Article

Not peer-reviewed version

On the Irrationality of the π-Normalized Odd Zeta Values

Archan Chattopadhyay *

Posted Date: 19 January 2026

doi: 10.20944/preprints202601.1390.v1

Keywords: odd zeta values; proof of irrationality; integer linear forms; Diophantine approximation; probabilistic methods in number theory

 doi:10.20944/preprints202601.1390.v1

Article

On the Irrationality of the π-Normalized Odd Zeta Values

Archan Chattopadhyay

Independent Researcher; archanc@alum.iisc.ac.in

Abstract

We prove the irrationality of a family of normalized odd zeta values of the form $\dfrac{\zeta(2n+1)}{\pi^{2n+1}}, \quad n \in \mathbb{N}, \quad n \geq 3$. Our approach is based on constructing explicit integer linear forms in the quantities $I_n = 4(4^n - 1)\left[\dfrac{\zeta(2n)\zeta(2n+2)}{\zeta(2n+1)^2} - 1\right] - 1$, and applying a refinement of Dirichlet's approximation theorem. The construction of the I_n is probabilistic in origin. We prove that the sequence of denominators produced by successive rational approximations yields infinitely many nontrivial integer relations of the type $\Lambda_m^{(q)} = A_m^{(q)} I_n - B_m^{(q)}$, with $|\Lambda_m^{(q)}|$ (q being a parameter) decaying towards zero as m approaches infinity. This permits us to invoke a general irrationality criterion and thereby deduce that I_n is irrational for each $n \geq 3$. Consequently, each corresponding normalized odd zeta value is irrational. Our method combines ideas from probability theory, analytic combinatorics and Diophantine approximation, and complements earlier work of Apéry, Beukers, Rivoal, and Zudilin.

Keywords: odd zeta values; proof of irrationality; integer linear forms; Diophantine approximation; probabilistic methods in number theory

MSC: 11M06; 11J72; 11J20; 11Y60; 33E20; 60E05

---

1. Introduction

1.1. Background and Motivation

The study of the arithmetic properties of special values of the Riemann zeta function occupies a central place in analytic number theory. Euler's classical formulae [1]

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}, \quad n \in \mathbb{N}$$

express the even zeta values in terms of Bernoulli numbers and powers of π, thereby establishing both their transcendence and algebraic relations to π. In striking contrast, the arithmetic nature of the odd zeta values $\zeta(2n+1)$ remains largely mysterious. The pioneering breakthrough was Apéry's proof [2,3] of the irrationality of $\zeta(3)$, which not only settled a longstanding problem of Euler but also introduced a new paradigm for irrationality proofs through the construction of highly nontrivial linear recurrences. Subsequent refinements by Beukers [4] via integrals, and the landmark results of Rivoal [5] and Zudilin [6] showed that infinitely many odd zeta values must be irrational, and that at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational. Nevertheless, no specific irrationality result beyond Apéry's $\zeta(3)$ has been obtained, and the question of whether $\zeta(2n+1)$ is irrational for $n \geq 2$ remains open.

The goal of this paper is to prove irrationality results for a family of π-normalized odd zeta values, by reducing the problem to the irrationality of an explicitly defined correction term I_n, and applying a refined Diophantine approximation argument.

1.2. Main Theorem

We begin by stating the principal result.

 doi:10.20944/preprints202601.1390.v1

Theorem 1 (Main theorem)**.** *For every integer $n \geq 3$, the normalized odd zeta value*

$$\frac{\zeta(2n+1)}{\pi^{2n+1}}$$

is irrational. Equivalently, the quantity

$$I_n = 4(4^n - 1)\left[\frac{\zeta(2n)\zeta(2n+2)}{\zeta(2n+1)^2} - 1\right] - 1$$

is irrational for every $n \geq 3$.

1.3. Strategy of the Proof (Sketch)

We outline the logical structure of the proof before entering the technical details.

1. *Exact representation of $\zeta(2n+1)$.* We first develop a novel representation of $\zeta(2n+1)$:

$$\zeta(2n+1) = \sqrt{\frac{\zeta(2n)\zeta(2n+2)}{1 + \frac{1+I_n}{4(4^n-1)}}},$$

where the correction factor I_n is defined as the integral

$$I_n = \int_0^1 r_n'(x)\,dx,$$

and $r_n(x)$ is a function of polylogarithms:

$$r_n(x) = \frac{\left[\frac{\mathrm{Li}_{2n+2}(x)}{\mathrm{Li}_{2n}(x)} - \frac{\mathrm{Li}_{2n+1}(x)^2}{\mathrm{Li}_{2n}(x)^2}\right]\left[1 - \frac{\mathrm{Li}_{2n}(-x)^2}{\mathrm{Li}_{2n}(x)^2}\right]}{\left[\frac{\mathrm{Li}_{2n+1}(-x)}{\mathrm{Li}_{2n}(x)} - \frac{\mathrm{Li}_{2n+1}(x)\,\mathrm{Li}_{2n}(-x)}{\mathrm{Li}_{2n}(x)^2}\right]^2}.$$

2. *Reduction to irrationality of I_n.* Using Euler's expression for the even zeta values, the earlier representation transforms into

$$\zeta(2n+1) = \frac{(2\pi)^{2n+1}}{2(2n)!}\sqrt{\frac{-B_{2n}B_{2n+2}}{(2n+1)(2n+2)\left[1 + \frac{1+I_n}{4(4^n-1)}\right]}}.$$

From the above formula, we see that proving the irrationality of the normalized odd zeta values translates to proving the irrationality of I_n.

3. *Construction of integer linear forms.* We construct for each $m \in \mathbb{N}$ an asymmetric beta kernel $W_m^{(q)}(x)$ and its cumulative distribution function $S_m^{(q)}(x)$, enabling an exact integral identity linking I_n with a sequence of weighted moments $L_m^{(q)}$. Truncating the Maclaurin expansion of $r_n(x)$ yields integer linear forms $\Lambda_m^{(q)} = A_m^{(q)} I_n - B_m^{(q)}$, whose coefficients are made integral by a suitable least common multiple of denominators.
4. *Asymptotic decay and growth estimates.* Exponential decay of the kernel tail and remainder terms is established using Stirling asymptotics and L^p norms, while precise denominator growth is controlled through Legendre's formula and the Prime Number Theorem.
5. *Application of an irrationality criterion.* The balance of these estimates ensures that $|\Lambda_m^{(q)}| \to 0$ as $m \to \infty$, permitting application of a refined Dirichlet-type irrationality criterion to prove that I_n is irrational.
6. *Conclusion.* This yields the irrationality of the normalized odd zeta values for all $n \geq 3$.

doi:10.20944/preprints202601.1390.v1

2. Preparatory Results: Effective Closed-Form Expressions for $\zeta(2n+1)$

2.1. A Probabilistic Identity

Our starting device is the following one-parameter family of probability distributions.

Definition 1 (Generalized zeta distribution). *Fix real parameters $s > 1$ and $x \in [0,1]$. Define a discrete random variable X with probability mass function*

$$\mathrm{P}(X = k) = f_s(k; x) := \frac{1}{\mathrm{Li}_s(x)} \frac{x^k}{k^s}, \quad k \in \mathbb{N},$$

where $\mathrm{Li}_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s}$ *denotes the polylogarithm [7].*

Let $s = 2n$, and define the random variables $U := \frac{1}{X}$, $V := (-1)^X$. A routine computation of moments yields

$$\mathrm{E}(U) = \frac{1}{\mathrm{Li}_{2n}(x)} \sum_{k=1}^{\infty} \frac{x^k}{k^{2n+1}} = \frac{\mathrm{Li}_{2n+1}(x)}{\mathrm{Li}_{2n}(x)}, \tag{1}$$

$$\mathrm{E}(V) = \frac{1}{\mathrm{Li}_{2n}(x)} \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k^{2n}} = \frac{\mathrm{Li}_{2n}(-x)}{\mathrm{Li}_{2n}(x)}, \tag{2}$$

$$\mathrm{E}(U^2) = \frac{1}{\mathrm{Li}_{2n}(x)} \sum_{k=1}^{\infty} \frac{x^k}{k^{2n+2}} = \frac{\mathrm{Li}_{2n+2}(x)}{\mathrm{Li}_{2n}(x)}, \tag{3}$$

$$\mathrm{E}(V^2) = \frac{1}{\mathrm{Li}_{2n}(x)} \sum_{k=1}^{\infty} \frac{(-1)^{2k} x^k}{k^{2n}} = \frac{\mathrm{Li}_{2n}(x)}{\mathrm{Li}_{2n}(x)} = 1, \tag{4}$$

$$\mathrm{E}(UV) = \frac{1}{\mathrm{Li}_{2n}(x)} \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k^{2n+1}} = \frac{\mathrm{Li}_{2n+1}(-x)}{\mathrm{Li}_{2n}(x)}. \tag{5}$$

From these identities, the squared Pearson correlation coefficient $\rho_n(x)^2$ (Chapter 7.4 [8]) can be written in closed form as an explicit function of the polylogarithms $\mathrm{Li}_{2n}(\pm x)$, $\mathrm{Li}_{2n+1}(\pm x)$, and $\mathrm{Li}_{2n+2}(x)$:

$$\begin{aligned} \rho_n(x)^2 &= \frac{\mathrm{cov}(U,V)^2}{\mathrm{var}(U)\,\mathrm{var}(V)} \\ &= \frac{[\mathrm{E}(UV) - \mathrm{E}(U)\,\mathrm{E}(V)]^2}{[\mathrm{E}(U^2) - \mathrm{E}(U)^2][\mathrm{E}(V^2) - \mathrm{E}(V)^2]} \\ &= \frac{\left[\dfrac{\mathrm{Li}_{2n+1}(-x)}{\mathrm{Li}_{2n}(x)} - \dfrac{\mathrm{Li}_{2n+1}(x)\,\mathrm{Li}_{2n}(-x)}{\mathrm{Li}_{2n}(x)^2}\right]^2}{\left[\dfrac{\mathrm{Li}_{2n+2}(x)}{\mathrm{Li}_{2n}(x)} - \dfrac{\mathrm{Li}_{2n+1}(x)^2}{\mathrm{Li}_{2n}(x)^2}\right]\left[1 - \dfrac{\mathrm{Li}_{2n}(-x)^2}{\mathrm{Li}_{2n}(x)^2}\right]}. \end{aligned} \tag{6}$$

Setting $x = 1$, and noting that

$$\begin{aligned} \mathrm{Li}_s(1) &= \zeta(s), \\ \mathrm{Li}_s(-1) &= -\eta(s) = -\left(1 - \frac{1}{2^{s-1}}\right)\zeta(s) \end{aligned}$$

(where $\eta(s)$ is the Dirichlet eta function), we obtain the algebraic relation

$$\rho_n(1)^2 = \frac{\left[-\dfrac{\eta(2n+1)}{\zeta(2n)} + \dfrac{\zeta(2n+1)\eta(2n)}{\zeta(2n)^2}\right]^2}{\left[\dfrac{\zeta(2n+2)}{\zeta(2n)} - \dfrac{\zeta(2n+1)^2}{\zeta(2n)^2}\right]\left[1 - \dfrac{\eta(2n)^2}{\zeta(2n)^2}\right]}. \tag{7}$$

 doi:10.20944/preprints202601.1390.v1

Defining
$$r_n(x) := \frac{1}{\rho_n(x)^2}, \tag{8}$$
and solving for $\zeta(2n+1)$ yields the identity
$$\zeta(2n+1) = \sqrt{\frac{\zeta(2n)\zeta(2n+2)}{1 + \dfrac{r_n(1)}{4(4^n-1)}}}, \tag{9}$$
where $r_n(1)$ is a correction factor. Note that the numerator of $\rho_n(1)^2$ does not vanish; hence, $r_n(1)$ is well-defined.

2.2. Integral Representation of the Correction Factor

A key step towards effectivity is an explicit representation of $r_n(1)$ that avoids circular dependence on $\zeta(2n+1)$. To that end, we verify the following properties of $r_n(x)$ for $x \in [0,1]$:

Lemma 1 (Rational coefficients). *$r_n(x)$ can be expressed as a power series*
$$r_n(x) := \sum_{k=0}^{\infty} c_{n,k} x^k, \tag{10}$$
where the coefficients $c_{n,k}$ are rational numbers, and $c_{n,0} \equiv 1$. These coefficients satisfy the decay law
$$c_{n,k} = O\left(\frac{1}{k^{2n-2}}\right). \tag{11}$$

Proof. Let
$$A := \mathrm{Li}_{2n}(x), \quad B := \mathrm{Li}_{2n}(-x), \quad C := \mathrm{Li}_{2n+1}(x), \quad D := \mathrm{Li}_{2n+1}(-x), \quad E := \mathrm{Li}_{2n+2}(x).$$
Then, from (6),
$$r_n = \frac{FG}{H^2},$$
where
$$F := EA - C^2, \quad G := A^2 - B^2, \quad H := DA - CB.$$
The coefficient of x^k in F is (for $i, j \geq 1$)
$$[x^k]F = \sum_{i+j=k} \left(\frac{1}{i^{2n+2} j^{2n}} - \frac{1}{i^{2n+1} j^{2n+1}} \right).$$
Group the terms in symmetric pairs (i,j) and (j,i). The paired contribution is
$$\frac{1}{i^{2n+2} j^{2n}} + \frac{1}{j^{2n+2} i^{2n}} - \frac{2}{i^{2n+1} j^{2n+1}} = \frac{(j-i)^2}{i^{2n+2} j^{2n+2}}.$$
Therefore, $[x^k]F = \sum_{\substack{i+j=k \\ i<j}} \frac{(j-i)^2}{i^{2n+2} j^{2n+2}}$. Similarly,
$$[x^k]G = \sum_{i+j=k} \frac{1-(-1)^{i+j}}{i^{2n} j^{2n}} = \begin{cases} 0, & k \text{ even}, \\ 4 \displaystyle\sum_{\substack{i+j=k \\ i<j}} \frac{1}{i^{2n} j^{2n}}, & k \text{ odd}, \end{cases}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 January 2026 doi:10.20944/preprints202601.1390.v1

and

$$[x^k]H = \sum_{i+j=k} \frac{(-1)^i - (-1)^j}{i^{2n+1} j^{2n}} = \begin{cases} 0, & k \text{ even}, \\ 2 \sum_{\substack{i+j=k \\ i<j}} \frac{(-1)^i (j-i)}{i^{2n+1} j^{2n+1}}, & k \text{ odd}. \end{cases}$$

In particular, F, G, H all vanish to order exactly 3, and their leading coefficients are explicit rational numbers (powers of 2). Write

$$F(x) := x^3 f(x), \quad G(x) := x^3 g(x), \quad H(x) := x^3 h(x),$$

where $f, g, h \in \mathbb{Q}[[x]]$, and

$$f(0) = \frac{1}{2^{2n+2}}, \quad g(0) = \frac{1}{2^{2n-2}}, \quad h(0) = -\frac{1}{2^{2n}}.$$

Then, $r_n = \frac{fg}{h^2}$. Since $h(0) \neq 0$, h is a unit in $\mathbb{Q}[[x]]$, hence the right-hand side is a power series in $\mathbb{Q}[[x]]$. Therefore, there exist unique coefficients $c_{n,k} \in \mathbb{Q}$ such that

$$r_n(x) = \sum_{k=0}^{\infty} c_{n,k} x^k,$$

whose constant term is

$$c_{n,0} = \frac{f(0)g(0)}{h(0)^2} \equiv 1.$$

For the proof of the coefficient decay law, refer to (Theorem VI.14 [9]), which states that for a function $f(z)$ continuous in $|z| \leq 1$ and μ times continuously differentiable on $|z| = 1$, the coefficients satisfy

$$[z^k]f(z) = O\left(\frac{1}{k^\mu}\right).$$

Since $x\,\mathrm{Li}_s'(x) = \mathrm{Li}_{s-1}(x)$, for integer $s \geq 2$, $\mathrm{Li}_s(x)$ is C^∞ on $[0,1)$ and C^{s-2} at $x = 1$; the same holds for $r_n(x)$, which is C^{2n-2} at $x = 1$. Hence, μ assumes the value $2n-2$. This means that $r_n'(x)$ is well-defined on $[0,1]$ for $n \geq 2$. □

This brings us to the following: for every $n \geq 2$, the function $r_n(x)$ is continuous on $[0,1]$ (it is analytic on $[0,1)$ and extends continuously to $x = 1$). The coefficient decay law (11) implies

$$|c_{n,k}| \leq \frac{C_n}{k^{2n-2}},$$

for some $C_n > 0$. Now,

$$\sum_{k=0}^{\infty} |c_{n,k} x^k| \leq 1 + \sum_{k=1}^{\infty} \frac{C_n}{k^{2n-2}} = 1 + C_n \zeta(2n-2).$$

Since $1 + C_n\zeta(2n-2)$ converges for $n \geq 2$, by the Weierstrass M-test, $r_n(x)$ is uniformly convergent on $[0,1]$. We also have $r_n(0) = c_{n,0} \equiv 1$. Since $r_n(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$, the Fundamental Theorem of Calculus yields

$$r_n(1) - r_n(0) = \int_0^1 r_n'(x)\,dx.$$

Definition 2 (Integral representation of correction factor)**.** *We define*

$$r_n(1) = 1 + I_n, \tag{12}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 January 2026 doi:10.20944/preprints202601.1390.v1

where

$$I_n := \int_0^1 r_n'(x)\,dx. \tag{13}$$

For $n = 1$, it is meaningful to consider the integral $I_1 = \int_0^{1^-} r_1'(x)\,dx$. The derivative $r_n'(x)$ can be written explicitly in terms of derivatives of polylogarithms (hence again in terms of polylogarithms of lower order), so the integral is a single-variable function built from standard special functions.

2.3. *Reduction to Irrationality of I_n*

Having obtained an integral representation for the correction factor, we get from (9)

$$\zeta(2n+1) = \sqrt{\frac{\zeta(2n)\zeta(2n+2)}{1 + \dfrac{1+I_n}{4(4^n-1)}}}. \tag{14}$$

Using Euler's expression for the even zeta values, we finally arrive at

$$\zeta(2n+1) = \frac{(2\pi)^{2n+1}}{2(2n)!}\sqrt{\frac{-B_{2n}B_{2n+2}}{(2n+1)(2n+2)\left[1 + \dfrac{1+I_n}{4(4^n-1)}\right]}}. \tag{15}$$

Thus, the proof of Theorem 1 reduces to showing that I_n is irrational.

3. Construction of Integer Linear Forms

This section builds the Diophantine machinery.

3.1. *The Beta Kernel and the Exact Integral Identity*

Define the *asymmetric beta kernel* [10]

$$W_m^{(q)}(x) := \binom{(q+1)m}{m} x^m (1-x)^{qm}, \quad m, q \in \mathbb{N}. \tag{16}$$

The corresponding probability density function on $[0, 1]$ is

$$\tilde{W}_m^{(q)}(x) := \Omega_m^{(q)} W_m^{(q)}(x), \tag{17}$$

where the normalization $\Omega_m^{(q)} := (q+1)m + 1$. The cumulative distribution function is

$$S_m^{(q)}(x) := \int_0^x \tilde{W}_m^{(q)}(t)\,dt. \tag{18}$$

Set

$$L_m^{(q)} := \int_0^1 W_m^{(q)}(x)[r_n(x) - 1]\,dx. \tag{19}$$

Lemma 2 (Exact integral identity)**.** *We have*

$$\Omega_m^{(q)} L_m^{(q)} = I_n - \int_0^1 r_n'(x) S_m^{(q)}(x)\,dx. \tag{20}$$

Proof. Write $r_n(x) = r_n(0) + \int_0^x r_n'(t)\,dt = 1 + \int_0^x r_n'(t)\,dt$ and swap the integrals:

$$L_m^{(q)} = \int_0^1 W_m^{(q)}(x) \int_0^x r_n'(t)\,dt\,dx = \int_0^1 r_n'(t) \int_t^1 W_m^{(q)}(x)\,dx\,dt = \frac{1}{\Omega_m^{(q)}} \int_0^1 r_n'(t)[1 - S_m^{(q)}(t)]\,dt.$$

Multiply by $\Omega_m^{(q)}$ and use $I_n = \int_0^1 r_n'(t)\,dt$. □

3.2. Truncation and Denominator Control

From Lemma 1,
$$r_n(x) - 1 = \sum_{k=1}^{\infty} c_{n,k} x^k.$$
Multiplying by $W_m^{(q)}$ and integrating termwise (justified by uniform convergence on $[0,1]$) gives
$$L_m^{(q)} = \sum_{k=1}^{\infty} c_{n,k} \binom{(q+1)m}{m} \mathrm{B}(m+k+1, qm+1) = \sum_{k=1}^{\infty} c_{n,k} \frac{((q+1)m)!\,(m+k)!}{m!\,((q+1)m+k+1)!} = \sum_{k=1}^{\infty} c_{n,k} F_{m,k}^{(q)}, \tag{21}$$
where
$$F_{m,k}^{(q)} := \frac{((q+1)m)!\,(m+k)!}{m!\,((q+1)m+k+1)!}, \tag{22}$$
and we used $\mathrm{B}(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ to simplify.

Fix a truncation parameter $K = K(m)$ and split
$$L_m^{(q)} = L_m^{(q)(\le K)} + L_m^{(q)(>K)}, \tag{23}$$
where $L_m^{(q)(\le K)} := \sum_{k=1}^{K} c_{n,k} F_{m,k}^{(q)}$ and $L_m^{(q)(>K)} := \sum_{k>K} c_{n,k} F_{m,k}^{(q)}$. Let $\mathrm{den}(x)$ be the denominator of the rational number x written in lowest terms. Define
$$D_m^{(q)} := \operatorname*{lcm}_{1\le k\le K} \mathrm{den}(c_{n,k} F_{m,k}^{(q)}). \tag{24}$$

Lemma 3 (Integer linear form)**.** *With*
$$A_m^{(q)} := D_m^{(q)} \in \mathbb{Z}, \tag{25}$$
$$B_m^{(q)} := \Omega_m^{(q)} D_m^{(q)} L_m^{(q)(\le K)} \in \mathbb{Z}, \tag{26}$$
we have
$$\Lambda_m^{(q)} := A_m^{(q)} I_n - B_m^{(q)} = \Omega_m^{(q)} D_m^{(q)} L_m^{(q)(>K)} + D_m^{(q)} \int_0^1 r_n'(x) S_m^{(q)}(x)\,dx. \tag{27}$$

Proof. Multiply (20) by $D_m^{(q)}$ and substitute $L_m^{(q)} = L_m^{(q)(\le K)} + L_m^{(q)(>K)}$. The choice (24) guarantees every $B_m^{(q)}$ is an integer (the least common multiple clears all denominators of $c_{n,k}$ for $k \le K$ and all factorial denominators up to $(q+1)m + K + 1$). □

Thus, $\Lambda_m^{(q)} = A_m^{(q)} I_n - B_m^{(q)}$ is our integer linear form in I_n.

4. Analytic Estimates for the Linear Forms

This section proves that the constructed linear forms tend to zero as $m \to \infty$.

4.1. Exponential Decay of the Kernel Tail

Using Stirling asymptotics and large-deviation estimates, we show that the truncated tail contributes an exponentially small error.

 doi:10.20944/preprints202601.1390.v1

Lemma 4 (Kernel tail)**.** *There exist parameters $\alpha, \gamma^{(q)}(\alpha), C_1(\alpha, m, q), C_2(\alpha, q) > 0$, such that for $m > C_2(\alpha, q)$,*

$$|\Omega_m^{(q)} L_m^{(q)(>K)}| \leq C_1(\alpha, m, q)\left[1 - \frac{C_2(\alpha, q)}{m}\right]^{-1} e^{-\gamma^{(q)}(\alpha)m}. \tag{28}$$

Proof. Set $k = \sigma m$, where $\sigma > 0$ is a scaling parameter. Using the Stirling estimate

$$\log N! = N \log N - N + O(\log N),$$

we get

$$\begin{aligned}\log F_{m,k}^{(q)} &= \log((q+1)m)! + \log(m+k)! - \log m! - \log((q+1)m + k + 1)! \\ &= m\phi^{(q)}(\sigma) + O(\log m),\end{aligned}$$

where

$$\phi^{(q)}(\sigma) := (1+q)\log(1+q) + (1+\sigma)\log(1+\sigma) - (1+q+\sigma)\log(1+q+\sigma) \tag{29}$$

is a rate function, which is strictly decreasing on $(0, \infty)$. Since $\phi^{(q)}(0) = 0$, this implies $\phi^{(q)}(\sigma) < 0$ for all $\sigma > 0$. Moreover, since $c_{n,k} = O\left(\frac{1}{k^{2n-2}}\right)$, $|c_{n,k}| \leq C_n$. Thus,

$$|c_{n,k} F_{m,k}^{(q)}| \leq C_n m^{O(1)} e^{m\phi^{(q)}(\sigma)}.$$

Let $K := \lceil \alpha m \rceil$, for some fixed real $\alpha > 0$. Then,

$$|L_m^{(q)(>K)}| \leq C_n m^{O(1)} \sum_{k > \alpha m} e^{m\phi^{(q)}(\sigma)} \leq C_n m^{O(1)} \int_{\alpha m}^{\infty} e^{m\phi^{(q)}\left(\frac{x}{m}\right)}\, dx = C_n m^{O(1)} \int_{\alpha}^{\infty} e^{m\phi^{(q)}(\sigma)}\, d\sigma.$$

Since

$$\frac{d}{d\sigma} e^{m\phi^{(q)}(\sigma)} = m\phi'^{(q)}(\sigma) e^{m\phi^{(q)}(\sigma)},$$

an integration by parts gives

$$\int_{\alpha}^{\infty} e^{m\phi^{(q)}(\sigma)}\, d\sigma = -\frac{e^{m\phi^{(q)}(\alpha)}}{m\phi'^{(q)}(\alpha)} + \frac{1}{m}\int_{\alpha}^{\infty} \frac{\phi''^{(q)}(\sigma)}{\phi'^{(q)}(\sigma)^2} e^{m\phi^{(q)}(\sigma)}\, d\sigma.$$

The remainder integral may be bounded as

$$\left|\int_{\alpha}^{\infty} \frac{\phi''^{(q)}(\sigma)}{\phi'^{(q)}(\sigma)^2} e^{m\phi^{(q)}(\sigma)}\, d\sigma\right| \leq \sup_{\sigma \geq \alpha}\left|\frac{\phi''^{(q)}(\sigma)}{\phi'^{(q)}(\sigma)^2}\right| \int_{\alpha}^{\infty} e^{m\phi^{(q)}(\sigma)}\, d\sigma = C_2(\alpha, q)\int_{\alpha}^{\infty} e^{m\phi^{(q)}(\sigma)}\, d\sigma,$$

where

$$C_2(\alpha, q) := \sup_{\sigma \geq \alpha}\left|\frac{\phi''^{(q)}(\sigma)}{\phi'^{(q)}(\sigma)^2}\right|.$$

Note that $C_2(\alpha, q)$ is finite, since for $\sigma \geq \alpha > 0$,

$$\phi'^{(q)}(\sigma) = \log\frac{1+\sigma}{1+q+\sigma} < 0, \quad \phi''^{(q)}(\sigma) = \frac{q}{(1+\sigma)(1+q+\sigma)} > 0;$$

 doi:10.20944/preprints202601.1390.v1

both are continuous on $[\alpha, \infty)$. Moreover, both $\phi'^{(q)}(\sigma)$ and $\phi''(q)(\sigma)$ tend to zero as $\sigma \to \infty$; an application of L'Hôpital's rule shows that this limit is $\frac{1}{q}$. So, $\frac{\phi''^{(q)}}{\phi'^{(q)2}}$ is continuous and bounded on $[\alpha, \infty)$. Therefore, we get

$$\int_\alpha^\infty e^{m\phi^{(q)}(\sigma)}\,d\sigma \le -\frac{e^{m\phi^{(q)}(\alpha)}}{m\phi'^{(q)}(\alpha)} + \frac{C_2(\alpha,q)}{m}\int_\alpha^\infty e^{m\phi^{(q)}(\sigma)}\,d\sigma,$$

or,

$$\left[1 - \frac{C_2(\alpha,q)}{m}\right]\int_\alpha^\infty e^{m\phi^{(q)}(\sigma)}\,d\sigma \le -\frac{e^{m\phi^{(q)}(\alpha)}}{m\phi'^{(q)}(\alpha)}.$$

Since $1 - \frac{C_2(\alpha,q)}{m} > 0$ for all $m > C_2(\alpha,q)$, we get

$$\int_\alpha^\infty e^{m\phi^{(q)}(\sigma)}\,d\sigma \le -\frac{e^{m\phi^{(q)}(\alpha)}}{m\phi'^{(q)}(\alpha)}\left[1 - \frac{C_2(\alpha,q)}{m}\right]^{-1} = \frac{e^{m\phi^{(q)}(\alpha)}}{m\log\frac{1+q+\alpha}{1+\alpha}}\left[1 - \frac{C_2(\alpha,q)}{m}\right]^{-1}.$$

Therefore,

$$|L_m^{(q)(>K)}| \le C_n m^{O(1)}\frac{e^{m\phi^{(q)}(\alpha)}}{\log\frac{1+q+\alpha}{1+\alpha}}\left[1 - \frac{C_2(\alpha,q)}{m}\right]^{-1}.$$

Defining

$$C_1(\alpha,m,q) := \frac{C_n\Omega_m^{(q)}m^{O(1)}}{\log\frac{1+q+\alpha}{1+\alpha}}$$

and $\gamma^{(q)}(\sigma) := -\phi^{(q)}(\sigma)$ (so $\gamma^{(q)}(\sigma)$ is positive and increasing on $(0,\infty)$), we get

$$|\Omega_m^{(q)}L_m^{(q)(>K)}| \le C_1(\alpha,m,q)\left[1 - \frac{C_2(\alpha,q)}{m}\right]^{-1} e^{-\gamma^{(q)}(\alpha)m}.$$

□

4.2. Control of the Remainder Term

Using finite difference operators and L^1/L^∞ bounds, we prove that the remainder integral also decays exponentially.

Lemma 5 (Remainder term)**.** *There exist parameters $C_3(m,q) > 0$, $0 < \delta < 1$ such that*

$$\left|\int_0^1 r_n'(x)S_m^{(q)}(x)\,dx\right| \le C_3(m,q)\left[\left(\frac{\delta}{m}\right)^m + \delta^m\right]. \tag{30}$$

Proof. We adopt the convention that all L^1/L^∞ norms are defined on $[0,1]$. Then, we have

$$\left|\int_0^1 r_n'(x)S_m^{(q)}(x)\,dx\right| \le \|r_n'\|_\infty\|S_m^{(q)}\|_1, \tag{31}$$

where

$$r_n'(x) = \sum_{k=1}^\infty kc_{n,k}x^{k-1}.$$

For $0 \le x \le 1$, $|kc_{n,k}x^{k-1}| \le \frac{C_n}{k^{2n-3}}$; therefore,

$$|r_n'(x)| \le C_n \sum_{k=1}^{\infty} \frac{1}{k^{2n-3}} = C_n \zeta(2n-3),$$

which particularly implies (for $n \ge 3$)

$$\|r_n'\|_\infty \le C_n \zeta(2n-3). \tag{32}$$

Now consider the *finite mth backward difference with step h* [11], applied to $S_m^{(q)}$:

$$\nabla_h^m S_m^{(q)}(x) := \sum_{k=0}^{m} \binom{m}{k} (-1)^k S_m^{(q)}(x-kh). \tag{33}$$

Appendix A proves the following properties of $\nabla_h^m S_m^{(q)}$:

$$\|\nabla_h^m S_m^{(q)}\|_1 \le h^m \|S_m^{(q)(m)}\|_1, \quad \|S_m^{(q)} - \nabla_h^m S_m^{(q)}\|_1 \le (hm)^m \|S_m^{(q)(m)}\|_1.$$

By definition, $S_m^{(q)(m)} = \tilde{W}_m^{(q)(m-1)}$. For the asymmetric beta density, using the Leibniz rule,

$$\frac{d^{m-1}}{dx^{m-1}}[x^m(1-x)^{qm}] = \sum_{k=0}^{m-1} \binom{m-1}{k} (m)_k (qm)_{m-1-k} (-1)^{m-1-k} x^{m-k} (1-x)^{qm-(m-1-k)},$$

where $(m)_k$ is the falling factorial. Taking absolute values and integrating:

$$\begin{aligned}
\int_0^1 \left| \frac{d^{m-1}}{dx^{m-1}}[x^m(1-x)^{qm}] \right| dx &\le \sum_{k=0}^{m-1} \binom{m-1}{k} (m)_k (qm)_{m-1-k} \, \mathrm{B}(m-k+1, (q-1)m+k+2) \\
&\le \sum_{k=0}^{m-1} \binom{m-1}{k} (m)_k (qm)_{m-1-k} \frac{\Gamma(m-k+1)\Gamma((q-1)m+k+2)}{\Gamma(qm+3)} \\
&= 2^{m-1} \frac{\Gamma(m+1)\Gamma(qm+1)}{\Gamma(qm+3)}.
\end{aligned}$$

Multiplying by the normalization $\Omega_m^{(q)} \binom{(q+1)m}{m}$ gives us $\tilde{W}_m^{(q)}$:

$$\|\tilde{W}_m^{(q)(m-1)}\|_1 \le \Omega_m^{(q)} \binom{(q+1)m}{m} 2^{m-1} \frac{m!\,(qm)!}{(qm+2)!} = \Omega_m^{(q)} 2^{m-1} \frac{((q+1)m)!}{(qm+2)!}.$$

Hence,

$$\|S_m^{(q)(m)}\|_1 \le \Omega_m^{(q)} 2^m \frac{((q+1)m)!}{(qm)!}.$$

From [12], we obtain the Stirling–Robbins bounds

$$\sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N+\frac{1}{12N+1}} \le N! \le \sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N+\frac{1}{12N}}.$$

Therefore,

$$\|S_m^{(q)(m)}\|_1 \le c_1(m,q) \left(1+\frac{1}{q}\right)^{qm} \left[\frac{2(q+1)m}{e}\right]^m \le c_1(m,q)[2(q+1)m]^m, \tag{34}$$

where

$$c_1(m,q) := \Omega_m^{(q)} \sqrt{\frac{q+1}{q}} \exp\left(\frac{1}{12(q+1)m} - \frac{1}{12qm+1}\right).$$

From the trivial rearrangement $S_m^{(q)}(x) = \nabla_h^m S_m^{(q)}(x) + [S_m^{(q)}(x) - \nabla_h^m S_m^{(q)}(x)]$, we obtain the norm inequality

$$\|S_m^{(q)}\|_1 \leq \|\nabla_h^m S_m^{(q)}\|_1 + \|S_m^{(q)} - \nabla_h^m S_m^{(q)}\|_1.$$

Applying the properties of $\nabla_h^m S_m^{(q)}$ gives

$$\|S_m^{(q)}\|_1 \leq c_1(m,q)[\{2(q+1)hm\}^m + \{2(q+1)hm^2\}^m].$$

Set $\delta := 2(q+1)hm^2$ for a fixed real $\delta \in (0,1)$. Then,

$$\|S_m^{(q)}\|_1 \leq c_1(m,q)\left[\left(\frac{\delta}{m}\right)^m + \delta^m\right].$$

Combining (31), (32) and the inequality above, we get

$$\left|\int_0^1 r_n'(x) S_m^{(q)}(x)\,dx\right| \leq C_3(m,q)\left[\left(\frac{\delta}{m}\right)^m + \delta^m\right], \tag{35}$$

where $C_3(m,q) := C_n\zeta(2n-3)c_1(m,q)$. $\square$

Finally, combining (27), (28), (30), we obtain

$$|\Lambda_m^{(q)}| \leq D_m^{(q)} C_1(\alpha,m,q)\left[1 - \frac{C_2(\alpha,q)}{m}\right]^{-1} e^{-\gamma^{(q)}(\alpha)m} + D_m^{(q)} C_3(m,q)\left[\left(\frac{\delta}{m}\right)^m + \delta^m\right]. \tag{36}$$

4.3. Growth of Denominators

From Lemma 1, the coefficient $c_{n,k}$ is a $\mathbb{Q}$-linear combination of coefficients of f, g, and the power series expansion of $\frac{1}{h^2}$. By the power series division recurrence rule, for $k \geq 1$,

$$c_{n,k} = [x^k]\frac{fg}{h^2} = \frac{1}{[x^0]h^2}\left([x^k]fg - \sum_{j=0}^{k-1}[x^j]\frac{fg}{h^2}\cdot[x^{k-j}]h^2\right).$$

We observe that the maximum coefficient index that $c_{n,k}$ depends on is k. Furthermore,

$$[x^k]fg = \sum_{i+j=k}[x^i]f\cdot[x^j]g$$

(and similarly for h^2). Since F, G, H vanish up to order 3, from the coefficient formulae in Lemma 1, we see that the maximum number that can occur in the denominator of $[x^k]FG$ (and $[x^k]H^2$) is $k-4$. Moreover, $[x^k]fg = [x^{k+6}]FG$ (and so for h^2). Hence, the maximum number that occurs in the denominator of $c_{n,k}$ is $k+2$. Thus, we may write

$$\mathrm{den}(c_{n,k}) \mid \mathrm{lcm}(1,2,\ldots,k+2)^\kappa,$$

where $\kappa = \kappa(n) > 0$.

Now recall Legendre's formula (Theorem 3.14 [13]), which gives the exponent of a prime p in $N!$:

$$\nu_p(N!) = \sum_{j=1}^{\lfloor \log_p N\rfloor}\left\lfloor\frac{N}{p^j}\right\rfloor.$$

For any integers $N, a \geq 0$, we have for each j,

$$\left\lfloor\frac{a}{p^j}\right\rfloor \leq \left\lfloor\frac{N+a}{p^j}\right\rfloor - \left\lfloor\frac{N}{p^j}\right\rfloor \leq \left\lfloor\frac{a}{p^j}\right\rfloor + 1.$$

Compute

$$
\begin{aligned}
-\nu_p(F_{m,k}^{(q)}) &= -\nu_p(((q+1)m)!) - \nu_p((m+k)!) + \nu_p(m!) + \nu_p(((q+1)m+k+1)!) \\
&= [\nu_p(((q+1)m+k+1)!) - \nu_p(((q+1)m)!)] - [\nu_p((m+k)!) - \nu_p(m!)] \\
&\le [\nu_p((k+1)!) + \lfloor \log_p((q+1)m+k+1) \rfloor] - \nu_p(k!) \\
&= \nu_p(k+1) + \lfloor \log_p((q+1)m+k+1) \rfloor.
\end{aligned}
$$

Since $\mathrm{den}(F_{m,k}^{(q)}) = \prod_p p^{\max(0, -\nu_p(F_{m,k}^{(q)}))}$, the above inequality results in

$$
\mathrm{den}(F_{m,k}^{(q)}) \mid (k+1) \prod_p p^{\lfloor \log_p((q+1)m+k+1) \rfloor} = (k+1)\,\mathrm{lcm}(1,2,\ldots,(q+1)m+k+1).
$$

Taking the least common multiple over $1 \le k \le K$, one gets a bound of the form

$$
D_m^{(q)} \mid \mathrm{lcm}(1,2,\ldots,K+2)^{\kappa+1}\,\mathrm{lcm}(1,2,\ldots,(q+1)m+K+1).
$$

With $K = \lceil \alpha m \rceil$ and $\mathrm{lcm}(1,2,\ldots,N) = e^{\psi(N)}$ (where $\psi(N)$ is the Chebyshev psi function), we obtain the Rosser–Schoenfeld bound ((2.29) [14])

$$
D_m^{(q)} \le \exp([(\kappa+1)(\alpha m+3) + (1+q+\alpha)m + 2][1+o(1)]) = c_2 \exp((1+q+\lambda\alpha)m[1+o(1)]), \quad (37)
$$

where $c_2 := \exp((3\kappa+5)[1+o(1)])$ and $\lambda := \kappa + 2$.

5. Proof of the Main Theorem

5.1. Parameter Selection

From (36), for exponential decay to zero, we must ensure that

$$
1 + q + \lambda\alpha < \min(\gamma^{(q)}(\alpha), -\log\delta). \quad (38)
$$

Lemma 6 (Existence of admissible parameters). *With $\lambda > 0$, there exist parameters $q \in \mathbb{N}, \delta \in (0,1), \alpha > 0$ which satisfy (38).*

Proof. We first solve the inequality

$$
1 + q + \lambda\alpha < \gamma^{(q)}(\alpha) = -(1+q)\log(1+q) - (1+\alpha)\log(1+\alpha) + (1+q+\alpha)\log(1+q+\alpha).
$$

Let

$$
g(\alpha) := 1 + q + \lambda\alpha + (1+q)\log(1+q) + (1+\alpha)\log(1+\alpha) - (1+q+\alpha)\log(1+q+\alpha).
$$

We want the set of $\alpha > 0$ with $g(\alpha) < 0$. We have

$$
g'(\alpha) = \lambda - \log\frac{1+q+\alpha}{1+\alpha}, \quad g''(\alpha) = \frac{q}{(1+\alpha)(1+q+\alpha)} > 0.
$$

So, g' is strictly increasing and g is strictly convex on $(0,\infty)$. Since $g(0) = 1+q > 0$ and the initial slope is $g'(0) = \lambda - \log(1+q)$, we have the following two cases:

1. If $\lambda \ge \log(1+q)$, then $g'(0) \ge 0$. Since g is convex and $g(0) > 0$, we then have $g(\alpha) > 0$ for all $\alpha > 0$. Hence, no positive α satisfies the inequality.

 doi:10.20944/preprints202601.1390.v1

2. If $\lambda < \log(1+q)$, then $g'(0) < 0$. By convexity, g' increases from $g'(0) < 0$ to $\lim_{\alpha\to\infty} g'(\alpha) = \lambda > 0$; so, there is exactly one $\alpha^* > 0$ with $g'(\alpha^*) = 0$. The value of α^* is

$$\alpha^* = \frac{q}{e^\lambda - 1} - 1,$$

and with this,

$$(g(\alpha^*))(\lambda, q) = (1+q)[1 + \log(1+q) - \lambda] - q\log\frac{q}{e^\lambda - 1}.$$

At small positive α, we have $g(\alpha) > 0$, while $g(\alpha) \to \infty$ as $\alpha \to \infty$; hence, for there to exist a pair of positive roots of $g(\alpha) = 0$, we require that $g(\alpha^*) < 0$. Differentiating with respect to λ, we find that $\partial_\lambda g(\alpha^*) = \frac{q}{e^\lambda - 1} - 1 > 0$; this means that $g(\alpha^*)$ is strictly increasing in λ on $(0, \log(1+q))$. Since $\lim_{\lambda\to 0^+} g(\alpha^*) = -\infty$ and $\lim_{\lambda\to\log(1+q)^-} g(\alpha^*) = 1+q > 0$, by continuity and strict monotonicity in λ, there exists a unique $\lambda^* \in (0, \log(1+q))$ such that $(g(\alpha^*))(\lambda^*, q) = 0$. Therefore, for $0 < \lambda < \lambda^*$, g has exactly two positive roots α_1, α_2 (with $0 < \alpha_1 < \alpha^* < \alpha_2$). Thus, the solution set of the strict inequality is the interval $\alpha_1 < \alpha < \alpha_2$.

From the second case, the requirement is $\lambda < \log(1+q)$, or $q > e^\lambda - 1$. Fix such a q. Then, there exists an interval (α_1, α_2) such that $1 + q + \lambda\alpha < \gamma^{(q)}(\alpha)$ for all $\alpha \in (\alpha_1, \alpha_2)$. Fix any α in this interval. Then, it suffices to choose δ small enough so that $1 + q + \lambda\alpha \le -\log\delta$; this guarantees (38). □

5.2. Irrationality Criterion

With (38) satisfied, $|\Lambda_m^{(q)}| \to 0$ as $m \to \infty$. This places us precisely within the framework of the following general criterion.

Theorem 2 (Irrationality criterion)**.** *Let $x \in \mathbb{R}$. Suppose there exist infinitely many integers $A_m \neq 0$ and B_m, such that*

$$0 < |\Lambda_m| = |A_m x - B_m| \le \epsilon_m, \tag{39}$$

where $\epsilon_m \to 0$ as $m \to \infty$. Then, x is irrational.

Proof. Assume that $x = \frac{p^*}{q^*} \in \mathbb{Q}$ is in lowest terms. Then,

$$|\Lambda_m| = \frac{|A_m p^* - B_m q^*|}{|q^*|}.$$

Hence, for every m, either $|\Lambda_m| = 0$ or $|\Lambda_m| \ge \frac{1}{|q^*|}$. By (39), $|\Lambda_m| \le \epsilon_m \to 0$ as $m \to \infty$. So, for all sufficiently large m, we have $\epsilon_m < \frac{1}{|q^*|}$, contradicting the dichotomy above. Therefore, x cannot be rational. □

5.3. Completion of the Proof

All that is left to be shown is that $\Lambda_m^{(q)}$ cannot be zero infinitely often. Indeed, if that were the case, then from (27), we would have for infinitely many m,

$$\int_0^1 r_n'(x) S_m^{(q)}(x)\, dx = -\Omega_m^{(q)} L_m^{(q)(>K)}.$$

By the kernel tail bound, the right-hand side tends to zero. Hence, along the infinite subsequence of m along which $\Lambda_m^{(q)} = 0$,

$$\lim_{m\to\infty} \int_0^1 r_n'(x) S_m^{(q)}(x)\, dx = 0.$$

Since $\lim_{m\to\infty} S_m^{(q)}(x) = u\left(x - \frac{1}{q+1}\right)$ (where $u(x)$ is the unit step function), we get

$$\int_0^1 r_n'(x)u\left(x - \frac{1}{q+1}\right) dx = r_n(1) - r_n\left(\frac{1}{q+1}\right),$$

implying $r_n(1) = r_n\left(\frac{1}{q+1}\right)$.

From Appendix B, we get

$$\left|I_n - \frac{16}{9}\left(\frac{4}{9}\right)^n\right| \le \frac{C'}{4^n},$$

for some $C' > 0$. The minimum permissible value of C' is

$$C'_{\min} := \sup_{n\ge1} 4^n \left|I_n - \frac{16}{9}\left(\frac{4}{9}\right)^n\right|.$$

Since this supremum is attained at $n = 1$, we get

$$C'_{\min} = 4\left|I_1 - \frac{64}{81}\right| = \frac{4\pi^6}{45\zeta(3)^2} - \frac{4468}{81} = 3.98159\ldots.$$

Hence, a safe choice for C' is $C' = 4$, and

$$r_n(1) \ge 1 + \frac{16}{9}\left(\frac{4}{9}\right)^n - \frac{1}{4^{n-1}}.$$

Note that $\frac{16}{9}\left(\frac{4}{9}\right)^n > \frac{1}{4^{n-1}}$ for $n \ge 2$, so that there is sufficient margin between $r_n(1)$ and 1. Now, using the coefficient formulae in Lemma 1, we find $c_{n,1} = \frac{16}{9}\left(\frac{4}{9}\right)^n$. Hence, in the vicinity of $x = 0$,

$$r_n(x) = 1 + \frac{16}{9}\left(\frac{4}{9}\right)^n x + O(x^2),$$

implying

$$r_n'(x) = \frac{16}{9}\left(\frac{4}{9}\right)^n + O(x).$$

Therefore, for sufficiently small x, r_n is strictly increasing. Hence, for large enough q, we get $r_n\left(\frac{1}{q+1}\right) < r_n(1)$, which contradicts the equality obtained earlier. We thus conclude that I_n is irrational for all $n \ge 3$. This completes the proof of Theorem 1.

Appendix A. Proof of $\|\nabla_h^m S_m^{(q)}\|_1 \le h^m \|S_m^{(q)(m)}\|_1$ and $\|S_m^{(q)} - \nabla_h^m S_m^{(q)}\|_1 \le (hm)^m \|S_m^{(q)(m)}\|_1$

Proof. We have

$$\nabla_h^1 S_m^{(q)}(x) = S_m^{(q)}(x) - S_m^{(q)}(x-h) = -\int_0^h S_m'^{(q)}(x - t_1)\, dt_1.$$

Applying induction, we arrive at the box-averaged representation

$$\begin{aligned}\nabla_h^m S_m^{(q)}(x) &= (-1)^m \int_0^h \cdots \int_0^h S_m^{(q)(m)}(x - t_1 - \cdots - t_m)\, dt_1 \cdots dt_m \\ &= (-1)^m \int_{[0,h]^m} S_m^{(q)(m)}(x - t)\, d^m t,\end{aligned}$$

where $t := t_1 + \cdots + t_m$ and $d^m t := dt_1 \cdots dt_m$.

Take absolute values and integrate in x. By Fubini's theorem, we get

$$\begin{aligned}
\|\nabla_h^m S_m^{(q)}\|_1 &= \int_0^1 \left| \int_{[0,h]^m} S_m^{(q)(m)}(x-t)\, d^m t \right| dx \\
&\leq \int_{[0,h]^m} \int_0^1 |S_m^{(q)(m)}(x-t)|\, dx\, d^m t \\
&\leq \int_{[0,h]^m} \|S_m^{(q)(m)}\|_1\, d^m t \\
&= \|S_m^{(q)(m)}\|_1 \int_{[0,h]^m} d^m t \\
&= h^m \|S_m^{(q)(m)}\|_1.
\end{aligned}$$

Now, consider the Taylor series of $S_m^{(q)}(x-kh)$ expanded to the first m terms:

$$S_m^{(q)}(x-kh) = \sum_{j=0}^{m-1} \frac{(-kh)^j}{j!} S_m^{(q)(j)}(x) + \frac{(-kh)^m}{(m-1)!} \int_0^1 (1-\theta)^{m-1} S_m^{(q)(m)}(x-kh\theta)\, d\theta.$$

The mth backward difference annihilates polynomials up to degree $m-1$; that is,

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} k^j = 0, \quad j = 0, 1, \ldots, m-1.$$

Hence, we get

$$\begin{aligned}
S_m^{(q)}(x) - \nabla_h^m S_m^{(q)}(x) &= -\sum_{j=0}^{m-1} \frac{(-h)^j}{j!} S_m^{(q)(j)}(x) \left[\sum_{k=1}^{m} (-1)^k \binom{m}{k} k^j \right] \\
&\quad - \frac{(-h)^m}{(m-1)!} \int_0^1 (1-\theta)^{m-1} \left[\sum_{k=1}^{m} (-1)^k \binom{m}{k} k^m S_m^{(q)(m)}(x-kh\theta) \right] d\theta \\
&= -\frac{(-h)^m}{(m-1)!} \int_0^1 (1-\theta)^{m-1} \left[\sum_{k=1}^{m} (-1)^k \binom{m}{k} k^m S_m^{(q)(m)}(x-kh\theta) \right] d\theta.
\end{aligned}$$

Take absolute values and integrate:

$$\|S_m^{(q)} - \nabla_h^m S_m^{(q)}\|_1 \leq \frac{h^m}{m!} \left[\sum_{k=1}^{m} \binom{m}{k} k^m \right] \|S_m^{(q)(m)}\|_1.$$

Since $\sum_{k=1}^{m} \binom{m}{k} k^m \leq (2m)^m$, and $2^m \leq m!$ for $m \geq 4$,

$$\|S_m^{(q)} - \nabla_h^m S_m^{(q)}\|_1 \leq (hm)^m \|S_m^{(q)(m)}\|_1$$

for those m. □

Appendix B. Order-of-Magnitude Estimation of I_n

From (14), we use the expression

$$I_n = 4(4^n - 1) \left[\frac{\zeta(2n)\zeta(2n+2)}{\zeta(2n+1)^2} - 1 \right] - 1.$$

 doi:10.20944/preprints202601.1390.v1

Let $s := 2n$ and

$$\Delta(s) := \log \frac{\zeta(s)\zeta(s+2)}{\zeta(s+1)^2} = \log \zeta(s) + \log \zeta(s+2) - 2\log \zeta(s+1).$$

Then,

$$I_n = 4(2^s - 1)[e^{\Delta(s)} - 1] - 1.$$

Define $z(s) := \zeta(s) - 1 = \sum_{k=2}^{\infty} \frac{1}{k^s}$. A direct calculation yields the exact identity

$$\Delta_z(s) := z(s) + z(s+2) - 2z(s+1) = \sum_{k=2}^{\infty} \frac{1}{k^s}\left(1 + \frac{1}{k^2} - \frac{2}{k}\right) = \sum_{k=2}^{\infty} \frac{1}{k^s}\left(1 - \frac{1}{k}\right)^2.$$

Using the Maclaurin series of the logarithm for small arguments,

$$\log(1+x) = x - \frac{1}{2}x^2 + O(x^3)$$

yields

$$\Delta(s) = \Delta_z(s) - \frac{1}{2}\Delta_{z^2}(s) + O(z(s)^3 + z(s+1)^3 + z(s+2)^3),$$

where $\Delta_{z^2}(s) := z(s)^2 + z(s+2)^2 - 2z(s+1)^2 = O\left(\frac{1}{2^{2s}}\right)$. Therefore,

$$\Delta(s) = \Delta_z(s) + O\left(\frac{1}{2^{2s}}\right).$$

Since $e^{\Delta(s)} - 1 = \Delta(s) + O(\Delta(s)^2)$, and $\Delta(s)^2 = O\left(\frac{1}{2^{2s}}\right)$, we get

$$I_n = 4 \cdot 2^s\left[\Delta(s) + O\left(\frac{1}{2^{2s}}\right)\right] - 1 = 4 \cdot 2^s \Delta_z(s) + O\left(\frac{1}{2^s}\right) - 1.$$

Now,

$$4 \cdot 2^s \Delta_z(s) = 1 + 4\sum_{k=3}^{\infty}\left(\frac{2}{k}\right)^s\left(1 - \frac{1}{k}\right)^2 = 1 + \frac{16}{9}\left(\frac{2}{3}\right)^s + O\left(\frac{1}{2^s}\right).$$

Therefore,

$$I_n = \frac{16}{9}\left(\frac{4}{9}\right)^n + O\left(\frac{1}{4^n}\right).$$

Since $0 < \rho_n(1)^2 < 1$, it implies $r_n(1) > 1$, and in particular, $I_n > 0$. This shows that I_n steadily decreases towards zero as $n \to \infty$.

References

1. E. C. Titchmarsh (rev. D. R. Heath-Brown), *The Theory of the Riemann Zeta-Function*, 2nd ed., Oxford University Press (1986).
2. R. Apéry, *Irrationalité de $\zeta 2$ et $\zeta 3$*, Astérisque, **61** (1979), 11–13. https://smf.emath.fr/system/files/filepdf/AS-61.pdf
3. A. van der Poorten, *A proof that Euler missed: Apéry's proof of the irrationality of $\zeta(3)$*, Math. Intell., **1** (1979), 195–203. https://doi.org/10.1007/BF03028234
4. F. Beukers, *A note on the irrationality of $\zeta(2)$ and $\zeta(3)$*, Bull. London Math. Soc., **11** (1979), 268–272. https://doi.org/10.1007/978-1-4757-4217-6_48
5. T. Rivoal, *La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C. R. Acad. Sci. Ser. I, Math., **331** (2000), 267–270. https://doi.org/10.1016/S0764-4442(00)01624-4

6. W. Zudilin, *One of the numbers* $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ *is irrational*, Uspekhi Mat. Nauk, **56**(4) (2001), 149–150. https://doi.org/10.1070/RM2001v056n04ABEH000427
7. L. Lewin, *Polylogarithms and Associated Functions*, North-Holland (1981).
8. S. M. Ross, *A First Course in Probability*, 9th ed., Pearson Education, Inc. (2014).
9. P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press (2009).
10. S. X. Chen, *Beta kernel estimators for density functions*, Comput. Stat. Data Anal., **31**(2) (1999), 131–145. https://doi.org/10.1016/S0167-9473(99)00010-9
11. N. E. Nørlund, *Vorlesungen über Differenzenrechnung*, Berlin: Springer-Verlag (1924).
12. H. Robbins, *A Remark on Stirling's Formula*, Amer. Math. Monthly, **62**(1) (1955), 26–29. https://doi.org/10.2307/2308012
13. T. M. Apostol, *Introduction to Analytic Number Theory*, New York: Springer-Verlag (1976).
14. J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math., **6**(1) (1962), 64–94. https://doi.org/10.1215/ijm/1255631807