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Alexander Migdal *

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Dual Theory of Decaying Turbulence: 1: Fermionic Representation

Alexander Migdal

Department of Physics, New York University Abu Dhabi, Saadiyat Island, Abu Dhabi, PO Box 129188, Abu Dhabi, United Arab Emirates; am10485@nyu.edu or sasha.migdal@gmail.com

Abstract: This is the first paper in a cycle investigating the exact solution of the loop equation for decaying unforced turbulence in three dimensions, which we have found in the previous work. In this paper, we prove the exact equivalence of this solution (the Euler ensemble) to the theory of Fermi particles on a ring. The continuum limit of this theory will be studied in the next papers.

Keywords: turbulence; fractal; anomalous dissipation; fixed point; velocity circulation; loop equations; Euler Phi; prime numbers

1. Physical Introduction. The energy flow and random vorticity structures

The decaying turbulence is an old subject addressed before within a weak turbulence framework (truncated perturbative expansion in the nonlinearity of the forced Navier-Stokes equation). Some phenomenological models were also fitted to the experiments: see the recent review of these models and the experimental data in [1].

This analysis is inadequate for the strong isotropic turbulence, which must be built from the first principles by solving the Navier-Stokes equations beyond the perturbation theory. The problem of universality of the strong turbulence with and without random forcing is the first question to ask when building such a theory.

The experimental data for the energy decay in turbulent flows, fitted in [1] suggest the decay of the dissipation rate $\partial_t E \sim t^{-1-n}$ with $n \approx 1.2$.

There is no consensus here: the fitted index varies from 1 to 1.5, which may reflect the absence of the universal scaling law or jumps between various regimes.

By dimensional counting, any power law for the vorticity correlation in coordinate space must have the form of $t^{-2}(t\nu/r^2)^{\lambda}$.

The index $\lambda = 1 - n$ must be **positive**; otherwise, there will be no growth at $\vec{r} \to 0$, hence, no anomalous dissipation. In the Fourier space, the energy spectrum must decay slower than k^{-3}, so the total integral diverges at large k.

The fit of decaying turbulence data in [1] corresponds to negative $\lambda = 1 - n \sim -0.2$, which contradicts the anomalous dissipation requirement. The large-scale numerical simulation of the Euler ensemble [2] yields positive $\lambda \sim +0.22$ in agreement with this requirement.

What could be the explanation of this paradox? Perhaps the data fit in [1] did not reach a true turbulent limit corresponding to our regime. It is also possible that the stochastic forces added to the Navier-Stokes equation in simulations pollute the decaying turbulence. By design, these forces were supposed to trigger and amplify the spontaneous stochasticity of the turbulent flow. However, in our theory [3], this natural stochasticity is related to a dual quantum system and is **discrete**.

The Gaussian forcing with a continuous wavelength spectrum can distort these quantum stochastic phenomena as these forces stir the flow "with a large dirty spoon" all over the space at every moment. With forcing turned off at some moment and the turbulence reaching the universal stage in its subsequent decay, the energy dissipation would occur in vorticity structures deep inside the volume by the pure turbulent dynamics we are studying.

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The following calculation supports this scenario. The energy balance in the pure Navier-Stokes reduces to the energy dissipation by the enstrophy in the bulk, compensated by the energy pumping by forces from the boundary (say, the large sphere around the flow).

The general identity, which follows from the Navier-Stokes if one multiplies both sides by $\vec{v}$ and averages over an infinite time interval, reads:

$$\int_V d^3r \left\langle \nu \vec{\omega}^2 \right\rangle = -\int_V d^3r \partial_\beta \left\langle v_\beta \left(p + 1/2 v_\alpha^2 \right) + \nu v_\alpha (\partial_\beta v_\alpha - \partial_\alpha v_\beta) \right\rangle \tag{1}$$

By the Stokes theorem, the right side reduces to the flow through the boundary ∂V of the integration region V. The left side is the dissipation in this volume, so we find:

$$\mathcal{E}_V = -\int_{\partial V} d\vec{\sigma} \cdot \left\langle \vec{v} \left(p + 1/2 v_\alpha^2 \right) + \nu \vec{\omega} \times \vec{v} \right\rangle \tag{2}$$

This identity holds for an arbitrary volume. The left side represents the viscous dissipation inside V, while the right represents the energy flow through the boundary ∂V.

If there is a finite collection of vortex structures in the bulk, we can expand this volume to an infinite sphere; in this case, the $\vec{\omega} \times \vec{v}$ term drops as there is no vorticity at infinity.

Furthermore, the velocity in the Biot-Savart law decreases as $|\vec{r}|^{-3}$ at infinity, so that only the $\vec{v}p$ term survives

$$\langle \mathcal{E}_V \rangle \to -\int_{\partial V} d\vec{\sigma} \cdot \langle \vec{v} p \rangle \tag{3}$$

This energy flow on the right side will stay finite in the limit of the expanding sphere in case the pressure grows as $p \to -\vec{f} \cdot \vec{r}$, where $\vec{f}$ is the local force at a given point on a large sphere.

$$\langle \mathcal{E} \rangle = \vec{f}_\alpha \lim_{R \to \infty} R^3 \int_{S_2} n_\alpha n_\beta \left\langle v_\beta (R\vec{n}) \right\rangle \tag{4}$$

Where did we lose the Kolmogorov energy flow? It is still there for any finite volume surrounding the vortex sheet

$$\langle \mathcal{E}_V \rangle = -\int_V d^3r \left\langle v_\beta \partial_\beta p + v_\alpha v_\beta \partial_\beta v_\alpha \right\rangle = \tag{5}$$

$$-\int_V d^3r \left\langle v_\alpha v_\beta \partial_\beta v_\alpha \right\rangle - \int_{\partial V} d\vec{\sigma} \cdot \langle \vec{v} p \rangle \tag{6}$$

The first term is the Kolmogorov energy flow inside the volume V, and the second is the energy flow through the boundary.

Without finite force $\vec{f}$ acting on the boundary, say, with periodic boundary conditions, the boundary integral would be absent, and we would recover the Kolmogorov relation.

In the conventional approach, based on the time averaging of the Navier-Stokes equations, the periodic Gaussian random force $\vec{f}(\vec{r})$ is added to the right side. In this case, with periodic boundary conditions

$$\langle \mathcal{E}_V \rangle = -\int_V d^3r \left\langle v_\beta \partial_\beta p - v_\beta f_\beta(\vec{r}) + v_\alpha v_\beta \partial_\beta v_\alpha \right\rangle = \int_V d^3r \left\langle v_\beta f_\beta(\vec{r}) \right\rangle \tag{7}$$

In the limit when the force becomes uniform in space, we recover another definition with $\mathcal{E} = \vec{f} \cdot \vec{P}$, where $\vec{P} = \int_V d^3r \vec{v}$ is a total momentum.

The turbulence phenomenon we study is a universal spontaneous stochasticity independent of the boundary conditions.

As long as there is an energy flow from the boundaries, the confined turbulence in the middle would dissipate this flow in singular vortex tubes. The spontaneous stochasticity results from the random distribution of these singular tubes (Kelvinons) inside the volume in the velocity flow picture [4]. In the dual picture of our recent theory [3], these are the random gaps in the momentum curve $\vec{P}(\theta)$.

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The relation between the energy pumping on the large sphere and the distribution of the vortex blobs in bulk follows from the Biot-Savart integral

$$\vec{v}(r) = -\vec{\nabla} \times \int d^3r' \frac{\vec{\omega}(r')}{4\pi|r - r'|} \tag{8}$$

On a large sphere ∂V with radius $R \to \infty$,

$$\lim_{R\to\infty} R^3\vec{v}(R\vec{n}) \propto \frac{1}{4\pi} \sum_{\text{blobs}} \int_{\text{blob}} d^3r'\vec{\omega}(r') \times (\vec{r}' - < \vec{r} >_{\text{blob}}); \tag{9}$$

Here $< \vec{r} >_{\text{blob}}$ is the geometric center of each blob. Substituting this into the identity (4), we directly relate the energy pumping with the forces at the boundary and the blob's dipole moments of vorticity.

No forcing inside the flow is needed for this energy pumping; the energy flow starts at the boundary and propagates to numerous singular vorticity blobs, where it is finally dissipated. The distribution of these vorticity blobs is all we need for the turbulence theory. The forcing is required only as a boundary condition at infinity.

These assumptions about confined turbulence as stochastic dynamics of isolated vortex structures were confirmed in a beautiful experimental work by William Irvine and collaborators at Chicago University ([5]). The energy was pumped in by vortex rings flying from eight corners of a large glass cube and colliding in the center, making a turbulent blob.

They measured the (approximately) Kolmogorov energy spectrum, proving that periodic boundary conditions were unnecessary.

The latest paper [6] also observed how the singular vortex structures move and reconnect inside this confined turbulence.

As for the decaying turbulence, these authors observed (William Irvine, private communication) two distinct decay regimes, not just a single power law like the old works [1]. This observation agrees with the results of [2], where regime changes were observed in numerical simulations of the Euler ensemble.

Another critical comment: with the velocity correlations **growing** with distance by the approximate K41 law, even the forcing at the remote boundary would influence the potential part of velocity in bulk. This boundary influence makes the energy cascade picture non-universal; it may depend upon the statistics of the random forcing. Only the statistics of the rotational part of velocity, i.e., vorticity, could reach some universal regime independent of the boundary conditions at infinity.

Unlike the potential part of velocity, the vorticity is localized in singular regions – tubes and sheets, sparsely filling the space, as observed in numerical simulations. The potential part of velocity drops in the loop equations, and the remaining stochastic motion of the velocity circulation is equivalent to the vorticity statistics. Therefore, our solutions [3] of the loop equations [7,8] describe the internal stochastization of the decaying turbulence by a dual discrete system. Measuring these internal stochastic phenomena challenges real experiments and numerical simulations alike.

2. Mathematical Introduction. The loop equation and its solution

In the previous paper, [3], we have found a family of exact solutions of the loop equation for decaying turbulence [7,8]. This family describes a **fixed trajectory** of solutions with the universal time decay factor. The solutions are formulated in terms of the Wilson loop or loop average

$$\Psi[\gamma, C] = \left\langle \exp\left(\frac{\imath\gamma}{\nu} \oint d\vec{C}(\theta) \cdot \vec{v}\left(\vec{C}(\theta)\right)\right)\right\rangle_{init}; \tag{10}$$

$$\Psi[\gamma, C] \Rightarrow \left\langle \exp\left(\frac{\imath\gamma}{\nu} \oint d\vec{C}(\theta)\vec{P}(\theta)\right)\right\rangle_{\mathbb{E}}; \tag{11}$$

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In the first equation (the definition), the averaging $\langle \dots \rangle$ goes over initial data for the solutions of the Navier-Stokes equation for velocity field $\vec{v}(\vec{r})$. In the second one (the solution), the averaging goes over the distribution of the random variable $\vec{P}(\theta)$ to be determined from the loop equation [3]. We choose in this paper the parametrization of the loop with $\xi = \frac{\theta}{2\pi}$ to match with the fermionic coordinates below (the parametrization is arbitrary, in virtue of parametric invariance of the loop dynamics).

The loop equation for the momentum loop $\vec{P}(\theta)$ follows from the Navier-Stokes equation for $\vec{v}$

$$\partial_t v_\alpha = \nu \partial_\beta \omega_{\beta\alpha} - v_\beta \omega_{\beta\alpha} - \partial_\alpha \left(p + \frac{v_\beta^2}{2} \right); \tag{12}$$

$$\partial_\alpha v_\alpha = 0; \tag{13}$$

$$\omega_{\beta\alpha} = \partial_\beta v_\alpha - \partial_\alpha v_\beta \tag{14}$$

After some loop calculus transformation, replacing velocity and vorticity by the functional derivatives by the loop $\vec{C}$ acting on a loop functional, we found the following momentum loop equation in [3,8]

$$\nu \partial_t \vec{P} = -\gamma^2 (\Delta \vec{P})^2 \vec{P} + \Delta \vec{P} \left(\gamma^2 \vec{P} \cdot \Delta \vec{P} + \imath \gamma \left(\frac{(\vec{P} \cdot \Delta \vec{P})^2}{\Delta \vec{P}^2} - \vec{P}^2 \right) \right); \tag{15}$$

$$\vec{P}(\theta) \equiv \frac{\vec{P}(\theta^+) + \vec{P}(\theta^-)}{2}; \tag{16}$$

$$\Delta \vec{P}(\theta) \equiv \vec{P}(\theta^+) - \vec{P}(\theta^-); \tag{17}$$

The momentum loop has a discontinuity $\Delta \vec{P}(\theta)$ at every parameter $0 < \theta \leq 1$, making it a fractal curve in complex space $\mathbb{C}_d$. The details can be found in [3,8]. We will skip the arguments t, θ in these loop equations, as there is no explicit dependence of these equations on either of these variables.

This Anzatz represents a plane wave in loop space, solving the loop equation for the Wilson loop due to the lack of direct dependence of the loop operator on the shape of the loop.

The superposition of these plane wave solutions would solve the **Cauchy problem in loop space**: find the stochastic function $\vec{P}_0(\theta)$ at $t = 0$, providing the initial distribution of the velocity field. Formally, the initial distribution $W_0[P]$ of the momentum field $\vec{P}(\theta)$ is given by inverse functional Fourier transform.

$$W_0[P] = \int DC \delta^3(\vec{C}[0]) \Psi[C, \gamma]_{t=0} \exp\left(-\frac{\imath\gamma}{\nu} \int d\vec{C}(\theta) \cdot \vec{P}(\theta) \right) \tag{18}$$

This functional integral was computed in [3,8] for a special stochastic solution of the Navier-Stokes equation: the global rotation with Gaussian random rotation matrix.

Though this special solution does not describe isotropic turbulence, it helps understand the mathematical properties of the loop technology. In particular, it shows the significance of the discontinuities of the momentum loop $\vec{P}(\theta)$.

Rather than solving the Cauchy problem, we are looking for an attractor: the fixed trajectory for $\vec{P}(\theta, t)$ with some universal probability distribution related to the decaying turbulence statistics.

The following transformation reveals the hidden scaling invariance of decaying turbulence

$$\vec{P} = \sqrt{\frac{\nu}{2(t + t_0)}} \frac{\vec{F}}{\gamma} \tag{19}$$

The new vector function $\vec{F}$ satisfies an equation

$$2\partial_\tau \vec{F} = \left(1 - (\Delta \vec{F})^2\right) \vec{F} + \Delta \vec{F} \left(\gamma^2 \vec{F} \cdot \Delta \vec{F} + \imath \gamma \left(\frac{(\vec{F} \cdot \Delta \vec{F})^2}{\Delta \vec{F}^2} - \vec{F}^2 \right) \right); \tag{20}$$

$$\tau = \log(t + t_0) \tag{21}$$

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This equation is invariant under translations of the new variable $\tau = \log(t + t_0)$, corresponding to the rescaling/translation of the original time.

$$t \Rightarrow \lambda t + (\lambda - 1)t_0 \tag{22}$$

There are two consequences of this invariance.

- There is a fixed point for $\vec{F}$.
- The approach to this fixed point is exponential in τ, which is power-like in original time.

Both of these properties were used in [3]: the first one was used to find a fixed point, and the second one was used to derive the spectral equation for the anomalous dimensions λ_i of decay $t^{-\lambda_i}$ of the small deviations from these fixed points. In this paper, we only consider the fixed point, leaving the exciting problem of the spectrum of anomalous dimensions for future research.

3. The big and small Euler ensembles

Let us remember the basic properties of the fixed point for $\vec{F}$ in [3]. It is defined as a limit $N \to \infty$ of the polygon $\vec{F}_0 \dots \vec{F}_N = \vec{F}_0$ with the following vertices

$$\vec{F}_k = \frac{\left\{\cos(\alpha_k), \sin(\alpha_k), i\cos\left(\frac{\beta}{2}\right)\right\}}{2\sin\left(\frac{\beta}{2}\right)}; \tag{23}$$

$$\theta_k = \frac{k}{N}; \; \beta = \frac{2\pi p}{q}; \; N \to \infty; \tag{24}$$

$$\alpha_{k+1} = \alpha_k + \sigma_k \beta; \; \sigma_k = \pm 1, \; \beta \sum \sigma_k = 2\pi p r; \tag{25}$$

The parameters $\hat{\Omega}, p, q, r, \sigma_0 \dots \sigma_N = \sigma_0$ are random, making this solution for $\vec{F}(\theta)$ a fixed manifold rather than a fixed point.

It is a fixed point of (20) with the discrete version of discontinuity and principal value:

$$\Delta \vec{F} \equiv \vec{F}_{k+1} - \vec{F}_k; \tag{26}$$

$$\vec{F} \equiv \frac{\vec{F}_{k+1} + \vec{F}_k}{2} \tag{27}$$

Both terms of the right side (20) vanish; the term proportional to $\Delta\vec{F}$ and the term proportional to $\vec{F}$. Otherwise, we would have $\vec{F} \parallel \Delta\vec{F}$, leading to zero vorticity [3]. The ensemble of all the different solutions is called the big Euler ensemble. The integer numbers $\sigma_k = \pm 1$ came as the solution of the loop equation, and the requirement of the rational $\frac{p}{q}$ came from the periodicity requirement.

We can use integration (summation) by parts to write the circulation as follows (in virtue of periodicity):

$$\oint d\vec{C}(\theta) \cdot \vec{P}(\theta) = -\oint d\vec{P}(\theta) \cdot \vec{C}(\theta); \tag{28}$$

$$\sum_k \Delta C_k \vec{P}_k = -\oint \Delta\vec{P}_k \cdot \vec{C}_k; \tag{29}$$

We assign equal weights to all elements of this set; we call this conjecture the ergodic hypothesis. This prescription is similar to assigning equal weights to each triangulation of curved space with the same topology in dynamically triangulated quantum gravity [9]. Mathematically, this is the most symmetric weight assignment, and there are general expectations that various discrete theories converge into the same symmetry classes of continuum theories in the statistical limit. This method works remarkably well in two dimensions, providing the same correlation functions as continuum gravity (Liouville theory).

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The fractions $\frac{p}{q}$ with fixed denominator are counted by Euler totient function $\varphi(q)$ [10]

$$\varphi(q) = \sum_{\substack{p=1\\(p,q)}}^{q-1} 1 = q \prod_{p|q} \left(1 - \frac{1}{p}\right); \tag{30}$$

$$\tag{31}$$

In some cases, one can analytically average over spins σ in the big Euler ensemble, reducing the problem to computations of averages over the small Euler ensemble $\mathcal{E}(N) : N, p, q, r$ with the measure induced by averaging over the spins in the big Euler ensemble.

In this paper, we perform this averaging over σ analytically, without any approximations, reducing it to a trace of a certain quantum mechanical system with Fermi particles.

4. The Markov chain and its Fermionic representation

Here is a new representation of the Euler ensemble, leading us to the exact analytic solution below.

We start by replacing independent random variables σ with fixed sum by a Markov process, as suggested in [3]. We start with n random values of $\sigma_i = 1$ and remaining $N - n$ values of $\sigma_i = -1$. Instead of averaging over all of these values simultaneously, we follow a Markov process of picking $\sigma_N, \dots \sigma_1$ one after another. At each step, there will be $M = N, \dots 0$ remaining σ. We get a transition $n \Rightarrow n - 1$ with probability $\frac{n}{M}$ and $n \Rightarrow n$ with complementary probability.

Multiplying these probabilities and summing all histories of the Markov process is equivalent to the computation of the product of the Markov matrices

$$\prod_{M=1}^{N} Q(M); \tag{32}$$

$$Q(M)|n\rangle = \frac{M-n}{M}|n\rangle + \frac{n}{M}|n-1\rangle; \tag{33}$$

This Markov process will be random until $n = 0$. After that, all remaining σ_k will have negative signs and be taken with probability 1, keeping $n = 0$.

The expectation value of some function $X(\{\sigma\})$ reduces to the matrix product

$$\mathbb{P}[X] = \sum_{n=0}^{N_+} \langle n| \left(\prod_{M=1}^{N} \hat{Q}(M) \right) \cdot X \cdot |N_+\rangle; \tag{34}$$

$$\hat{Q}(M) \cdot X|n\rangle = \frac{n}{M} X(\sigma_M \to 1)|n-1\rangle + \frac{M-n}{M} X(\sigma_M \to -1)|n\rangle \tag{35}$$

Here $N_+ = (N + \sum \sigma_l)/2 = (N + qr)/2$ is the number of positive sigmas. The operator $\hat{Q}(M)$ sets in $X|n\rangle$ the variable σ_M to 1 with probability $\frac{n}{M}$ and to -1 with complementary probability. The generalization of the Markov matrix $Q(M)$ to the operator $\hat{Q}(M)$ will be presented shortly.

Once the whole product is applied to X, all the sigma variables in all terms will be specified so that the result will be a number.

This Markov process is implemented as a computer code in [2], leading to a fast simulation with $O(N^0)$ memory requirement.

Now, we observe that quantum Fermi statistics can represent the Markov chain of Ising variables. Let us construct the operator $\hat{Q}(M)$ with Fermionic creation and annihilation operators, with occupation numbers $\nu_k = (1 + \sigma_k)/2 = (0, 1)$. These operators obey (anti)commutation relations, and they create/annihilate $\sigma = 1$ state as follows (with Kronecker delta $\delta[n] \equiv \delta_{n,0}$):

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$$\left[a_i, a_j^\dagger\right]_+ = \delta_{ij}; \tag{36}$$
$$\left[a_i, a_j\right]_+ = \left[a_i^\dagger, a_j^\dagger\right]_+ = 0; \tag{37}$$
$$a_n^\dagger|\sigma_1, \dots, \sigma_N\rangle = \delta[\sigma_n + 1]|\sigma_1, \dots, \sigma_n \to 1, \dots \sigma_N\rangle; \tag{38}$$
$$a_n|\sigma_1, \dots, \sigma_N\rangle = \delta[\sigma_n - 1]|\sigma_1, \dots, \sigma_n \to -1, \dots \sigma_N\rangle; \tag{39}$$
$$\hat{v}_n = a_n^\dagger a_n; \tag{40}$$
$$\hat{v}_n|\sigma_1, \dots, \sigma_N\rangle = \delta[\sigma_n - 1]|\sigma_1, \dots, \sigma_N\rangle \tag{41}$$

The number $n(M)$ of positive sigmas $\sum_{l=1}^M \delta[\sigma_l - 1]$ coincides with the occupation number of these Fermi particles.

$$\hat{n}(M) = \sum_{l=1}^{M} \hat{v}_l; \tag{42}$$

This relation leads to the representation

$$\hat{Q}(M) = \hat{v}_M \frac{\hat{n}(M)}{M} + (1 - \hat{v}_M)\frac{M - \hat{n}(M)}{M}; \tag{43}$$

The variables σ_l can also be expressed in terms of this operator algebra by using

$$\hat{\sigma}_l = 2\hat{v}_l - 1. \tag{44}$$

The Wilson loop in (10) can now be represented as an average over the small Euler ensemble $\mathcal{E}(N)$ of a quantum trace expression

$$\Psi[\gamma, C] = \frac{\left\langle \operatorname{tr}\left(\hat{Z}(qr) \exp\left(\frac{\imath\gamma}{\nu}\sum_l \Delta\vec{C}_l \cdot \hat{\Omega} \cdot \vec{\mathcal{P}}_l(t)\right) \prod_{M=1}^{N} \hat{Q}(M)\right)\right\rangle_{\hat{\Omega},\mathcal{E}(N)}}{\left\langle \operatorname{tr}\left(\hat{Z}(qr) \prod_{M=1}^N \hat{Q}(M)\right)\right\rangle_{\mathcal{E}(N)}}, \tag{45}$$
$$\hat{Z}(s) = \oint \frac{d\omega}{2\pi} \exp\left(\imath\omega\left(\sum_l \hat{\sigma}_l - s\right)\right); \tag{46}$$
$$\Delta\vec{C}_l = \vec{C}\left(\frac{l+1}{N}\right) - \vec{C}\left(\frac{l}{N}\right), \tag{47}$$
$$\vec{\mathcal{P}}_l(t) = \sqrt{\frac{\nu}{2(t+t_0)}}\frac{\vec{\mathcal{F}}_l}{\gamma}, \quad \hat{\Omega} \in O(3), \tag{48}$$
$$\vec{\mathcal{F}}_l = \frac{\{\cos(\hat{\alpha}_l), \sin(\hat{\alpha}_l), 0\}}{2\sin\left(\frac{\beta}{2}\right)}, \tag{49}$$
$$\mathcal{E}(N): \quad p, q, r \in \mathbb{Z} \quad \text{with} \quad 0 < p < q < N, \mathbf{gcd}(p,q) = 1, \quad -N \le qr \le N, \tag{50}$$
$$\hat{\alpha}_l = \beta \sum_{k=1}^{l-1} (2\hat{v}_k - 1); \tag{51}$$

The last component of the vector $\vec{\mathcal{F}}_l$ is set to 0 as this component does not depend on l and yields zero in the sum over the loop $\sum_l \Delta\vec{C}_l = 0$.

The proof of equivalence to the combinatorial formula with an average over $\sigma_l = \pm 1$ can be given using the following Lemma (obvious for a physicist).

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Lemma 1. *The operators $\hat{\nu}_l$ all commute with each other.*

Proof. Using commutation relations, we can write

$$\hat{\nu}_l \hat{\nu}_n = a_l^\dagger (\delta_{ln} - a_n^\dagger a_l) a_n = a_l^\dagger a_n \delta_{ln} - a_l^\dagger a_n^\dagger a_l a_n \tag{52}$$

Interchanging indexes l, n in this relation, we see that the first term does not change due to Kronecker delta, and the second term does not change because $a_l^\dagger, a_n^\dagger$ anti-commute, as well as a_l, a_n, so the second term is symmetric as well. Therefore, $\hat{\nu}_l \hat{\nu}_n = \hat{\nu}_n \hat{\nu}_l$ □

Quantum Trace Theorem. *The trace formula* (45) *equals the expectation value of the momentum loop ansatz* (11), (19), (23) *in the big Euler ensemble.*

Proof. As all the operators $\hat{\nu}_l$ commute with each other, the operators $\hat{Q}(M)$ can be applied in arbitrary order to the states $\Sigma = |\sigma_1, \dots \sigma_N\rangle$ involved in the trace. The same is true about individual terms in the circulation in the exponential of the Wilson loop. These terms $\vec{\mathcal{F}}_l$ involve the operators $\hat{\alpha}_l$, which commute with each other and with each $\hat{Q}(M)$. Thus, we can use the ordered product of the operators $\hat{G}_l = \hat{Q}(l) \exp\left(\imath\omega\hat{\sigma}_l + \frac{i\gamma}{\nu}\Delta\vec{C}_l \cdot \vec{\mathcal{P}}_l(t)\right)$. Each of the operators $\hat{G}_l$ acting in turn on arbitrary state Σ will create two terms with $\delta[\sigma_l \pm 1]$. The exponential in $\hat{G}_l$ will involve $\hat{\sigma}_k, k \leq l$. As a result of the application of the operator $\hat{Z}_l = \prod_{k=1}^{l} \hat{G}_k$ to the state vector Σ we get 2^l terms with $\Sigma \prod_{k=1}^{l} \delta[\sigma_k - \eta_k], \eta_k = \pm 1$. The factors $\hat{Z}_l$ will involve only $\hat{\sigma}_k, k \leq l$, which are all reduced to $\eta_k, k \leq l$ in virtue of the product of the Kronecker deltas. Multiplying all operators $\hat{G}_M$ will lead to superposition $\hat{\Pi}_N$ of 2^N terms, each with product $\prod_{M=1}^{N} \delta[\sigma_M - \eta_M]$ with various choices of the signs $\eta_i = \pm 1$ for each i. Furthermore, the product of Kronecker deltas will project the total sum of 2^N combinations of the states Σ in the trace tr ... to a single term corresponding to a particular history $\eta_1, \dots \eta_N$ of the Markov process. The product of Kronecker deltas in each history will be multiplied by the same state vector Σ, by the product of Markov transition probabilities, and by the exponential $\exp\left(\frac{i\gamma}{\nu} \sum_l \Delta\vec{C}_l \cdot \hat{\Omega} \cdot \vec{P}_l(t)\right)$ with the operators $\hat{\sigma}$ in $\vec{\mathcal{P}}_k(t)$ replaced by numbers η leading to the usual numeric $\vec{P}_l(t)$. The transition probabilities of the Markov process are designed to reproduce combinatorial probabilities of random sigmas, adding up to one after summation over histories [11]. The integration over ω will produce $\delta\left[\sum_l \hat{\eta}_l - s\right]$. This delta function will reduce the trace to the required sum over all histories of the Markov process with a fixed $\sum_l \eta_l$. □

5. Conclusion

We have found a third vertex of the triangle of equivalent theories: the decaying turbulence in three-dimensional space, the fractal curve in complex space, and Fermi particles on a ring. By degrees of freedom, this is a one-dimensional Fermi-gas in the statistical limit $N \to \infty$. However, there is no local Hamiltonian in this quantum partition function, just a trace of certain products of operators in Fock space. So, an algebraic (or quantum statistical) problem remains to find the continuum limit of this theory of the fermion ring.

This problem is addressed in the next article of this series.

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