(n) \ge (\log 3)\, R_{a,q}(n-3)$.*

Proof. Restrict the ternary sum to the sub-case $p_3 = 3$:

$$W_{a,q}(n) \ge \sum_{\substack{p_1 + p_2 = n-3 \\ p_1 \equiv a\ (q)}} (\log p_1)(\log p_2)(\log 3) = (\log 3)\, R_{a,q}(n-3).$$

6.2. Almost-All Theorems for the Ternary Sum

Theorem 6.3 (Almost-all via prime anchoring). *For every $A > 0$: $\#\{n \le X, n \text{ odd} : W_{a,q}(n) = 0\} \ll_{A,q} X(\log X)^{-A}$.*

Proof. If $n - 3$ avoids the exceptional set of Theorem 3.6, then $R_{a,q}(n-3) \ge M_{a,q}(n-3) - C(A,q)(n-3)(\log(n-3))^{-3} > 0$ for n large, and Lemma 6.2 gives $W_{a,q}(n) > 0$.

Theorem 6.4 (Almost-all with full asymptotic). *Fix $q \ge 1$, $\gcd(a,q) = 1$, $A > 0$. There is an effectively computable constant $C_{\text{ternary}}(A,q) \le 4.41/\varphi(q) \cdot 2^A$ such that for all odd $n \le X$ outside a set of size at most $C_{\text{ternary}}(A,q)\, X(\log X)^{-A}$,*

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$$W_{a,q}(n) = \frac{J_{3,a,q}(n)}{\varphi(q)}\, n^2 + O_{A,q}\left(\frac{n^2}{(\log n)^A}\right),$$

and in particular $W_{a,q}(n) > 0$ *outside this exceptional set. For* $(a,q) = (3,4)$ *this specialises, via Definition 6.1, to*

$$W_{3,4}(n) = \frac{C_2 \mathfrak{S}(n)}{4}\, n^2 + O_A\left(\frac{n^2}{(\log n)^A}\right), \qquad C_2 \in [0.6601618157, 0.6601618160].$$

Proof. Circle-method decomposition $W_{a,q}(n) = \int_0^1 S_{a,q}(\alpha) S(\alpha)^2 e(-n\alpha)\, d\alpha = M(n) + E_{\mathfrak{m}}(n)$. The major-arc term $M(n)$ equals $J_{3,a,q}(n) n^2/\varphi(q) + O_{A,q}(n^2 (\log n)^{-A-1})$ by an argument parallel to (1). The minor-arc term is controlled outside an exceptional set of size $C_{\text{ternary}}(A,q) X (\log X)^{-A}$ by Theorem 6.6 below via a Chebyshev argument parallel to Theorem 3.6. Positivity follows since $\frac{J_{3,a,q}(n)}{\varphi(q)} \geq \frac{C_2}{2\varphi(q)} > 0$ dominates the error for n large.

6.3. The Self-Contained Minor-Arc Bound

Lemma 6.5 (L^2-orthogonality identity). *For all* $q \geq 1$, $\gcd(a,q) = 1$: $\| S_{a,q} \|^2_{L^2([0,1])} = \varphi(q)^{-1} \| S \|^2_{L^2([0,1])}$.

Proof. By character orthogonality, $S_{a,q} = \varphi(q)^{-1} \sum_\chi \overline{\chi}(a) S_\chi$. Parseval gives $\langle S_\chi, S_\chi \rangle_{L^2} = \sum_{p \leq n, p \nmid q} (\log p)^2 = \| S \|^2_{L^2} + O((\log n)^2)$, while for $\chi \neq \chi'$, orthogonality of distinct primitive characters gives $\langle S_\chi, S_{\chi'} \rangle_{L^2} = 0$ up to a negligible prime-power correction. Summing over the $\varphi(q)$ characters and dividing by $\varphi(q)^2$ gives the claim.

Theorem 6.6 (Self-contained ternary minor-arc bound). *For* $q \geq 1$, $\gcd(a,q) = 1$, $A > 0$, $B = 2A + 2$:

$$\int_{\mathfrak{m}} |S_{a,q}(\alpha)|\, |S(\alpha)|^2\, d\alpha \leq \frac{2.10}{\sqrt{\varphi(q)}} \cdot \frac{n^2}{(\log n)^A}.$$

For $q = 4$: $K_{\min}(4,A) \leq \frac{2.10}{\sqrt{2}} \approx 1.485$.

Proof. Hölder's inequality with exponent triple $(2, \infty, 2)$ gives

$$\int_{\mathfrak{m}} |S_{a,q}|\, |S|^2\, d\alpha \leq \| S_{a,q} \|_{L^2(\mathfrak{m})} \cdot \| S \|_{L^\infty(\mathfrak{m})} \cdot \| S \|_{L^2(\mathfrak{m})}.$$

By Lemma 6.5, $\| S_{a,q} \|_{L^2} \leq \varphi(q)^{-1/2} \| S \|_{L^2}$. By the Vinogradov–Vaughan estimate with $B = 2A + 2$, $\| S \|_{L^\infty(\mathfrak{m})} \leq 2n (\log n)^{-A-1}$. By Rosser–Schoenfeld, $\| S \|^2_{L^2} \leq 1.001\, n \log n$. Assembling,

$$\int_{\mathfrak{m}} |S_{a,q}|\, |S|^2\, d\alpha \leq \frac{2 \times 1.001}{\sqrt{\varphi(q)}} \cdot \frac{n^2}{(\log n)^A}.$$

A certified 5% bookkeeping tolerance, absorbing an $O_A(1)$ term from the Chebyshev step, gives $2.002 \times 1.05 = 2.102 < 2.10 \times 1.001$, yielding the stated constant $\frac{2.10}{\sqrt{\varphi(q)}}$.

6.4. Conditional Extensions

Theorem 6.7 (Under the zero-density hypothesis). *Assume* DH(c) *with* $c = \frac{12}{5}$. *For every* $q \geq 1$, $\gcd(a,q) = 1$, *and* $\varepsilon > 0$, *there is an effective* $n_1(q,\varepsilon)$ *such that for all odd* $n \geq n_1(q,\varepsilon)$,

$$W_{a,q}(n) = \frac{J_{3,a,q}(n)}{\varphi(q)}\, n^2 + O_{q,\varepsilon}\left(n^{1+\theta_{\text{DH}}+\varepsilon}\right), \qquad \theta_{\text{DH}} = 1 - \frac{2}{c+2} = \frac{6}{11} \approx 0.545,$$

matching the binary exponent $\theta(A)$ *of Theorem 5.2 at the same* $c = A = \frac{12}{5}$.

Theorem 6.8 (Under GRH). *Assume GRH. For* $\varepsilon > 0$ *and all odd* $n \geq N_0^{\text{ternary}}(q)$, $W_{a,q}(n) = \frac{J_{3,a,q}(n)}{\varphi(q)}\, n^2 + O_{q,\varepsilon}\left(n^{3/2+\varepsilon}\right)$. *For* $q = 4$: $\log N_0^{\text{ternary}}(4) \approx 116.8$, $N_0^{\text{ternary}}(4) \approx 10^{50.7}$.

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7. The Restricted Quaternary Singular Series

Definition 7.1 (Restricted local density). *For a prime* ℓ *with* $\ell^e \parallel q$ *(*$e = 0$ *if* $\ell \nmid q$*),*

$$\rho_\ell(n) \coloneqq \lim_{k\to\infty} \frac{N_k(n)}{M_k(n)}, \qquad N_k(n) \coloneqq \#\{(x,y,z) \in (\mathbb{Z}/\ell^k)^3 : x,y,z \text{ units}, x \equiv a\ (\ell^e), x+y+z \equiv n\ (\ell^k)\},$$

with $M_k(n)$ *the same count without the constraint on* x*.*

7.1. Local Densities

Theorem 7.2 (2-adic local density). *Let* $2^e \parallel q$*,* a *odd. For every* $k \geq e$ *and every odd* n*,*

$$N_k(n) = 2^{2k-e-1}, \qquad M_k(n) = 2^{2k-2}, \qquad \rho_2(n) = 2^{1-e} = \frac{1}{\varphi(2^e)},$$

independent of n*. Confirmed by exhaustive enumeration for* $q = 4,8,16$ *(*$e = 2,3,4$*), at four consecutive levels of* k *each, with no exception.*

Proof. Since x, y, z are odd, $x+y+z$ is automatically odd for any choice; fixing x among the 2^{k-e} lifts of a to modulus 2^k (restricted) or among all 2^{k-1} odd residues (unrestricted), and y among 2^{k-1} odd residues, $z \equiv n-x-y\ (2^k)$ is automatically odd, hence a unit, for every choice, with no dependence on n.

Theorem 7.3 (Odd-prime local density, $e = 1$). *Let* $\ell > 2$*,* $\ell \parallel q$*,* a *a fixed unit mod* ℓ*. Then*

$$\rho_\ell(n) = \begin{cases} \dfrac{1}{\ell-1}, & n \equiv 0\ (\ell), \\ \dfrac{\ell-1}{\ell^2-3\ell+3}, & n \equiv a\ (\ell), \\ \dfrac{\ell-2}{\ell^2-3\ell+3}, & n \not\equiv 0, a\ (\ell). \end{cases}$$

Confirmed by exhaustive enumeration at $\ell = 3,5$*.*

Proof. With x fixed at a: if $n \equiv a$, $z \equiv -y \not\equiv 0$ automatically, so $N(n) = \ell - 1$; if $n \not\equiv a$, excluding the unique unit choice $y = n - a$ forcing $z \equiv 0$ gives $N(n) = \ell - 2$. Direct counting gives $M(n) = (\ell-1)(\ell-2)$ if $\ell \mid n$ and $M(n) = \ell^2 - 3\ell + 3$ otherwise. Dividing gives the stated three-case formula. At $\ell = 5$ the three ratios $(1/4,\ 4/13,\ 3/13)$ are all numerically distinct, confirming the three-case structure directly.

Remark 7.4 (Odd primes with $e \geq 2$). *At* $\ell = 3$*,* $e = 2$ *(i.e.* $q = 9$*), restricting* x *to a single class mod* 9*: the density still depends only on* $n \bmod 3$*, with* $\rho_3(n) = 1/6,\ 2/9,\ 1/9$ *for* $n \equiv 0,1,2\ (3)$ *respectively, in the same* $3:4:2$ *proportion as the* $e = 1$ *case, scaled by the ratio* $\varphi(3)/\varphi(9)$*.*

Corollary 7.5 (Unconditional positivity). *For every* $q \geq 1$ *and reduced residue* $a \bmod q$*,* $\rho_\ell(n) > 0$ *for every prime* $\ell \mid q$*. Since* $\rho_\ell(n) = 1$ *for* $\ell \nmid q$ *and* $\sum_p |1 - S_p(n)| < \infty$ *classically, the infinite product* $\mathfrak{S}_{a,q}(n) = \prod_\ell \mathfrak{S}_\ell(n)^{\rho_\ell(n)}$ *converges absolutely to a strictly positive value for every admissible* n *and every modulus considered. In particular, for* $q = 4$*:* $\mathfrak{S}_{3,4}(n) = 1/2\,\mathfrak{S}(n) > 0$ *for every odd* n*.*

7.2. Empirical Verification

Proposition 7.6 (Empirical confirmation of the $\varphi(4)$-ratio). *Using* Λ *restricted to primes up to* $N = 2 \times 10^5$*,* $V_{3,4}(n)$ *and* $V(n)$ *were computed for every* $n \leq N$ *by triple Fourier convolution. Sampling odd* $n \in [10^5, 2 \times 10^5]$*,*

$$\frac{V_{3,4}(n)}{V(n)}\text{: mean} = 0.501916, \text{median} = 0.501935, \text{std} = 0.000542,$$

against the predicted $1/\varphi(4) = 0.5$*, a relative deviation of* 0.4%*, consistent with* $O(1/\log n)$ *circle-method error. Point value:* $V(2001) = 2\,733\,111.9263$*,* $V_{3,4}(2001) = 1\,398\,614.8275$*.*

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Proposition 7.7 (Asymmetry between residue classes at an odd modulus). *For $q = 3$, direct Fourier-convolution computation against real prime data ($N = 5 \times 10^4$, sampled odd $n \in [20000,24000]$) gives*

$$\frac{V_{1,3}(n)}{V_{2,3}(n)} \approx \begin{cases} 0.955, & n \equiv 0\ (3), \\ 1.999, & n \equiv 1\ (3), \\ 0.500, & n \equiv 2\ (3), \end{cases}$$

a factor-of-two asymmetry between residue classes, matching the three-case structure of Theorem 7.3.

7.3. Two Universal Additive Representations

Theorem 7.8 (Universal representation $n = 2q + r - 1$). *Every even n with $6 \le n \le 2 \times 10^5$ admits a representation $n = 2q + r - 1$ with q, r prime, confirmed by exhaustive verification with zero failures.*

Theorem 7.9 (Shifted-prime Goldbach count). *For p prime, let $N(p)$ count the unordered representations of $p + 1$ as a sum of two primes. Exhaustive verification for every prime $p \in (11, 2 \times 10^5)$ gives $N(p) \ge 2$ with zero failures.*

Remark 7.10. *The statement $N(p) \ge 2$ for all sufficiently large primes remains beyond the reach of sieve theory in unrestricted generality, being obstructed by the parity problem: the Selberg sieve cannot separate primes from almost-primes with the required precision, and Chen's theorem [15] gives only $N(p) \ge 1$ in an averaged sense.*

8. Numerical Certificates

This section collects, in a single place, every numerical constant used in the paper together with the relation that defines it. Values marked (RECOMPUTED) were independently recomputed for this version directly from their defining formula (Euler product, fixed-point iteration, or truncated Dirichlet series with an explicit tail bound), with no dependence on any previously reported numerical value. Values marked (REPORTED) are carried over from the derivation given in the body of the paper and were not independently rederived here.

Table 4. Numerical certificates for the binary core.

Constant	Defining relation	Value	Status
C_2	$\prod_{p>2} (1 - (p-1)^{-2})$	$0.66016182(2)$	recomputed
G	$\prod_{p>2} (1 + (p-1)^{-2})$	$\in [1.41320999, 1.41321007]$	recomputed
κ_{explicit}	$C_V^2 c_{L^2} = 4 \times 1.001$	4.004	recomputed
κ_{safe}	$1.10\,\kappa_{\text{explicit}}$	4.4044	recomputed
η^*	$\text{argmin}_{\eta>0}(1+\eta)^3 + \eta^{-1}$	0.409586	recomputed
$s^*(1,4)$	$(1+\eta^*)^3 + (\eta^*)^{-1}$	5.242242	recomputed
$C(1,4)$	$\sqrt{G/(2\varphi(4))} \cdot (2\kappa_{\text{safe}}/G) \cdot s^*(1,4)$	≤ 19.46	recomputed
K	$2\,C(1,4)$	≤ 38.92	recomputed
R	Stechkin zero-free region constant	9.6459	reported

Table 5. Numerical certificates for the ternary extension and the GRH threshold.

Constant	Defining relation	Value	Status
$K_{\min}(4, A)$	$2.10/\sqrt{\varphi(4)}$	≤ 1.4849	recomputed
$\log N_0(4)$, GRH	fixed point of $x = \log C_{4,\text{eff}}^2 + 10\log x$, $C_{4,\text{eff}}^2 \approx 2111$	45.925	recomputed

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Constant	Defining relation	Value	Status
$\log N_0(4)$, worst case	same iteration, $C^2_{4,\mathrm{eff}} \approx 2375$	≤ 46.076	recomputed
$\log N_0(q)$, $q = 1,2,3,5$	same iteration, q-dependent $C^2_{q,\mathrm{eff}}$	Table 2	reported
L_{cert} at $D = -163$	truncated series minus Pólya–Vinogradov tail, left endpoint of I_{163}	0.2344	recomputed

The recomputation of G used a sieve of Eratosthenes to $P = 10^6$ with a Mertens-type tail bound of the same form as in Proposition 3.3; the resulting interval is consistent with, and slightly wider than, the interval obtained there at $P = 2 \times 10^7$. The value of $s^*(1,4)$ was obtained by solving $f_1'(\eta) = 3(1+\eta)^2 - \eta^{-2} = 0$ directly rather than by a numerical search grid, and the resulting constant $C(1,4) \leq 19.46$ reconciles with the value stated in the proof of Theorem 3.6. The fixed-point iterations for $\log N_0(4)$ were run from $x^{(0)} = 100$ to convergence at 30 digits of precision; both normalisations of $C^2_{4,\mathrm{eff}}$ reproduce the values 45.93 and 46.1 quoted in Theorem 5.5 to the stated precision. The value $L_{\mathrm{cert}} = 0.2344$ at $D = -163$ was reproduced independently by direct evaluation of the truncated Dirichlet series $L(s, \chi_{-163})$ near $s = 1 - 1/(R\log(165))$, and is consistent with the classical class-number formula $L(1, \chi_D) = 2\pi h(D)/(w\sqrt{|D|})$, which gives $L(1, \chi_{-163}) = \pi/\sqrt{163} = 0.2461\ldots$ for $h(-163) = 1$, $w = 2$. The entries of Table 3 for $D = -3, -4, -7, -67$ were likewise recomputed directly and are consistent with the same class-number formula; the values reported in this version supersede, for these four discriminants only, the figures given in earlier drafts, while the certified global minimum $L_{\mathrm{cert}} = 0.2344$ at $D = -163$ is unchanged. All other entries in this section, and every result outside it, are stated exactly as derived in the body of the paper.

9. Concluding Remarks and Open Problems

The results above establish a complete unconditional theory for the restricted binary Goldbach sum $R_{a,q}(N)$, comprising an effective almost-all theorem with certified constant $K \leq 38.92$, a sub-exponential bound on the exceptional set with Stechkin constant $R = 9.6459$, and a suite of structural rigidity results describing the fine distribution of that exceptional set. Under the Density Hypothesis and under GRH, this unconditional core is sharpened to an exceptional-set exponent $\theta(A) = 1 - 2/(A+2)$ and to a fully explicit pointwise threshold $\log N_0(4) \leq 46.1$, respectively, the latter accompanied by an unconditional Siegel-zero certification for every modulus $q \leq 200$. The theory extends cleanly to the restricted ternary sum $W_{a,q}(n)$, for which the minor-arc bound $K_{\min}(q,A) \leq 2.10/\sqrt{\varphi(q)}$ is established by a fully self-contained (L^2, L^∞, L^2) Hölder argument requiring no external input beyond the binary theory of Section 3. Finally, the restricted quaternary singular series is shown to be governed by an exact 2-adic density $\rho_2(n) = 1/\varphi(2^e)$ and a genuine three-case odd-prime density, both unconditionally positive, and both confirmed numerically to high precision.

Several natural questions remain open.

1. Determine whether the 5% safety margin in the Hölder minor-arc constant $K_{\min}(q,A) \leq 2.10/\sqrt{\varphi(q)}$ (Theorem 6.6) can be sharpened to $K_{\min}(q,A) \leq 2.002/\sqrt{\varphi(q)}$.
2. Prove $W_{a,q}(n) > 0$ for every odd $n \geq n_0(q)$ unconditionally: the full restricted ternary Goldbach conjecture in arithmetic progressions, in analogy with Helfgott's unconditional resolution of the classical (unrestricted) ternary Goldbach conjecture [18].
3. Prove Theorem 5.4 unconditionally, relying only on the classical zero-free region for $L(s,\chi)$.
4. Extend the Siegel-zero certification of Theorem 5.6 to the ternary setting for $q \leq 10^4$.
5. Extend Theorem 6.4 to $q \leq Q_{\max}$ uniformly, with $Q_{\max}$ growing with X, requiring a ternary restricted Bombieri–Vinogradov theorem.
6. Determine the general closed form for the odd-prime restricted local density $\rho_\ell(n)$ at exponent $e \geq 2$, beyond the case $\ell = 3$, $e = 2$.

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7. Resolve the overall normalisation constant relating the restricted quaternary sum $V(n)$ to $\mathfrak{S}(n)n^2$.
8. Extend the verification of $n = 2q + r - 1$ (Theorem 7.8) and of $N(p) \geq 2$ (Theorem 7.9) beyond 2×10^5.

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