

Article

Not peer-reviewed version

Adaptive Fuzzy Control with Predefined-Time Convergence for High-Order Nonlinear Systems Facing Input Delay and Unmodeled Dynamics

[Mohamed Kharrat](#) and [Paolo Mercorelli](#) *

Posted Date: 3 February 2026

doi: 10.20944/preprints202602.0230.v1

Keywords: nonlinear systems; adaptive control; predefined-time stability; unmodeled dynamics; input delay



Preprints.org is a free multidisciplinary platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This open access article is published under a [Creative Commons CC BY 4.0 license](#), which permit the free download, distribution, and reuse, provided that the author and preprint are cited in any reuse.

Disclaimer/Publisher's Note: The statements, opinions, and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions, or products referred to in the content.

Article

Adaptive Fuzzy Control with Predefined-Time Convergence for High-Order Nonlinear Systems Facing Input Delay and Unmodeled Dynamics

Mohamed Kharrat ¹ and Paolo Mercorelli ^{2,*}

¹ Mathematics Department, College of Science, Jouf University, Sakaka, Saudi Arabia

² Institute for Production Technology and Systems, Leuphana University of Lüneburg, Lüneburg, 21335, Germany

* Correspondence: paolo.mercorelli@leuphana.de

Abstract

This work addresses the design of a predefined-time adaptive fuzzy control scheme for high-order nonlinear systems with nonstrict-feedback structures, subject to unmodeled dynamics and input time delay. To mitigate the influence of unmodeled dynamics, a predefined-time auxiliary dynamic signal is incorporated into the controller design. Meanwhile, the adverse effects caused by input delay are handled by integrating a Padé approximation with the introduction of an intermediate state variable. Fuzzy logic systems are utilized to approximate the unknown nonlinear terms present in the system dynamics. Based on a recursive backstepping framework and a power-type Lyapunov function formulation, an adaptive fuzzy tracking controller with predefined-time convergence characteristics is constructed. A detailed stability analysis demonstrates that the closed-loop system achieves practical predefined-time convergence, and appropriate selection of design parameters guarantees that the tracking errors remain confined within a small bounded region around the origin. Finally, the effectiveness and advantages of the proposed control strategy are validated through a numerical example and a practical example.

Keywords: nonlinear systems; adaptive control; predefined-time stability; unmodeled dynamics; input delay

MSC: 93C10; 93C40; 37N35

1. Introduction

The control of high-order nonlinear systems has attracted sustained attention due to their broad applicability in complex engineering processes. Compared with strict-feedback systems, high-order nonlinear systems exhibit a more general structure, allowing them to describe intricate nonlinear dynamics more accurately. However, conventional linearization techniques often lead to uncontrollable or unobservable modes, which significantly complicates controller design [1–3]. To address these challenges, a constructive control framework based on the adding one power integrator technique was developed, enabling effective stabilization and tracking of high-order nonlinear systems. This approach has been widely extended to adaptive control designs for uncertain systems. Nevertheless, many existing methods rely on precise knowledge of system nonlinearities, an assumption that is difficult to satisfy in practical applications due to modeling uncertainties and external disturbances. To overcome this limitation, intelligent adaptive control schemes employing fuzzy logic systems and neural networks have been introduced. Owing to their strong learning and approximation capabilities, these methods can estimate unknown nonlinear functions online, thereby improving robustness and tracking performance for high-order nonlinear systems with unknown dynamics. Several studies have explored specific adaptive control strategies to handle these challenges in different contexts [4,5]. For example, an adaptive control strategy for high-order nonlinear multi-agent systems operating

under event-triggered communication protocols has been reported to decrease communication load while maintaining coordinated tracking accuracy [6]. To overcome the well-known complexity growth associated with backstepping methods, an adaptive dynamic surface control approach incorporating parameter estimation has been proposed for high-order strict-feedback nonlinear systems [7]. Furthermore, an adaptive neural control framework with dynamic memory and event-triggered mechanisms has been introduced for high-order stochastic nonlinear systems subject to delayed output constraints, effectively addressing stochastic disturbances and constraint requirements in a unified manner [8]. Additionally, adaptive tracking control schemes for high-order nonlinear systems with time-varying delays and asymmetric output constraints have been investigated, ensuring robust tracking performance in the presence of delay effects and nonuniform constraint limits [9].

Due to modeling inaccuracies, external disturbances, measurement noise, and inevitable modeling simplifications, unmodeled dynamics frequently exist in practical control systems. These neglected dynamics, such as unknown high-order system behaviors, actuator dynamics, and unaccounted nonlinear effects, introduce additional uncertainties that are not represented in the nominal model [10–12]. As a consequence, the presence of unmodeled dynamics can degrade tracking accuracy, slow system responses, and in severe cases, lead to instability of the controlled plants [13,14]. Therefore, considerable research attention has been devoted to the study of nonlinear systems with unmodeled dynamics, with the objective of developing robust and adaptive control strategies that can effectively compensate for these uncertainties while guaranteeing stability and satisfactory control performance. Several studies have explored specific adaptive control strategies to address unmodeled dynamics in various contexts [15–17]. As an example, adaptive control scheme has been reported for nonlinear systems with input time delays, unknown dead-zone characteristics, and unmodeled dynamics, aiming to simultaneously counteract delay effects, actuator nonlinearities, and modeling inaccuracies [18]. To enhance approximation accuracy and fault-handling capability in fractional-order systems, neural network-based adaptive controller has been introduced for nonlinear dynamics affected by actuator faults and unmodeled uncertainties [19]. Moreover, event-triggered fuzzy adaptive finite-time control strategy has been proposed for stochastic nonlinear systems with unmodeled dynamics, enabling rapid convergence while alleviating communication and computational overhead [20]. Furthermore, adaptive control approach for pure-feedback nonlinear systems under full-state time-varying constraints and unmodeled dynamics has been developed to ensure closed-loop stability and continuous satisfaction of state constraints despite structural uncertainties and evolving limitations [21].

Input delay is an inevitable factor that adversely affects the control accuracy of practical systems due to limitations in actuators, communication channels, and hardware implementation [22]. The existence of input delay can significantly degrade transient response and tracking performance, and may even result in instability if not properly addressed. To mitigate the adverse effects of input delay, various compensation and approximation-based control strategies have been proposed [23,24]. Among them, the Padé approximation method has been widely adopted to transform the delayed input into an equivalent delay-free form, which greatly simplifies controller design. By incorporating Padé approximation into adaptive control frameworks, the influence of input delay in nonlinear systems can be effectively reduced, leading to improved control performance and enhanced closed-loop stability. Several studies have explored specific adaptive control strategies to address input delays in various contexts [25–27]. For instance, a finite-time adaptive prescribed performance control has been developed for nonlinear systems with input delay to ensure rapid convergence while compensating for delayed control actions [28]. Adaptive tracking control for nonlinear systems affected by input delay and dynamic uncertainties has been investigated by employing multi-dimensional Taylor network approximations to enhance modeling accuracy and robustness [29]. Neural network-based adaptive reinforcement learning approaches have been explored to achieve optimized backstepping tracking control for nonlinear systems with input delay, enabling online performance improvement through learning mechanisms [30]. In addition, adaptive neural control has been studied for non-strict feedback stochastic nonlinear systems with input delay to handle randomness and delayed inputs while

guaranteeing stability and satisfactory tracking behavior [31].

Finite-time control has attracted considerable attention because it guarantees system convergence within a finite duration, providing faster response than asymptotic stabilization methods. It is well known that the convergence time associated with finite-time control schemes is typically a function of the system's initial conditions, which limits their effectiveness in applications requiring a uniform convergence duration. To address this limitation, fixed-time control was proposed, where the convergence time is explicitly decoupled from the initial state of the system [32,33]. By suitably selecting the control laws, fixed-time strategies guarantee that all system trajectories reach the desired equilibrium within a prescribed bounded time interval. Nevertheless, the convergence time in fixed-time control still depends on several design parameters, and the theoretically derived upper bound is often overly conservative, resulting in a mismatch between analytical estimates and the observed convergence behavior in practical implementations [34–36]. In many practical applications, such as aircraft attitude control, multi-agent coordination, and cooperative robotic systems, stability must be achieved within a prescribed time interval. Neither finite-time control nor fixed-time control can guarantee convergence within an exact predefined duration. To eliminate the dependence of convergence time on initial conditions, predefined-time control was introduced, wherein the settling time is exclusively determined by a tunable control parameter rather than the system's initial state [37–39]. This property enables the convergence duration to be explicitly specified a priori, rendering predefined-time control highly attractive for applications with stringent time-performance constraints. In recent years, a variety of adaptive predefined-time control frameworks have been reported for different classes of nonlinear systems [40–43]. As representative examples, predefined-time adaptive fuzzy control scheme has been developed for nonlinear systems subject to input saturation and delayed constraints, guaranteeing convergence within a designer-assigned time interval even in the presence of actuator limitations [44]. Moreover, adaptive predefined-time control approach has been proposed for stochastic switched nonlinear systems with full-state error constraints and input quantization, ensuring prescribed convergence behavior under switching dynamics and quantization effects [45]. Additionally, adaptive fuzzy predefined-time control strategy has been investigated for stochastic nonlinear systems experiencing actuator and sensor faults, thereby achieving fault-tolerant operation while preserving predefined-time stability [46]. Furthermore, adaptive fuzzy predefined-time tracking controller has been designed for nonstrict-feedback high-order nonlinear systems with input quantization, attaining accurate tracking performance within a predefined time bound despite system complexity and quantization-induced nonlinearities [47].

The motivation for this work stems from the persistent challenges encountered in controlling high-order nonlinear systems under realistic operating conditions. Despite significant advancements in adaptive control, many practical systems continue to suffer from uncertainties arising from unmodeled dynamics, input delays, actuator nonlinearities, and structural complexities. Existing control strategies, including finite-time and fixed-time approaches, often fail to guarantee precise convergence within a user-specified duration, limiting their applicability in safety-critical and time-sensitive applications such as aerospace systems, multi-agent coordination, and cooperative robotics. Moreover, while intelligent adaptive techniques using fuzzy logic systems and neural networks have improved robustness and approximation capabilities, integrating these methods with rigorous time-performance guarantees remains an open problem. These gaps inspired the present research to develop a predefined-time adaptive fuzzy control framework that simultaneously addresses high-order nonlinear dynamics, input delays, and unmodeled uncertainties, with the objective of achieving accurate, reliable, and fast tracking within a rigorously prescribed time interval. This work is further motivated by the need for a control approach that combines the flexibility of intelligent approximation with the predictability of predefined-time convergence, offering practical utility in complex, uncertain, and delayed nonlinear systems. Based on the above analysis, this study is devoted to the design of a predefined-time control framework for high-order nonlinear systems affected by dynamic uncertainties and input time delays. In contrast to the existing literature, the key contributions of this work can be outlined as follows

- (i) Relative to the control schemes presented in [31,32], this work investigates a predefined-time adaptive fuzzy control problem for high-order nonstrict-feedback nonlinear systems in the presence of unmodeled dynamics and input time delays. In contrast to finite-time and fixed-time control approaches reported in [33–36], the proposed method guarantees system stabilization within a designer-assigned time horizon. Specifically, the convergence duration is directly governed by a tunable design parameter, enabling its explicit selection prior to controller implementation to satisfy desired performance specifications. To enhance practical feasibility, a Padé approximation technique is employed to address the adverse effects of input delays. This delay-compensation mechanism ensures that the closed-loop system retains stability and satisfactory tracking performance even when control actions are subject to delays. Consequently, the developed control framework is applicable to a wide range of high-order nonlinear systems where input delay effects are unavoidable.
- (ii) A dynamic control signal based on the predefined-time control principle is designed to address the effects of unmodeled dynamics, including unknown high-order system behaviors and actuator nonlinearities. The control law ensures that the closed-loop system achieves practical predefined-time stability, meaning that system states converge to a small neighborhood around the origin within the specified time. By selecting appropriate control parameters, the tracking performance can be effectively improved, and the impact of modeling uncertainties, external disturbances, and unmodeled dynamics can be minimized. This provides both theoretical assurance and practical feasibility for high-precision control of complex nonlinear systems.

The remainder of this paper is organized as follows. Section 2 presents the problem formulation along with the required preliminary concepts. The design of the adaptive tracking controller and the associated stability analysis are detailed in Section 3. Simulation results and a representative practical application are provided in Section 4 to demonstrate the performance of the proposed method. Finally, Section 5 concludes the paper and outlines potential directions for future investigation.

2. Problem Formulation and Preliminaries

Consider a class of high-order nonlinear systems with a nonstrict-feedback structure, which can be expressed as

$$\begin{cases} \dot{\xi} = q(\xi, x), \\ \dot{x}_i = x_{i+1}^{\sigma_i} + f_i(x) + \Delta_i(x, \xi, t), \quad i = 1, 2, \dots, h-1, \\ \dot{x}_h = u^{\sigma_h}(t - \tau) + f_h(x) + \Delta_h(x, \xi, t), \\ y = x_1. \end{cases} \quad (1)$$

where the system state vector is defined as $x = x_h = [x_1, \dots, x_h]^T \in \mathbb{R}^h$, with the partial state vector $x_i = [x_1, \dots, x_i]^T \in \mathbb{R}^i$. The constants $\sigma_i \geq 1$ are odd integers. The functions $f_i(x)$ denote unknown smooth nonlinearities satisfying $f_i(0) = 0$. The variable ξ represents unmeasured internal state dynamics, where the ξ -subsystem corresponds to unmodeled dynamics. The terms $\Delta_i(\cdot)$ describe unknown nonlinear disturbances acting on the system. It is assumed that $q(\cdot)$ and $\Delta_i(\cdot)$ are uncertain but locally Lipschitz continuous. The scalar variables $u \in \mathbb{R}$ and $y \in \mathbb{R}$ denote the control input and system output, respectively. τ represents an unknown input delay, which is assumed to be a positive constant.

To address the input delay appearing in the high-order nonlinear system (1), the Padé approximation technique is employed, following the approach in [22]. By virtue of the delay property of the Laplace transform, the following relationship holds:

$$\mathcal{L}\{u^{\sigma_h}(t - \tau)\} = e^{-\tau v} \mathcal{L}\{u^{\sigma_h}(t)\} = \frac{\exp(-\tau v/2)}{\exp(\tau v/2)}, \quad (2)$$

where v denotes the Laplace variable. Using the first-order Padé approximation, one has

$$\exp(-\tau v) \mathcal{L}\{u^{\sigma_h}(t)\} \approx \frac{1 - \frac{\tau v}{2}}{1 + \frac{\tau v}{2}} \mathcal{L}\{u^{\sigma_h}(t)\}, \quad (3)$$

where $\mathcal{L}\{u(t)\}$ is the Laplace transform of $u(t)$.

Remark 1: Due to the inherent limitation of the Padé approximation, the proposed approach is applicable to systems with small input delays. When τ is sufficiently small, the approximation error $e^{-\tau v} - \frac{1 - \frac{\tau v}{2}}{1 + \frac{\tau v}{2}}$ is negligible. The extension to large or time-varying delays will be investigated in future work.

To facilitate controller design, an auxiliary state x_{h+1} is introduced such that

$$\frac{1 - \frac{\tau v}{2}}{1 + \frac{\tau v}{2}} \mathcal{L}\{u^{\sigma_h}(t)\} = \mathcal{L}\{x_{h+1}(t)\} - \mathcal{L}\{u^{\sigma_h}(t)\}. \quad (4)$$

Rearranging the above equation yields

$$2\mathcal{L}\{u^{\sigma_h}(t)\} = \mathcal{L}\{x_{h+1}(t)\} + \frac{\tau v}{2} \mathcal{L}\{x_{h+1}(t)\}. \quad (5)$$

Applying the inverse Laplace transform gives

$$\dot{x}_{h+1} = \frac{4}{\tau} u^{\sigma_h} - \frac{2}{\tau} x_{h+1}. \quad (6)$$

Define $\lambda = 2/\tau$, then the above equation can be rewritten as

$$\dot{x}_{h+1} = 2\lambda u^{\sigma_h} - \lambda x_{h+1}. \quad (7)$$

Following the aforementioned transformations, the delayed system in (1) can be reformulated into an equivalent delay-free augmented representation given by

$$\begin{cases} \dot{\xi} = q(\xi, x), \\ \dot{x}_\iota = x_{\iota+1}^{\sigma_\iota} + f_\iota(x) + \Delta_\iota(x, \xi, t), \quad \iota = 1, \dots, h-1, \\ \dot{x}_h = x_{h+1} - u^{\sigma_h} + f_h(x) + \Delta_h(x, \xi, t), \\ \dot{x}_{h+1} = -\lambda x_{h+1} + 2\lambda u^{\sigma_h}, \\ y = x_1. \end{cases} \quad (8)$$

The objective of this study is to develop a predefined-time adaptive control strategy based on fuzzy logic systems for system (1), such that all closed-loop signals remain bounded within a designer-specified time interval, and the output tracking error converges to a sufficiently small neighborhood of the origin.

Assumption 1 [13]: For each $\iota = 1, 2, \dots, h$, $\exists$ known, smooth, and nonnegative functions $\Psi_{\iota,1}(\cdot)$ and $\Psi_{\iota,2}(\cdot)$ satisfying

$$|\Delta_\iota(x, \xi, t)| \leq \Psi_{\iota,1}(|x_\iota|) + \Psi_{\iota,2}(|\xi|). \quad (9)$$

Moreover, without imposing any restriction on generality, the function $\Psi_{\iota,2}(\cdot)$ is assumed to satisfy $\Psi_{\iota,2}(0) = 0$.

Assumption 2 [14]: Consider the ξ -subsystem $\dot{\xi} = q(\xi, x)$ associated with system (1). There exist class $\mathcal{K}_\infty$ functions $\underline{\mathcal{L}}(\cdot)$, $\overline{\mathcal{L}}(\cdot)$, and $\eta(\cdot)$, together with positive constants a and ω , such that an exponentially input-to-state practically stable (Exp-ISpS) Lyapunov function $V(\xi)$ can be constructed satisfying

$$\underline{\mathcal{L}}(\xi) \leq V(\xi) \leq \overline{\mathcal{L}}(\xi), \quad (10)$$

and

$$\frac{\partial V(\xi)}{\partial \xi} q(\xi, x) \leq -aV(\xi) + \eta(|x_1|) + \omega. \quad (11)$$

Assumption 3 [48]: Define σ_i , for $i = 1, 2, \dots, h$, as positive odd integers, and let $\sigma = \max_{i=1,2,\dots,h} \{\sigma_i\}$. The odd integers σ_i satisfy the inequality

$$\frac{\sigma + 1}{\sigma_i} \geq \sigma - \sigma_{i+1} + 1, \quad i = 1, 2, \dots, h - 1. \quad (12)$$

Assumption 4 [36]: The reference trajectory $y_r(t)$, together with its derivative of order h , is assumed to be smooth and uniformly bounded for all time.

Lemma 1 [49]: Given arbitrary real scalars $\mathcal{P}$ and $\mathcal{N}$, and any prescribed positive constants $\tilde{c}$, $\tilde{d}$, and ε , the following result holds

$$|\mathcal{P}|^{\tilde{c}} |\mathcal{N}|^{\tilde{d}} \leq \frac{\tilde{c}}{\tilde{c} + \tilde{d}} \varepsilon |\mathcal{P}|^{\tilde{c} + \tilde{d}} + \frac{\tilde{d}}{\tilde{c} + \tilde{d}} \varepsilon^{-\frac{\tilde{c}}{\tilde{d}}} |\mathcal{N}|^{\tilde{c} + \tilde{d}}. \quad (13)$$

Lemma 2 [39]: Assume that conditions (10) and (11) are satisfied. Under these conditions, the function $V(\xi)$ qualifies as an exponentially input-to-state practically stable (Exp-ISpS) Lyapunov function for the subsystem $\dot{\xi} = q(\xi, x)$. Let g_0^- and g_0^+ be positive constants satisfying $g_0^- = \left(\frac{3}{\delta_0}\right)^{\frac{\xi}{2}} \frac{\pi}{\xi T_d}$, $g_0^+ = \frac{l_0^{\frac{\xi}{2}} \pi}{\xi T_d}$ and define $g_0 = g_0^- + g_0^+ \in (0, g)$. For any initial time $t_0 \geq 0$ and initial condition $\xi(t_0) = \tau_0$, and for any continuous function $\bar{\eta}(\cdot)$ satisfying $\bar{\eta}(x_1) \geq \eta(|x_1|)$, there exists a predefined time $T_0 = T_0(g_0^-, g_0^+, r_0, \tau_0) \geq 0$, a positive constant $\omega = \bar{\omega} + g_0^+ \frac{\xi}{2} \left(\frac{2}{2-\xi}\right)^{\frac{\xi-2}{\xi}}$, and a nonnegative function $B(t_0, t)$ defined for all $t \geq t_0$, such that the auxiliary signal $r(t)$ satisfies

$$\dot{r} = -g_0^- r^{1+\frac{\xi}{2}} - g_0^+ r^{1-\frac{\xi}{2}} + \bar{\eta}(x_1) + \bar{\omega}, \quad r(t_0) = r_0. \quad (14)$$

Moreover, the function $B(t_0, t)$ satisfies $B(t_0, t) = 0$, $\forall t \geq t_0 + T_0$, and the Lyapunov function is bounded by

$$V(\xi(t)) \leq r(t) + B(t_0, t). \quad (15)$$

The solution of the system exists for all $t \geq t_0$. Without loss of generality, the function $\bar{\eta}(x_1)$ is chosen as $\bar{\eta}(x_1) = x_1^2 \lambda(x_1^2)$ under which the auxiliary dynamic signal can be rewritten as

$$\dot{r} = -g_0^- r^{1+\frac{\xi}{2}} - g_0^+ r^{1-\frac{\xi}{2}} + x_1^2 \lambda(x_1^2) + \bar{\omega}, \quad r(t_0) = r_0. \quad (16)$$

where $\lambda(\cdot)$ denotes a smooth nonnegative function.

Lemma 3 [39]: For arbitrary real-valued functions $\mathcal{J}_1$ and $\mathcal{J}_2$, any odd integer $\sigma > 1$, and a given constant $k > 0$, one has

$$|\mathcal{J}_1^\sigma - \mathcal{J}_2^\sigma| \leq \sigma |\mathcal{J}_1 - \mathcal{J}_2| (\mathcal{J}_1^{\sigma-1} + \mathcal{J}_2^{\sigma-1}), \quad (17)$$

$$|\mathcal{J}_1 + \mathcal{J}_2|^k \leq c_k (|\mathcal{J}_1|^k + |\mathcal{J}_2|^k). \quad (18)$$

where $c_k = 1$ for $k < 1$, and $c_k = 2^{k-1}$ for $k \geq 1$. In this work, the exponent is selected as $k = \sigma_i - 1$. For notational convenience, both cases are unified as

$$\|\mathcal{J}_1 + \mathcal{J}_2\|^k \leq 2^k (|\mathcal{J}_1|^k + |\mathcal{J}_2|^k). \quad (19)$$

Lemma 4 [50]: Let $\omega > 1$ and $\mu > 0$ be constants, and define the set $\Omega_\chi = \{\chi \in \mathbb{R} \mid |\chi| < \varrho\mu\}$, $\varrho = \operatorname{arctanh}\left(\omega \sqrt{\frac{1}{\omega}}\right)$. Then, for all $\chi \notin \Omega_\chi$, the inequality $1 - \omega \tanh^\omega\left(\frac{\chi}{\mu}\right) \leq 0$ is satisfied.

Lemma 5 [39]: For any $\chi \in \mathbb{R}$ and any constant $l > 0$, the following inequality holds $0 \leq |\chi| - \chi \tanh(\frac{\chi}{l}) \leq p_0 l$, $p_0 = 0.2785$.

Lemma 6 [39]: Let $f(\chi)$ be an unknown continuous function defined on a compact set Ω . For any prescribed constant $\varepsilon > 0$, there exists a fuzzy logic system (FLS) of the form $W^T \Phi(\chi)$ such that

$$\sup_{\chi \in \Omega} |f(\chi) - W^T \Phi(\chi)| \leq \varepsilon. \quad (20)$$

where $W = [W_1, W_2, \dots, W_M]^T$ denotes the adjustable weight vector, and ε represents the minimum approximation error. Moreover, $M > 1$ is the number of fuzzy inference rules. The fuzzy basis function vector is defined as $\Phi(\chi) = \frac{[\Phi_1(\chi), \Phi_2(\chi), \dots, \Phi_M(\chi)]^T}{\sum_{L=1}^M \Phi_L(\chi)}$, where each basis function $\Phi_L(\chi)$ is selected as a Gaussian membership function given by

$$\Phi_L(\chi) = \exp \left[-\frac{(\chi - \mu_L)^T (\chi - \mu_L)}{G_L^2} \right], \quad L = 1, 2, \dots, M, \quad (21)$$

with μ_L and G_L denoting the center and width parameters of the Gaussian function, respectively.

Lemma 7 [39]: Let $Z = [z_1, z_2, \dots, z_h]^T$ be the input vector, and let $\Phi(Z) = [\Phi_1(Z), \Phi_2(Z), \dots, \Phi_l(Z)]^T$ denote the corresponding fuzzy basis function vector. For any positive integer $\iota \leq h$, define $\Xi_\iota = [z_1, z_2, \dots, z_\iota]^T$. Then, the following inequality holds:

$$\|\Phi(Z)\|^2 \leq \|\Phi(\Xi_\iota)\|^2. \quad (22)$$

Consider the nonlinear dynamical system

$$\dot{\chi} = f(\chi), \quad (23)$$

where $\chi \in \mathbb{R}^h$ denotes the state vector, the origin $\chi = 0$ is an equilibrium point, and $f: \mathbb{R}^h \rightarrow \mathbb{R}^h$ is a nonlinear mapping.

Definition 1 [39]: Let $T_d > 0$ and $X > 0$ be given constants. If the state trajectory satisfies $\|\chi(t)\| < X$ for all $t > T_d$, then the equilibrium point at the origin is said to be practically predefined-time stable. The constant T_d is referred to as the predefined time.

Lemma 8 [39]: Assume that there exists a Lyapunov function $V(\chi)$ such that

$$\dot{V} \leq -\frac{\pi}{\zeta T_d} V^{1+\frac{\zeta}{2}} - \frac{\pi}{\zeta T_d} V^{1-\frac{\zeta}{2}} + B, \quad (24)$$

where $0 < \zeta < 1$, and T_d and B are strictly positive constants. Under these conditions, the function V guarantees practical predefined-time stability, and the corresponding settling time is bounded above by $2T_d$.

3. Controller Design and Stability Analysis

To facilitate the construction of the desired control law, the following error coordinate transformations are introduced

$$\begin{cases} z_1 = x_1 - y_r, \\ z_\iota = x_\iota - \alpha_{\iota-1}, \quad \iota = 2, 3, \dots, h-1, \\ z_n = x_n - \alpha_{h-1} + x_{h+1}/\lambda, \end{cases} \quad (25)$$

where $\alpha_{\iota-1}$ denotes the virtual control signal to be designed subsequently.

Step 1: From (1) and (25), the time derivative of z_1 is obtained as

$$\dot{z}_1 = x_2^{\sigma_1} + f_1(x) + \Delta_1(x, \xi, t) - \dot{y}_r. \quad (26)$$

To analyze the stability of the first subsystem, consider the following power-type Lyapunov function

$$V_1 = \frac{z_1^{\sigma-\sigma_1+2}}{\sigma-\sigma_1+2} + \frac{\tilde{\Theta}_1^2}{2\tau_1} + r^{\delta_0}, \quad (27)$$

where $\hat{\Theta}_1$ is the estimate of the unknown parameter Θ_1^* and $\tilde{\Theta}_1 = \Theta_1^* - \hat{\Theta}_1$ denotes the corresponding estimation error. The design parameters τ_1 and δ_0 are chosen as positive constants. By invoking Assumption 1 together with (16), the time derivative of the Lyapunov function V_1 satisfies

$$\begin{aligned} \dot{V}_1 \leq & z_1^{\sigma-\sigma_1+1} (x_2^{\sigma_1} + f_1(x) - \dot{y}_r) + z_1^{\sigma-\sigma_1+1} \Psi_{1,1}(|x_1|) \\ & + z_1^{\sigma-\sigma_1+1} \Psi_{1,2}(|\xi|) - \frac{1}{\tau_1} \tilde{\Theta}_1 \dot{\Theta}_1 - \frac{g_0^-}{\delta_0} r^{1+\frac{\zeta}{2}} - \frac{g_0^+}{\delta_0} r^{1-\frac{\zeta}{2}} \\ & + \frac{1}{\delta_0} x_1^2 \lambda(x_1^2) + \frac{\bar{\omega}}{\delta_0}. \end{aligned} \quad (28)$$

Applying Lemma 5, the term $z_1^{\sigma-\sigma_1+1} \Psi_{1,1}(\|x_1\|)$ can be upper bounded as

$$z_1^{\sigma-\sigma_1+1} \Psi_{1,1}(|x_1|) \leq z_1^{\sigma-\sigma_1+1} \hat{\Psi}_{1,1}(x_1) + l_{1,1}^*, \quad (29)$$

where $l_{1,1} > 0$ is a design constant, $\hat{\Psi}_{1,1}(x_1) = \Psi_{1,1}(|x_1|) \tanh\left(\frac{z_1^{\sigma-\sigma_1+1} \Psi_{1,1}(\|x_1\|)}{l_{1,1}}\right)$, and $l_{1,1}^* = 0.2785 l_{1,1}$. By utilizing Assumption 2 together with Lemmas 2 and 5, and following estimation techniques similar to those developed in [26], the following upper bound can be derived:

$$z_1^{\sigma-\sigma_1+1} \Psi_{1,2}(|\xi|) \leq z_1^{\sigma-\sigma_1+1} \hat{\Psi}_{1,2}(x_1, r) + l_{1,2}^* + \frac{1}{4} z_1^{2(\sigma-\sigma_1+1)} + m_1(t_0, t), \quad (30)$$

where $l_{1,2} > 0$ is a design parameter, and $\mathcal{L}^{-1}(\cdot)$ denotes the inverse of the $\mathcal{L}(\cdot)$ function introduced in Assumption 2. The estimated function $\hat{\Psi}_{1,2}(x_1, r)$ is defined as $\hat{\Psi}_{1,2}(x_1, r) = (\Psi_{1,2} \circ \mathcal{L}^{-1})(2r) \tanh\left(\frac{z_1^{\sigma-\sigma_1+1} (\Psi_{1,2} \circ \mathcal{L}^{-1})(2r)}{l_{1,2}}\right)$, with $l_{1,2}^* = 0.2785 l_{1,2}$. Moreover, the composite function $(\Psi_{1,2} \circ \mathcal{L}^{-1})(r(t) + B(t_0, t)) = \Psi_{1,2}[\mathcal{L}^{-1}(r(t) + B(t_0, t))]$, and $m_1(t_0, t) = \left((\Psi_{1,2} \circ \mathcal{L}^{-1})(2M(t_0, t))\right)^2$, where $m_1(t_0, t) \geq 0$ for all $t \geq t_0 + T_0$.

Remark 2: It is worth noting that the term $\frac{x_1^2 \lambda(x_1^2)}{\delta_0 z_1^{\sigma-\sigma_1+1}}$ is discontinuous at $z_1 = 0$, which prevents its direct approximation using fuzzy logic systems. To overcome this difficulty, a smooth hyperbolic tangent function $\tanh^{\sigma-\sigma_1+2}\left(\frac{z_1}{\mu}\right)$ with a given constant $\mu > 0$ is introduced. As a result, the expression $z_1^{\sigma-\sigma_1+2} \tanh^{\sigma-\sigma_1+2}\left(\frac{z_1}{\mu}\right)$ becomes well defined and continuous at $z_1 = 0$.

By using (29) and (30) into (28), one has

$$\begin{aligned} \dot{V}_1 \leq & z_1^{\sigma-\sigma_1+1} (x_2^{\sigma_1} + \bar{f}_1(x) - \dot{y}_r) - \frac{1}{2} z_1^{2(\sigma-\sigma_1+1)} - \frac{1}{\tau_1} \tilde{\Theta}_1 \dot{\Theta}_1 \\ & - \frac{g_0^-}{\delta_0} r^{1+\frac{\zeta}{2}} - \frac{g_0^+}{\delta_0} r^{1-\frac{\zeta}{2}} + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2}\left(\frac{z_1}{\mu}\right)\right) \frac{x_1^2 \lambda(x_1^2)}{\delta_0} \\ & + l_{1,1}^* + l_{1,2}^* + m_1(t_0, t) + \frac{\bar{\omega}}{\delta_0}, \end{aligned} \quad (31)$$

where $\bar{f}_1(x) = f_1(x) + \hat{\Psi}_{1,1}(x_1) + \hat{\Psi}_{1,2}(x_1, r) + (\sigma - \sigma_1 + 2) z_1^{\sigma-\sigma_1+1} \tanh^{\sigma-\sigma_1+2}\left(\frac{z_1}{\mu}\right) \frac{x_1^2 \lambda(x_1^2)}{\delta_0} + \frac{3}{4} z_1^{\sigma-\sigma_1+1}$. Based on Lemma 6, the unknown smooth nonlinear function $f_1(\cdot)$ is approximated using a fuzzy logic system (FLS) expressed in the form

$$\bar{f}_1(Z_1) = W_1^{*T} \Phi_1(Z_1) + \varepsilon_1(Z_1), \quad (32)$$

where $Z_1 = [x_1, x_2, \dots, x_h]^T$ is the FLS input vector, W_1^* denotes the ideal weight vector, and $\varepsilon_1(Z_1)$ represents the fuzzy approximation error satisfying $\|\varepsilon_1(Z_1)\| \leq \varepsilon_1^*$, with $\varepsilon_1^* > 0$ being a known constant.

By applying Young's inequality together with Lemma 7, the following bounds can be established:

$$\begin{aligned} z_1^{\sigma-\sigma_1+1} f_1 &\leq |z_1|^{\sigma-\sigma_1+1} (\|W_1^*\| \|\Phi_1(Z_1)\| + \varepsilon_1^*) \\ &\leq |z_1|^{\sigma-\sigma_1+1} (\|W_1^*\| \|\Phi_1(\chi_1)\| + \varepsilon_1^*) \\ &\leq \frac{1}{2d_1^2} z_1^{2(\sigma-\sigma_1+1)} \Theta_1^* \Phi_1^T(\chi_1) \Phi_1(\chi_1) + \frac{d_1^2}{2} + \frac{1}{2} z_1^{2(\sigma-\sigma_1+1)} + \frac{\varepsilon_1^{*2}}{2}, \end{aligned} \quad (33)$$

where $\Theta_1^* = \|W_1^*\|^2$, $d_1 > 0$ is a design constant, and $\varepsilon_1^* > 0$ denotes an unknown bound satisfying $\|\varepsilon_1\| \leq \varepsilon_1^*$.

By using (26) into (24), one has

$$\begin{aligned} \dot{V}_1 &\leq z_1^{\sigma-\sigma_1+1} \left[\frac{1}{2d_1^2} z_1^{\sigma-\sigma_1+1} \hat{\Theta}_1 \Phi_1^T(\chi_1) \Phi_1(\chi_1) + x_2^{\sigma_1} - \alpha_1^{\sigma_1} + \alpha_1^{\sigma_1} - \dot{y}_r \right] \\ &\quad - \frac{g_0^-}{\delta_0} r^{1+\frac{\zeta}{2}} - \frac{g_0^+}{\delta_0} r^{1-\frac{\zeta}{2}} + \frac{\hat{\Theta}_1}{\tau_1} \left[\frac{\tau_1}{2d_1^2} z_1^{2(\sigma-\sigma_1+1)} \Phi_1^T(\chi_1) \Phi_1(\chi_1) - \hat{\Theta}_1 \right] \\ &\quad + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2} \left(\frac{z_1}{\mu} \right) \right) \frac{x_1^2 \lambda(x_1^2)}{\delta_0} + l_{1,1}^* + l_{1,2}^* + m_1(t_0, t) \\ &\quad + \frac{d_1^2}{2} + \frac{\varepsilon_1^{*2}}{2} + \frac{\bar{\omega}}{\delta_0}. \end{aligned} \quad (34)$$

Furthermore, by applying Young's inequality, for any $\iota = 1, 2, \dots, h$, one has

$$2(\mathfrak{e} + \tilde{\mathfrak{e}}) \tilde{\Theta}_\iota \hat{\Theta}_\iota \leq -(\mathfrak{e} + \tilde{\mathfrak{e}}) \tilde{\Theta}_\iota^2 + (\mathfrak{e} + \tilde{\mathfrak{e}}) \Theta_\iota^{*2}. \quad (35)$$

The virtual control law α_1 and the adaptive update law for the parameter estimate $\hat{\Theta}_1$ are designed as

$$\alpha_1 = - \left(\frac{1}{2d_1^2} z_1^{\sigma-\sigma_1+1} \hat{\Theta}_1 \Phi_1^T(\chi_1) \Phi_1(\chi_1) + \Gamma_1 z_1^{\sigma_1} + \frac{\beta_1 \pi}{\zeta T_d} z_1^{(1+\frac{\zeta}{2})((\sigma+1)-(\sigma-\sigma_1+1))} - \dot{y}_r \right)^{\frac{1}{\sigma_1}}, \quad (36)$$

$$\dot{\hat{\Theta}}_1 = \tau_1 \frac{1}{2d_1^2} z_1^{2(\sigma-\sigma_1+1)} \Phi_1^T(\chi_1) \Phi_1(\chi_1) - 2(\mathfrak{e} + \tilde{\mathfrak{e}}) \hat{\Theta}_1, \quad (37)$$

where $\beta_1 = \frac{(3n)^{\zeta/2}}{(\sigma-\sigma_1+2)(1+\zeta/2)}$, $\bar{\beta}_1 = \frac{1}{(\sigma-\sigma_1+2)(1-\zeta/2)}$, $\Gamma_1 = \frac{\bar{\beta}_1 \pi}{\zeta T_d} + 1$, $\mathfrak{e} = \frac{(2-\zeta)(3n)^{\zeta/2} \pi}{(2+\zeta/2)\tau^{\zeta/2} \kappa \zeta T_d}$, $\tilde{\mathfrak{e}} = \frac{(2-\zeta)\pi^2}{2(2-\zeta/2)\tau^{\zeta/2} \kappa \zeta T_d}$, and the initial condition satisfies $\hat{\Theta}_1(0) \geq 0$.

By using (35)-(37) into (34), one has

$$\begin{aligned} \dot{V}_1 &\leq z_1^{\sigma-\sigma_1+1} (x_2^{\sigma_1} - \alpha_1^{\sigma_1}) - \Gamma_1 z_1^{\sigma+1} - \beta_1 \frac{\pi}{\zeta T_d} z_1^{(\sigma+1)(1+\zeta/2)} - (\mathfrak{e} + \tilde{\mathfrak{e}}) \frac{1}{\tau_1} \tilde{\Theta}_1^2 \\ &\quad - \frac{g_0^-}{\delta_0} r^{1+\frac{\zeta}{2}} - \frac{g_0^+}{\delta_0} r^{1-\frac{\zeta}{2}} + (\mathfrak{e} + \tilde{\mathfrak{e}}) \frac{1}{\tau_1} \Theta_1^{*2} \\ &\quad + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2} \left(\frac{z_1}{\mu} \right) \right) \frac{x_1^2 \lambda(x_1^2)}{\delta_0} + l_{1,1}^* + l_{1,2}^* + m_1(t_0, t) \\ &\quad + \frac{d_1^2}{2} + \frac{\varepsilon_1^{*2}}{2} + \frac{\bar{\omega}}{\delta_0}. \end{aligned} \quad (38)$$

Using Lemmas 1 and 3, one has

$$\begin{aligned} z_1^{\sigma-\sigma_1+1}(x_2^{\sigma_1} - \alpha_1^{\sigma_1}) &\leq \sigma_1 |z_1|^{\sigma-\sigma_1+1} |x_2 - \alpha_1| \left(|x_2|^{\sigma_1-1} + |\alpha_1|^{\sigma_1-1} \right) \\ &\leq |z_1|^{\sigma+1} + \Delta_{11} |z_2|^{\sigma+1} + \Delta_{12} |z_2|^{(\sigma+1)/\sigma_1}, \end{aligned} \quad (39)$$

where $\Delta_{11} = \frac{2^{\sigma_1-1} \sigma_1^2}{\sigma+1} \left(\frac{\sigma+1}{\sigma-\sigma_1+1} \frac{1}{\sigma_1 2^{\sigma_1}} \right)^{-\frac{\sigma-\sigma_1+1}{\sigma_1}}$ is a positive constant related to the given constants σ and σ_1 and $\Delta_{12} = \frac{\sigma_1^2(2^{\sigma_1-1}+1)}{\sigma+1} \left(\frac{\sigma+1}{\sigma-\sigma_1+1} \frac{1}{\sigma_1(2^{\sigma_1}+2)} \right)^{-\frac{\sigma-\sigma_1+1}{\sigma_1}} \alpha^{\frac{(\sigma_1-1)(\sigma+1)}{\sigma_1}}$ is a nonnegative function. By using (39) into (38), we have

$$\begin{aligned} \dot{V}_1 &\leq -\beta_1 \frac{\pi}{\zeta T_d} z_1^{(\sigma+1)(1+\zeta/2)} - \bar{\beta}_1 \frac{\pi}{\zeta T_d} z_1^{(\sigma+1)(1-\zeta/2)} - \Gamma_1 z_1^{\sigma+1} \\ &\quad - (\epsilon + \tilde{\epsilon}) \frac{1}{\tau_1} \tilde{\Theta}_1^2 - \frac{g_0^-}{\delta_0} r_1^{1+\zeta/2} - \frac{g_0^+}{\delta_0} r_1^{1-\zeta/2} + \Delta_{11} z_2^{\sigma+1} + \Delta_{12} z_2^{(\sigma+1)/\sigma_1} \\ &\quad + \frac{\bar{\omega}}{\delta_0} + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2} \frac{z_1}{\mu} \right) \frac{x_1^2 \lambda(x_1^2)}{\delta_0} + l_{1,1}^* + l_{1,2}^* + m_1(t_0, t) \\ &\quad + (\epsilon + \tilde{\epsilon}) \frac{1}{\tau_1} \Theta_1^{*2} + \frac{d_1^2}{2} + \frac{\epsilon_1^{*2}}{2} + \frac{\bar{\beta}_1 \pi}{\zeta T_d} z_1^{(1-\frac{\zeta}{2})(\sigma+1)} \end{aligned} \quad (40)$$

Applying Lemma 1, one has

$$z_1^{(\sigma+1)(1-\zeta/2)} \leq \frac{\zeta T_d \bar{\Gamma}_1}{\bar{\beta}_1 \pi} z_1^{(\sigma+1)} + \frac{\zeta}{2} \left(\frac{\zeta T_d \bar{\Gamma}_1}{(1-\zeta/2) \bar{\beta}_1 \pi} \right)^{1-2/\zeta}. \quad (41)$$

Next, the time derivative of V_1 can be calculated as

$$\begin{aligned} \dot{V}_1 &\leq -\beta_1 \frac{\pi}{\zeta T_d} z_1^{(\sigma+1)(1+\zeta/2)} - \bar{\beta}_1 \frac{\pi}{\zeta T_d} z_1^{(\sigma+1)(1-\zeta/2)} \\ &\quad - (\epsilon + \tilde{\epsilon}) \frac{1}{\tau_1} \tilde{\Theta}_1^2 - \frac{g_0^-}{\delta_0} r_1^{1+\zeta/2} - \frac{g_0^+}{\delta_0} r_1^{1-\zeta/2} + \Delta_{11} z_2^{\sigma+1} + \Delta_{12} z_2^{(\sigma+1)/\sigma_1} \\ &\quad + \frac{\bar{\omega}}{\delta_0} + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2} \frac{z_1}{\mu} \right) \frac{x_1^2 \lambda(x_1^2)}{\delta_0} + l_{1,1}^* + l_{1,2}^* + m_1(t_0, t) \\ &\quad + (\epsilon + \tilde{\epsilon}) \frac{1}{\tau_1} \Theta_1^{*2} + \frac{d_1^2}{2} + \frac{\epsilon_1^{*2}}{2} + \frac{\bar{\beta}_1 \pi}{\zeta T_d} \left(\frac{\zeta T_d \bar{\Gamma}_1}{(1-\zeta/2) \bar{\beta}_1 \pi} \right)^{1-2/\zeta}. \end{aligned} \quad (42)$$

Setting $\mathcal{P} = z_\iota$, $\mathcal{N} = j$, $\tilde{c} = \mathcal{D}(\sigma - \sigma_\iota + 2)$, $\tilde{d} = \mathcal{D}(\sigma + 1) - \mathcal{D}(\sigma - \sigma_\iota + 2)$, $\epsilon = \frac{\sigma+1}{\sigma-\sigma_\iota+2}$, one obtains

$$-z_\iota^{\mathcal{D}(\sigma+1)} \leq -z_\iota^{\mathcal{D}(\sigma-\sigma_\iota+2)} j^{\mathcal{D}(\sigma_\iota-1)} + \frac{\sigma_\iota-1}{\sigma+1} \left[\frac{\sigma+1}{\sigma-\sigma_\iota+2} \right]^{-(\sigma-\sigma_\iota+2)/(\sigma_\iota-1)} j^{\mathcal{D}(\sigma+1)}, \quad (43)$$

where $\mathcal{D} > 0$ and $j > 0$ are positive constants.

For (43), let us set $\iota = 1, 2, \dots, \hbar$. Defining $\mathcal{D} = 1 + \zeta/2$ and $j = 1$, we obtain

$$-z_\iota^{(\sigma+1)(1+\zeta/2)} \leq -z_\iota^{(\sigma-\sigma_\iota+2)(1+\zeta/2)} + \frac{\sigma_\iota-1}{\sigma+1} \left[\frac{\sigma+1}{\sigma-\sigma_\iota+2} \right]^{-(\sigma-\sigma_\iota+2)/(\sigma_\iota-1)}. \quad (44)$$

Similarly, for $\iota = 1, 2, \dots, \hbar$, if we choose $\mathcal{D} = 1 - \zeta/2$ and $j = 1$, we have

$$-z_\iota^{(\sigma+1)(1-\zeta/2)} \leq -z_\iota^{(\sigma-\sigma_\iota+2)(1-\zeta/2)} + \frac{\sigma_\iota-1}{\sigma+1} \left[\frac{\sigma+1}{\sigma-\sigma_\iota+2} \right]^{-(\sigma-\sigma_\iota+2)/(\sigma_\iota-1)}. \quad (45)$$

By using (43)–(45) into (42), one has

$$\begin{aligned} \dot{V}_1 \leq & -\beta_1 \frac{\pi}{\zeta T_d} z_1^{(\sigma-\sigma_1+2)(1+\zeta/2)} - \bar{\beta}_1 \frac{\pi}{\zeta T_d} z_1^{(\sigma-\sigma_1+2)(1-\zeta/2)} - (\mathfrak{e} + \bar{\mathfrak{e}}) \frac{1}{\tau_1} \bar{\Theta}_1^2 - \frac{g_0^-}{\delta_0} r_1^{1+\zeta/2} - \frac{g_0^+}{\delta_0} r_1^{1-\zeta/2} \\ & + \Delta_{11} z_2^{\sigma+1} + \Delta_{12} z_2^{(\sigma+1)/\sigma_1} + M_1 + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2} \frac{z_1}{\mu}\right) \frac{x_1^2 \lambda(x_1^2)}{\delta_0}, \end{aligned} \quad (46)$$

where $M_1 = (\beta_1 + \bar{\beta}_1) \frac{\pi}{\zeta T_d} \frac{\sigma_1-1}{\sigma+1} \left[\frac{\sigma+1}{\sigma-\sigma_1+2} \right]^{-(\sigma-\sigma_1+2)/(\sigma_1-1)} + l_{1,1}^* + l_{1,2}^* + m_1(t_0, t) + (\mathfrak{e} + \bar{\mathfrak{e}}) \frac{1}{\tau_1} \bar{\Theta}_1^{*2} + \frac{d_1^2}{2} + \frac{\varepsilon_1^{*2}}{2} + \frac{\omega}{\delta_0} + \beta_1 \frac{\pi}{2T_d} \left(\frac{\zeta T_d \Gamma_1}{(1-\zeta/2)\beta_1 \pi} \right)^{1-2/\zeta}$.

Step ι ($\iota = 2, 3, \dots, \hbar - 1$): By using (1) and (18), the derivative of z_ι is

$$\dot{z}_\iota = x_{\iota+1}^{\sigma_\iota} + f_\iota(x) + \Delta_\iota - \dot{\alpha}_{\iota-1}, \quad (47)$$

where

$$\begin{aligned} \dot{\alpha}_{\iota-1} = & \sum_{\kappa=1}^{\iota-1} \frac{\partial \alpha_{\iota-1}}{\partial x_\kappa} x_{\kappa+1} + \sum_{\kappa=1}^{\iota-1} \frac{\partial \alpha_{\iota-1}}{\partial x_\kappa} f_\kappa + \sum_{\kappa=1}^{\iota-1} \frac{\partial \alpha_{\iota-1}}{\partial x_\kappa} \Delta_\kappa \\ & + \sum_{\kappa=1}^{\iota-1} \frac{\partial \alpha_{\iota-1}}{\partial \bar{\Theta}_\kappa} \dot{\bar{\Theta}}_\kappa + \sum_{\kappa=0}^{\iota-1} \frac{\partial \alpha_{\iota-1}}{\partial y^{(\kappa)}} r^{(\kappa+1)} + \frac{\partial \alpha_{\iota-1}}{\partial r} \dot{r}. \end{aligned} \quad (48)$$

Consider the following Lyapunov function as

$$V_\iota = V_{\iota-1} + \frac{z_\iota^{\sigma-\sigma_\iota+2}}{\sigma - \sigma_\iota + 2} + \frac{\bar{\Theta}_\iota^2}{2\tau_\iota}, \quad (49)$$

where $\bar{\Theta}_\iota$ is the estimate of Θ_ι^* , $\bar{\Theta}_\iota = \Theta_\iota^* - \bar{\Theta}_\iota$, and $\tau_\iota > 0$ is a designed parameter.

Let $\bar{\Delta}_\iota$ be defined as $\bar{\Delta}_\iota = \Delta_\iota - \sum_{\kappa=1}^{\iota-1} \frac{\partial \alpha_{\iota-1}}{\partial x_\kappa} \Delta_\kappa$. By invoking Assumption 2 together with Lemmas 2 and 5, and by applying arguments similar to those presented in Step 1, one has

$$\begin{aligned} z_\iota^{\sigma-\sigma_\iota+1} \bar{\Delta}_\iota \leq & z_\iota^{\sigma-\sigma_\iota+1} \hat{\Psi}_{\iota,1}(x_1, \dots, x_\iota) + z_\iota^{\sigma-\sigma_\iota+1} \hat{\Psi}_{\iota,2}(x_1, \dots, x_\iota, r) \\ & + l_{\iota,1}^* + l_{\iota,2}^* + \frac{1}{4} z_\iota^{2(\sigma-\sigma_\iota+1)} + m_\iota(t_0, t), \end{aligned} \quad (50)$$

where $l_{\iota,1} > 0$, $l_{\iota,2} > 0$, $\hat{\Psi}_{\iota,1} = \left(\Psi_{\iota,1} + \sum_{\kappa=1}^{\iota-1} \left| \frac{\partial \alpha_{\iota-1}}{\partial x_\kappa} \right| \Psi_{\iota,1} \right) \tanh \left(\frac{z_\iota^{\sigma-\sigma_\iota+1}}{l_{\iota,1}} \left(\Psi_{\iota,1} + \sum_{\kappa=1}^{\iota-1} \left| \frac{\partial \alpha_{\iota-1}}{\partial x_\kappa} \right| \Psi_{\iota,1} \right) \right)$, $l_{\iota,1}^* = 0.2785 l_{\iota,1}$, $\Psi_{\iota,2}(x_1, x_2, \dots, x_\iota, r) = \Psi_{\iota,2} \circ \mathcal{L}^{-1}(2r) + \sum_{\kappa=1}^{\iota-1} \left| \frac{\partial \alpha_{\iota-1}}{\partial x_\kappa} \right| \Psi_{\iota,1} \Psi_{\iota,2} \circ \mathcal{L}^{-1}(2r)$, $\hat{\Psi}_{\iota,2}(x_1, x_2, \dots, x_\iota, r) = \Psi_{\iota,2}(x_1, x_2, \dots, x_\iota, r) \tanh \left(\frac{z_\iota^{\sigma-\sigma_\iota+1}}{l_{\iota,2}} \Psi_{\iota,2}(x_1, x_2, \dots, x_\iota, r) \right)$, $m_\iota(t_0, t) = \sum_{\kappa=1}^{\iota} (\Psi_{\kappa,2} \circ \mathcal{L}^{-1}(2M(t_0, t)))^2$, $m_\iota(t_0, t) \geq$

0 for all $t \geq t_0 + T_0$, $l_{i,2}^* = 0.2785 l_{i,2}$.

Using the results given in (47)–(50), the time derivative of V_i can be expressed as

$$\begin{aligned} \dot{V}_i \leq & - \sum_{\kappa=1}^{\iota-1} \frac{\beta_{\kappa} \pi}{\varsigma T_d} z_{\kappa}^{(1+\varsigma/2)(\sigma-\sigma_{\kappa}+2)} - \sum_{\kappa=1}^{\iota-1} \beta_{\kappa} \frac{\bar{\beta}_{\kappa} \pi}{\varsigma T_d} z_{\kappa}^{(1-\varsigma/2)(\sigma-\sigma_{\kappa}+2)} - \sum_{\kappa=1}^{\iota-1} \frac{\mathfrak{e} + \tilde{\mathfrak{e}}}{\tau_{\kappa}} \tilde{\Theta}_{\kappa}^2 \\ & + \Delta_{(\iota-1)1} z_i^{\sigma+1} + \Delta_{(\iota-1)2} z_i^{(\sigma+1)/\sigma_i} - \frac{1}{\tau_i} \tilde{\Theta}_i \dot{\tilde{\Theta}}_i - \frac{1}{2} z_i^{2(\sigma-\sigma_i+1)} \\ & - g_0 \delta_0 r_1^{+(1+\varsigma)/2} - g_0 \delta_0 r_1^{-(1-\varsigma)/2} + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2} \frac{z_1}{\mu} \right) x_1^2 \lambda(x_1) \delta_0 \\ & + l_{i,1}^* + l_{i,2}^* + m_i(t_0, t) + M_{i-1} \\ & + z_i^{\sigma-\sigma_i+1} \left(x_{i+1}^{\sigma_i} + \bar{f}_i(x) - \sum_{\kappa=1}^{\iota-1} \frac{\partial \alpha_{i-1}}{\partial x_{\kappa}} x_{\kappa+1} - \sum_{\kappa=1}^{\iota-1} \frac{\partial \alpha_{i-1}}{\partial x_{\kappa}} f_{\kappa} \right. \\ & \left. - \sum_{\kappa=1}^{\iota-1} \frac{\partial \alpha_{i-1}}{\partial \tilde{\Theta}_{\kappa}} \dot{\tilde{\Theta}}_{\kappa} - \sum_{\kappa=0}^{\iota-1} \frac{\partial \alpha_{i-1}}{\partial y^{(\kappa)}} r^{(\kappa+1)} - \frac{\partial \alpha_{i-1}}{\partial r} \dot{r} \right) \end{aligned} \quad (51)$$

where $\bar{f}_i(x) = f_i(x) + \frac{3}{4} z_i^{\sigma-\sigma_i+1} + \hat{\Psi}_{i,1}(x_1, \dots, x_i) + \hat{\Psi}_{i,2}(x_1, \dots, x_i, r)$.

Now, $\bar{f}_i$ can be estimated through a fuzzy logic system (FLS) approximation of the form

$$\bar{f}_i(Z_i) = W_i^{*T} \Phi_i(Z_i) + \varepsilon_i(Z_i), \quad \|\varepsilon_i(Z_i)\| \leq \varepsilon_i^*, \quad (52)$$

where $Z_i = [x_1, \dots, x_h, \hat{\Theta}_1, \dots, \hat{\Theta}_{i-1}]^T$ and $\varepsilon_i^* > 0$ is a constant.

With the help of Young's inequality and Lemma 7, one has

$$\begin{aligned} z_i^{\sigma-\sigma_i+1} f_i & \leq |z_i|^{\sigma-\sigma_i+1} \left(\|W_i^*\| \|\Phi_i(Z_i)\| + \varepsilon_i^* \right) \\ & \leq \frac{1}{2d_i^2} z_i^{2(\sigma-\sigma_i+1)} \Theta_i^* \Phi_i^T(\chi_i) \Phi_i(\chi_i) + \frac{d_i^2}{2} + \frac{1}{2} z_i^{2(\sigma-\sigma_i+1)} + \frac{(\varepsilon_i^*)^2}{2}, \end{aligned} \quad (53)$$

where $\Theta_i^* = \|W_i^*\|^2$, $\chi_i = [x_1, \dots, x_i]^T$, $d_i > 0$ is a design parameter, and $\varepsilon_i^* > 0$ satisfies $\|\varepsilon_i\| \leq \varepsilon_i^*$.

The virtual controller α_i and the adaptive law for $\tilde{\Theta}_i$ are designed as

$$\begin{aligned} \alpha_i = & - \frac{1}{\sigma_i} \left(\frac{1}{2d_i^2} z_i^{2(\sigma-\sigma_i+1)} \Theta_i \Phi_i^T(\chi_i) \Phi_i(\chi_i) + \Delta_{(\iota-1)1} z_i^{\sigma_i} + \Delta_{(\iota-1)2} z_i^{\sigma_i} \right. \\ & - \sum_{\kappa=1}^{\iota-1} \frac{\partial \alpha_{i-1}}{\partial x_{\kappa}} x_{\kappa+1} - \sum_{\kappa=1}^{\iota-1} \frac{\partial \alpha_{i-1}}{\partial x_{\kappa}} f_{\kappa} - \sum_{\kappa=1}^{\iota-1} \frac{\partial \alpha_{i-1}}{\partial \tilde{\Theta}_{\kappa}} \dot{\tilde{\Theta}}_{\kappa} \\ & \left. - \sum_{\kappa=0}^{\iota-1} \frac{\partial \alpha_{i-1}}{\partial y^{(\kappa)}} r^{(\kappa+1)} - \frac{\partial \alpha_{i-1}}{\partial r} \dot{r} + \Gamma_i z_i^{\sigma_i} + \frac{\beta_i \pi}{\varsigma T_d} z_i^{(1+\varsigma/2)(\sigma+1)-(\sigma-\sigma_i+1)} \right)^{\frac{1}{\sigma_i}}, \end{aligned} \quad (54)$$

$$\dot{\tilde{\Theta}}_i = \tau_i \frac{1}{2d_i^2} z_i^{2(\sigma-\sigma_i+1)} \Phi_i^T(\chi_i) \Phi_i(\chi_i) - 2(\mathfrak{e} + \tilde{\mathfrak{e}}) \tilde{\Theta}_i, \quad (55)$$

where $\beta_i = \frac{(3n)^{\frac{\varsigma}{2}}}{(\sigma - \sigma_i + 2)(1 + \varsigma/2)}$, $\bar{\beta}_i = \frac{1}{(\sigma - \sigma_i + 2)(1 - \varsigma/2)}$, and $\Gamma_i = \frac{\bar{\beta}_i \pi}{\varsigma T_d} + 1$. Moreover, $\bar{\sigma}_i = \frac{\sigma+1}{\sigma_{i-1}} - (\sigma - \sigma_i + 1)$, $\tilde{\Theta}_i(0) \geq 0$, and σ_i is a nonnegative constant under Assumption 3. In addition,

$$\Delta_{(\iota-1)1} = \frac{\sigma_{i-1}^{2\sigma_{i-1}-1}}{\sigma+1} \left[\frac{\sigma+1}{\sigma_{i-1} 2^{\sigma_{i-1}} (\sigma - \sigma_{i-1} + 1)} \right]^{-\frac{\sigma - \sigma_{i-1} + 1}{\sigma_{i-1}}},$$

and $\Delta_{(l-1)2} = \frac{\sigma_{l-1}^2(2^{\sigma_{l-1}-1}+1)}{\sigma+1} \alpha^{(\sigma_{l-1}-1)(\sigma+1)/\sigma_{l-1}} \left[\frac{\sigma+1}{\sigma_{l-1}(\sigma-\sigma_{l-1}+1)(2\sigma_{l-1}+2)} \right]^{-\frac{\sigma-\sigma_{l-1}+1}{\sigma_{l-1}}}$.

By using (53)–(55) into (51), one has

$$\begin{aligned} \dot{V}_l \leq & - \sum_{\kappa=1}^{l-1} \frac{\beta_{\kappa} \pi}{\zeta T_d} z_{\kappa}^{(1+\zeta/2)(\sigma-\sigma_{\kappa}+2)} - \sum_{\kappa=1}^{l-1} \beta_{\kappa} \frac{\pi \zeta}{T_d} z_{\kappa}^{(1-\zeta/2)(\sigma-\sigma_{\kappa}+2)} - \sum_{\kappa=1}^l \frac{\mathfrak{e} + \tilde{\mathfrak{e}}}{\tau_{\kappa}} \tilde{\Theta}_{\kappa}^2 \\ & + z_l^{\sigma-\sigma_l+1} (x_{l+1}^{\sigma_l} - \alpha_l^{\sigma_l}) - \Gamma_l z_l^{\sigma+1} - \frac{\beta_l \pi}{\zeta T_d} z_l^{(1+\zeta/2)(\sigma+1)} - \frac{\beta_l \pi}{\zeta T_d} z_l^{(1-\zeta/2)(\sigma+1)} \\ & + \frac{(\mathfrak{e} + \tilde{\mathfrak{e}})}{2\tau_l} \Theta_l^{*2} - \frac{g_0}{\delta_0} r_1^{(1+\zeta/2)} - \frac{\bar{g}_0}{\delta_0} r_1^{(1-\zeta/2)} + \frac{\beta_l \pi}{\zeta T_d} z_l^{(1-\zeta/2)(\sigma+1)} + m_l(t_0, t) \\ & + (1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2}(z_1/\mu)) x_1^2 \lambda(x_1^2) + M_{l-1} + l_{l,1}^* + l_{l,2}^* + \frac{d_l^2}{2} + \frac{(\varepsilon_l^*)^2}{2}. \end{aligned} \quad (56)$$

By Lemma 1 and Lemma 3, one has

$$z_l^{\sigma-\sigma_l+1} (x_{l+1}^{\sigma_l} - \alpha_l^{\sigma_l}) \leq z_l^{\sigma+1} + \Delta_{l1} z_{l+1}^{\sigma+1} + \Delta_{l2} z_{l+1}^{\frac{\sigma_l+1}{\sigma_l}}. \quad (57)$$

where $\Delta_{l1} = \frac{2^{\sigma_l-1} \sigma_l^2}{\sigma+1} \left(\frac{\sigma+1}{(\sigma-\sigma_l+1)\sigma_l 2^{\sigma_l}} \right)^{-\frac{\sigma-\sigma_l+1}{\sigma_l}}$, which is a positive constant related to the given constants σ and σ_l , and $\Delta_{l2} = \frac{\sigma_l^2(2^{\sigma_l-1}+1)}{\sigma+1} \left(\frac{\sigma+1}{(\sigma-\sigma_l+1)\sigma_l(2^{\sigma_l+1}+2)} \right)^{-\frac{\sigma-\sigma_l+1}{\sigma_l}} \alpha^{\frac{(\sigma_l-1)(\sigma+1)}{\sigma_l}}$, where α is a nonnegative function. Furthermore, by Lemma 1, we have

$$z_l^{(1-\zeta/2)(\sigma+1)} \leq \frac{\zeta T_d \bar{\Gamma}_l}{\bar{\beta}_l \pi} z_l^{\sigma+1} + \frac{\zeta}{2} \left(\frac{\zeta T_d \bar{\Gamma}_l}{(1-\zeta/2)\bar{\beta}_l \pi} \right)^{1-2/\zeta}, \quad (58)$$

where $\Gamma_l = \Gamma_l - 1$.

By using (57) and (58) into (56), one has

$$\begin{aligned} \dot{V}_l \leq & - \sum_{\kappa=1}^{l-1} \frac{\beta_{\kappa} \pi}{\zeta T_d} z_{\kappa}^{(1+\zeta/2)(\sigma-\sigma_{\kappa}+2)} - \sum_{\kappa=1}^{l-1} \frac{\bar{\beta}_{\kappa} \pi}{\zeta T_d} z_{\kappa}^{(1-\zeta/2)(\sigma-\sigma_{\kappa}+2)} - \sum_{\kappa=1}^l \frac{\mathfrak{e} + \tilde{\mathfrak{e}}}{\tau_{\kappa}} \tilde{\Theta}_{\kappa}^2 \\ & - \frac{g_0}{\delta_0} r_1^{1+\zeta/2} - \frac{\bar{g}_0}{\delta_0} r_1^{1-\zeta/2} + \Delta_{l1} z_{l+1}^{\sigma+1} + \Delta_{l2} z_{l+1}^{\frac{\sigma_l+1}{\sigma_l}} + M_{l-1} + l_{l,1}^* + l_{l,2}^* \\ & + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2} \frac{z_1}{\mu} \right) \frac{x_1^2 \lambda(x_1^2)}{\delta_0} + \frac{\bar{\beta}_l \pi}{2 T_d} \left(\frac{\zeta T_d \bar{\Gamma}_l}{(1-\zeta/2)\bar{\beta}_l \pi} \right)^{1-2/\zeta} \\ & + m_l(t_0, t) + \frac{(\mathfrak{e} + \tilde{\mathfrak{e}}) \Theta_l^{*2}}{2\tau_l} + \frac{d_l^2}{2} + \frac{(\varepsilon_l^*)^2}{2}. \end{aligned} \quad (59)$$

Using (44) and (45) into (59), one has

$$\begin{aligned} \dot{V}_l \leq & \sum_{\kappa=1}^{l-1} \frac{\beta_{\kappa} \pi}{\zeta T_d} z_{\kappa}^{(1+\zeta/2)(\sigma-\sigma_{\kappa}+2)} - \sum_{\kappa=1}^{l-1} \frac{\bar{\beta}_{\kappa} \pi}{\zeta T_d} z_{\kappa}^{(1-\zeta/2)(\sigma-\sigma_{\kappa}+2)} - \sum_{\kappa=1}^l \frac{\mathfrak{e} + \tilde{\mathfrak{e}}}{\tau_{\kappa}} \tilde{\Theta}_{\kappa}^2 \\ & - \frac{g_0}{\delta_0} r_1^{1+\zeta/2} - \frac{\bar{g}_0}{\delta_0} r_1^{1-\zeta/2} + \Delta_{l1} z_{l+1}^{\sigma+1} + \Delta_{l2} z_{l+1}^{\frac{\sigma_l+1}{\sigma_l}} + M_l \\ & + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2} \frac{z_1}{\mu} \right) \frac{x_1^2 \lambda(x_1^2)}{\delta_0}, \end{aligned} \quad (60)$$

where $M_l = m_l(t_0, t) + \frac{(\epsilon + \tilde{\epsilon})\Theta_l^{*2}}{2\tau_l} + \frac{\tilde{\beta}_l\pi}{2T_d} \left(\frac{\varsigma T_d \tilde{\Gamma}_l}{(1-\varsigma/2)\tilde{\beta}_l\pi} \right)^{1-2/\varsigma} + \frac{(\beta_l + \tilde{\beta}_l)\pi}{\varsigma T_d} \frac{\sigma_l - 1}{\sigma + 1} \left[\frac{\sigma + 1}{\sigma - \sigma_l + 2} \right]^{-\frac{\sigma - \sigma_l + 2}{\sigma_l - 1}} + I_{l,1}^* + I_{l,2}^* + \frac{d_l^2}{2} + \frac{(\epsilon_l^*)^2}{2} + M_{l-1}$.

Step h : By using (8) and (25), one has

$$\begin{aligned} \dot{z}_h &= \dot{x}_h - \dot{\alpha}_{h-1} + \frac{\dot{x}_{h+1}}{\lambda} \\ &= x_{h+1} - u^{\sigma_h} - \dot{\alpha}_{h-1} + f_h(x) + \Delta_h + \frac{1}{\lambda}(-\lambda x_{h+1} + 2\lambda u^{\sigma_h}), \\ &= u^{\sigma_h} + f_h(x) + \Delta_h - \dot{\alpha}_{h-1}, \end{aligned} \quad (61)$$

where $\dot{\alpha}_{h-1}$ is defined analogously to $\dot{\alpha}_{l-1}$ in Step l .

Consider the following Lyapunov function as

$$V_h = V_{h-1} + \frac{z_h^{\sigma - \sigma_h + 2}}{\sigma - \sigma_h + 2} + \frac{\tilde{\Theta}_h^2}{2\tau_h}, \quad (62)$$

where $\hat{\Theta}_h$ is the estimation of Θ_h^* , $\tilde{\Theta}_h = \Theta_h^* - \hat{\Theta}_h$, and $\tau_h > 0$ is a designed positive parameter.

Define $\Delta_h = \bar{\Delta}_h - \sum_{\kappa=1}^{h-1} \frac{\partial \alpha_{h-1}}{\partial x_\kappa} \Delta_\kappa$. Similar to Step 1, one has

$$\begin{aligned} z_h^{\sigma - \sigma_h + 1} \bar{\Delta}_h &\leq z_h^{\sigma - \sigma_h + 1} \hat{\Psi}_{h,1}(x_1, \dots, x_h) + z_h^{\sigma - \sigma_h + 1} \hat{\Psi}_{h,2}(x_1, \dots, x_h, r) \\ &\quad + I_{h,1}^* + I_{h,2}^* + \frac{1}{4} z_h^{2(\sigma - \sigma_h + 1)} + m_h(t_0, t), \end{aligned} \quad (63)$$

where $l_{h,1} > 0$, $l_{h,2} > 0$, $\hat{\Psi}_{h,1} = \left(\Psi_{h,1} + \sum_{\kappa=1}^{h-1} \left| \frac{\partial \alpha_{h-1}}{\partial x_\kappa} \right| \Psi_{h,1} \right) \tanh \left(\frac{z_h^{\sigma - \sigma_h + 1}}{l_{h,1}} \left(\Psi_{h,1} + \sum_{\kappa=1}^{h-1} \left| \frac{\partial \alpha_{h-1}}{\partial x_\kappa} \right| \Psi_{h,1} \right) \right)$, $I_{h,1}^* = 0.2785 l_{h,1}$, $\Psi_{h,2}(x_1, x_2, \dots, x_h, r) = \Psi_{h,2} \circ \mathcal{L}^{-1}(2r) + \sum_{\kappa=1}^{h-1} \left| \frac{\partial \alpha_{h-1}}{\partial x_\kappa} \right| \Psi_{h,1} \Psi_{h,2} \circ \mathcal{L}^{-1}(2r)$, $\hat{\Psi}_{h,2}(x_1, x_2, \dots, x_h, r) = \Psi_{h,2}(x_1, x_2, \dots, x_h, r) \tanh \left(\frac{z_h^{\sigma - \sigma_h + 1}}{l_{h,2}} \Psi_{h,2}(x_1, x_2, \dots, x_h, r) \right)$, $m_h(t_0, t) = \sum_{\kappa=1}^h (\Psi_{\kappa,2} \circ \mathcal{L}^{-1}(2M(t_0, t)))^2$, $m_h(t_0, t) \geq 0$ for all $t \geq t_0 + T_0$.

By using (63) and differentiating V_h , one has

$$\begin{aligned} \dot{V}_h &\leq - \sum_{\kappa=1}^{h-1} \frac{\beta_\kappa \pi}{\varsigma T_d} z_\kappa^{(1+\varsigma/2)(\sigma - \sigma_\kappa + 2)} - \sum_{\kappa=1}^{h-1} \frac{\tilde{\beta}_\kappa \pi}{\varsigma T_d} z_\kappa^{(1-\varsigma/2)(\sigma - \sigma_\kappa + 2)} - \sum_{\kappa=1}^{h-1} \frac{\epsilon + \tilde{\epsilon}}{\tau_\kappa} \tilde{\Theta}_\kappa^2 \\ &\quad - \frac{1}{\tau_h} \tilde{\Theta}_h \dot{\Theta}_h - \frac{g_0}{\delta_0} r_1^{+(1+\varsigma)/2} - \frac{\bar{g}_0}{\delta_0} r_1^{-(1-\varsigma)/2} \\ &\quad + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma - \sigma_1 + 2} \frac{z_1}{\mu} \right) x_1^2 \lambda(x_1) \delta_0 \\ &\quad + I_{h,1}^* + I_{h,2}^* + m_h(t_0, t) + M_{h-1} + z_h^{\sigma - \sigma_h + 1} \left(u^{\sigma_h} + \bar{f}_h(x) \right) \end{aligned} \quad (64)$$

where $\bar{\sigma}_h = \frac{\sigma + 1}{\sigma_{h-1}} - (\sigma - \sigma_h + 1)$ denotes a nonnegative constant and

$$\begin{aligned} \bar{f}_h(x) &= f_h(x) + \Delta_{(h-1)1} z_h^{\sigma_h} + \Delta_{(h-1)2} z_h^{\bar{\sigma}_h} + \hat{\Psi}_{h,1}(x) + \hat{\Psi}_{h,2}(x, r) + \frac{3}{4} z_h^{\sigma - \sigma_h + 1} \\ &\quad - \sum_{\kappa=1}^{h-1} \frac{\partial \alpha_{h-1}}{\partial x_\kappa} x_{\kappa+1} - \sum_{\kappa=1}^{h-1} \frac{\partial \alpha_{h-1}}{\partial x_\kappa} f_\kappa - \sum_{\kappa=1}^{h-1} \frac{\partial \alpha_{h-1}}{\partial \hat{\Theta}_\kappa} \dot{\Theta}_\kappa \\ &\quad - \sum_{\kappa=0}^{h-1} \frac{\partial \alpha_{h-1}}{\partial y^{(\kappa)}} r^{(\kappa+1)} - \frac{\partial \alpha_{h-1}}{\partial r} \dot{r}. \end{aligned} \quad (65)$$

Now, $\bar{f}_h$ can be approximated by the FLS as

$$\bar{f}_h(Z_h) = W_h^{*T} \Phi_h(Z_h) + \varepsilon_h(Z_h), \quad \|\varepsilon_h(Z_h)\| \leq \varepsilon_h^*, \quad (66)$$

where $Z_h = [x_1, \dots, x_h, \hat{\Theta}_1, \dots, \hat{\Theta}_{h-1}]^T$ and $\varepsilon_h^* > 0$ is a constant.

With the help of Young's inequality and Lemma 7, one has

$$\begin{aligned} z_h^{\sigma-\sigma_h+1} f_h &\leq |z_h|^{\sigma-\sigma_h+1} \left(\|W_h^*\| \|\Phi_h(Z_h)\| + \varepsilon_h^* \right) \\ &\leq \frac{1}{2d_h^2} z_h^{2(\sigma-\sigma_h+1)} \Theta_h^* \Phi_h^T(\chi_h) \Phi_h(\chi_h) + \frac{d_h^2}{2} + \frac{1}{2} z_h^{2(\sigma-\sigma_h+1)} + \frac{(\varepsilon_h^*)^2}{2}, \end{aligned} \quad (67)$$

By using (67) into (65), one has

$$\begin{aligned} \dot{V}_h &\leq - \sum_{\kappa=1}^{h-1} \frac{\beta_\kappa \pi}{\varsigma T_d} z_\kappa^{(1+\varsigma/2)(\sigma-\sigma_\kappa+2)} - \sum_{\kappa=1}^{h-1} \frac{\bar{\beta}_\kappa \pi}{\varsigma T_d} z_\kappa^{(1-\varsigma/2)(\sigma-\sigma_\kappa+2)} \\ &\quad - \sum_{\kappa=1}^{h-1} \frac{\mathfrak{e} + \tilde{\mathfrak{e}}}{\tau_\kappa} \tilde{\Theta}_\kappa^2 - \frac{1}{\tau_h} \tilde{\Theta}_h \dot{\Theta}_h - \frac{g_0}{\delta_0} r_1^{+(1+\varsigma)/2} - \frac{\bar{g}_0}{\delta_0} r_1^{-(1-\varsigma)/2} \\ &\quad + M_{h-1} + l_{h,1}^* + l_{h,2}^* + m_h(t_0, t) + \frac{d_h^2}{2} + \frac{(\varepsilon_h^*)^2}{2} \\ &\quad + z_h^{\sigma-\sigma_h+1} \left(u^{\sigma_h}(t) + \frac{1}{2d_h^2} z_h^{2(\sigma-\sigma_h+1)} \Theta_h^* \Phi_h^T(\chi_h) \Phi_h(\chi_h) \right) \\ &\quad + \frac{1}{\tau_h} \tilde{\Theta}_h \left[\frac{1}{2d_h^2} z_h^{2(\sigma-\sigma_h+1)} \Phi_h^T(\chi_h) \Phi_h(\chi_h) - \dot{\Theta}_h \right] \\ &\quad + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2} \left(\frac{z_1}{\mu} \right) \right) \frac{x_2^2 \lambda(x_2)}{\delta_0}. \end{aligned} \quad (68)$$

The real control u and the adaptive law $\dot{\Theta}_h$ are formulated as

$$u = - \left(\frac{1}{2d_h^2} z_h^{\sigma-\sigma_h+1} \hat{\Theta}_h \Phi_h^T(\chi_h) \Phi_h(\chi_h) + \Gamma_h z_h^{\sigma_h} \frac{\beta_h \pi}{\varsigma T_d} z_h^{(1+\varsigma/2)(\sigma+1)-(\sigma-\sigma_h+1)} \right)^{\frac{1}{\sigma_i}}, \quad (69)$$

$$\dot{\Theta}_h = \tau_h \frac{1}{2d_h^2} z_h^{2(\sigma-\sigma_h+1)} \Phi_h^T(\chi_h) \Phi_h(\chi_h) - 2(\mathfrak{e} + \tilde{\mathfrak{e}}) \hat{\Theta}_h, \quad (70)$$

where $\beta_h = \frac{(3n)^{\frac{\varsigma}{2}}}{(\sigma - \sigma_i + 2)(1 + \varsigma/2)}$, $\bar{\beta}_h = \frac{1}{(\sigma - \sigma_h + 2)(1 - \varsigma/2)}$, and $\Gamma_h = \frac{\bar{\beta}_h \pi}{\varsigma T_d} + 1$, and $\hat{\Theta}_h(0) \geq 0$.

By using (69) into (70), one has

$$\begin{aligned} \dot{V}_h &\leq - \sum_{\kappa=1}^{h-1} \frac{\beta_\kappa \pi}{\varsigma T_d} z_\kappa^{(1+\varsigma/2)(\sigma-\sigma_\kappa+2)} - \sum_{\kappa=1}^{h-1} \frac{\bar{\beta}_\kappa \pi}{\varsigma T_d} z_\kappa^{(1-\varsigma/2)(\sigma-\sigma_\kappa+2)} - \sum_{\kappa=1}^h \frac{\mathfrak{e} + \tilde{\mathfrak{e}}}{\tau_\kappa} \tilde{\Theta}_\kappa^2 \\ &\quad - \Gamma_h z_h^{\sigma+1} - \frac{\beta_h \pi}{\varsigma T_d} z_h^{(1+\varsigma/2)(\sigma+1)} - \frac{\bar{\beta}_h \pi}{\varsigma T_d} z_h^{(1-\varsigma/2)(\sigma+1)} \\ &\quad + \frac{\mathfrak{e} + \tilde{\mathfrak{e}}}{\tau_h} \tilde{\Theta}_h^2 - \frac{g_0}{\delta_0} r_1^{(1+\varsigma/2)} - \frac{\bar{g}_0}{\delta_0} r_1^{(1-\varsigma/2)} + \frac{\beta_h \pi}{\varsigma T_d} z_h^{(1-\varsigma/2)(\sigma+1)} + m_h(t_0, t) \\ &\quad + (1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2}(z_1/\mu)) \frac{x_1^2 \lambda(x_1^2)}{\delta_0} + M_{h-1} + l_{h,1}^* + l_{h,2}^* + \frac{d_h^2}{2} + \frac{(\varepsilon_h^*)^2}{2}. \end{aligned} \quad (71)$$

From Lemma 1, we have

$$z_h^{(1-\varsigma/2)(\sigma+1)} \leq \frac{\varsigma T_d \Gamma_h}{\bar{\beta}_h \pi} z_h^{\sigma+1} + \frac{\varsigma}{2} \left(\frac{\varsigma T_d \Gamma_h}{(1-\omega/2)\bar{\beta}_h \pi} \right)^{1-2/\varsigma}. \quad (72)$$

By using (44), (45), and (72) into (71), one has

$$\begin{aligned} \dot{V}_h \leq & - \sum_{\kappa=1}^{h-1} \frac{\beta_{\kappa} \pi}{\varsigma T_d} z_{\kappa}^{(1+\varsigma/2)(\sigma-\sigma_{\kappa}+2)} - \sum_{\kappa=1}^{h-1} \frac{\bar{\beta}_{\kappa} \pi}{\varsigma T_d} z_{\kappa}^{(1-\varsigma/2)(\sigma-\sigma_{\kappa}+2)} - \sum_{\kappa=1}^h \frac{\mathfrak{e} + \tilde{\mathfrak{e}}}{\tau_{\kappa}} \tilde{\Theta}_{\kappa}^2 \\ & - \frac{\bar{g}_0}{\delta_0} r_1^{+(1+\varsigma)/2} - \frac{\bar{g}_0}{\delta_0} r_1^{-(1-\varsigma)/2} + M_h + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2} \left(\frac{z_1}{\mu} \right) \right) \frac{x_2^2 \lambda(x_2)}{\delta_0}, \end{aligned} \quad (73)$$

where

$$\begin{aligned} M_h = m_h(t_0, t) + & \frac{(\mathfrak{e} + \tilde{\mathfrak{e}}) \Theta_h^{*2}}{2\tau_h} + \frac{\bar{\beta}_h \pi}{2T_d} \left(\frac{\varsigma T_d \bar{\Gamma}_h}{(1-\varsigma/2)\bar{\beta}_h \pi} \right)^{1-2/\varsigma} + \frac{(\beta_h + \bar{\beta}_h) \pi}{\varsigma T_d} \frac{\sigma_h - 1}{\sigma + 1} \left[\frac{\sigma + 1}{\sigma - \sigma_h + 2} \right]^{-\frac{\sigma - \sigma_h + 2}{\sigma_h - 1}} \\ & + I_{h,1}^* + I_{h,2}^* + \frac{d_h^2}{2} + \frac{(\varepsilon_h^*)^2}{2} + M_{h-1}. \end{aligned}$$

Consider the complete Lyapunov function as

$$V = \sum_{\kappa=1}^h \left(\frac{z_{\kappa}^{\sigma-\sigma_{\kappa}+2}}{\sigma - \sigma_{\kappa} + 2} \right) + \sum_{\kappa=1}^h \left(\frac{1}{2\tau_{\kappa}} \tilde{\Theta}_{\kappa}^2 \right) + \frac{r}{\delta_0}. \quad (74)$$

According to [39], there exists a positive constant Y^* such that the parameter estimation error satisfies $\|\tilde{\Theta}_l\| \leq Y^*$. Invoking Lemma 1 and choosing $\mathcal{P} = \tilde{\Theta}_l$, $\epsilon = \aleph = 1$, $\tilde{c} = 2 - \varsigma$, and $\tilde{d} = \varsigma$, one has

$$-\frac{2-\varsigma}{2} \tilde{\Theta}_l^2 \leq -(\tilde{\Theta}_l^2)^{1-\varsigma/2} + \frac{\varsigma}{2}, \quad (75)$$

$$-\frac{2-\varsigma}{2} \tilde{\Theta}_l^2 \leq -(\tilde{\Theta}_l^2)^{1+\varsigma/2} + \frac{\varsigma}{2} (Y^*)^4. \quad (76)$$

Based on the inequalities in [39], it follows that

$$-\sum_{\kappa=1}^h |x_{\kappa}|^{\check{L}} \leq -\left(\sum_{\kappa=1}^h |x_{\kappa}| \right)^{\check{L}}, \quad 0 < \check{L} \leq 1, \quad (77)$$

$$-\sum_{\kappa=1}^h |x_{\kappa}|^{\check{L}} \leq -h^{1-\check{L}} \left(\sum_{\kappa=1}^h |x_{\kappa}| \right)^{\check{L}}, \quad 1 < \check{L} \leq \infty, \quad (78)$$

where $x_{\kappa} \in \mathbb{R}$ for all $\kappa = 1, 2, \dots, h$.

By employing (75)–(78) and differentiating V , one obtains

$$\begin{aligned} \dot{V} \leq & -\frac{\pi}{\varsigma T_d} \left[\sum_{\kappa=1}^h \frac{z_{\kappa}^{\sigma-\sigma_{\kappa}+2}}{\sigma - \sigma_{\kappa} + 2} + \sum_{\kappa=1}^h \frac{\tilde{\Theta}_{\kappa}^2}{2\tau_{\kappa}} + \frac{r}{\delta_0} \right]^{1+\varsigma/2} - \frac{\pi}{\varsigma T_d} \left[\sum_{\kappa=1}^h \frac{z_{\kappa}^{\sigma-\sigma_{\kappa}+2}}{\sigma - \sigma_{\kappa} + 2} + \sum_{\kappa=1}^h \frac{\tilde{\Theta}_{\kappa}^2}{2\tau_{\kappa}} + \frac{r}{\delta_0} \right]^{1-\varsigma/2} \\ & + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2} (z_1/\mu) \right) \frac{x_2^2 \lambda(x_2)}{\delta_0} + \bar{B}_h \\ \leq & -\pi \omega_d^T V^{1+\omega/2} - \pi \omega_d^T V^{1-\omega/2} + \left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2} (z_1/\mu) \right) \frac{x_2^2 \lambda(x_2)}{\delta_0} + B_h, \end{aligned} \quad (79)$$

where $\bar{B}_h = \frac{h(3n)^{\zeta/2}\pi}{2^{2+\zeta/2}\tau_h^{1+\zeta/2}T_d}Y^{*4} + \frac{h\pi}{2^{2-\zeta/2}\tau_h^{1-\zeta/2}T_d} + B_h$.

Theorem 1: For the nonlinear system described in (1), suppose that Assumptions 1–4 hold. When the control inputs are designed according to (36), (54), and (69), and the adaptive update mechanisms are selected as in (37), (55), and (69), it follows that, starting from bounded initial states, every signal of the resulting closed-loop system remains bounded. Moreover, the tracking error is driven into a sufficiently small neighborhood of the origin within a user-prescribed finite time interval.

Proof: From the Lyapunov-based analysis, it follows that the terms $-\frac{\pi}{\zeta T_d}V^{1+\zeta/2}$ and $-\frac{\pi}{\zeta T_d}V^{1-\zeta/2}$ are negative definite. Moreover, $\bar{B}_h$ is a bounded constant. The sign of the remaining term,

$$\left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2}\left(\frac{z_1}{\mu}\right)\right) \frac{x_2^2 \lambda(x_2)}{\delta_0},$$

is governed by the value of z_1 . Accordingly, the analysis is divided into the following cases.

Case 1: Consider $z_1 \in \Omega_{z_1} = \{z_1 \mid |z_1| < 0.8814\mu\}$ for any $\mu > 0$. Since $z_1 = x_1 - y_r$, the boundedness of z_1 together with the bounded reference signal y_r guarantees that x_1 remains bounded. Furthermore, due to the smooth and nonnegative nature of x_2^2 , the term

$$\left(1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2}\left(\frac{z_1}{\mu}\right)\right) \frac{x_2^2 \lambda(x_2)}{\delta_0}$$

is bounded. Let Ψ_0 denote its upper bound. Consequently, the time derivative of the Lyapunov function satisfies

$$\dot{V} \leq -\frac{\pi}{\zeta T_d}V^{1+\zeta/2} - \frac{\pi}{\zeta T_d}V^{1-\zeta/2} + B^*, \quad (80)$$

where $B^* = \bar{B}_h + \Psi_0$.

Case 2: $z_1 \notin \Omega_{z_1}$. By Lemma 4, we have $1 - (\sigma - \sigma_1 + 2) \tanh^{\sigma-\sigma_1+2}\left(\frac{z_1}{\mu}\right) \leq 0$. Thus, the Lyapunov derivative satisfies

$$\dot{V} \leq -\frac{\pi}{\zeta T_d}V^{1+\zeta/2} - \frac{\pi}{\zeta T_d}V^{1-\zeta/2} + \bar{B}_h. \quad (81)$$

Combining both cases yields

$$\dot{V} \leq -\frac{\pi}{\zeta T_d}V^{1+\zeta/2} - \frac{\pi}{\zeta T_d}V^{1-\zeta/2} + B^*. \quad (82)$$

It follows from (82) that the time derivative of V satisfies the structural condition given in (24) of Lemma 8. As a consequence, the developed control scheme guarantees practical predefined-time stability of the tracking error for system (1). In particular, the Lyapunov function is ultimately bounded as $V < \frac{\varphi T_d B^*}{\pi}$, $\forall t > 2T_d$.

Figure 1 presents the block level illustration of the proposed control architecture, offering a clear overview of the overall control structure.

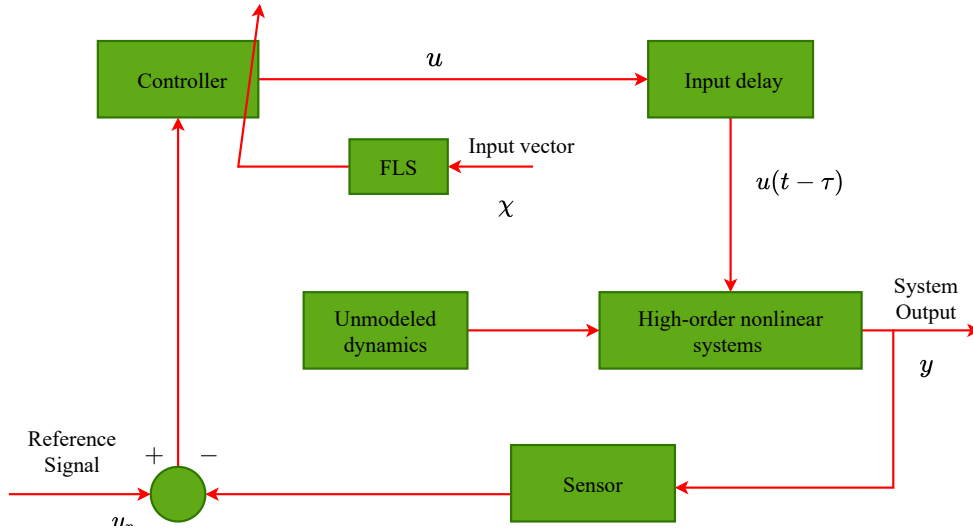


Figure 1. Block diagram depicting the structure of the developed control architecture.

4. Simulation Results

This section provides two illustrative examples to verify the effectiveness of the proposed control approach and to highlight its key characteristics.

Example 1: Consider the high-order nonlinear system as

$$\begin{cases} \dot{\xi} = -2\xi + 0.25x_1^2, \\ \dot{x}_1 = x_2^{\sigma_1} + x_2 \cos(x_1^2) + \frac{\sin(x_1 x_2)}{\sin^2(x_1) + 1} + \Delta_1, \\ \dot{x}_2 = u^{\sigma_2}(t - \tau(t)) + x_1^2 + x_1 x_2^2 + \Delta_2, \\ y = x_1, \end{cases} \quad (83)$$

where $f_1(x) = x_2 \cos(x_1^2) + \frac{\sin(x_1 x_2)}{\sin^2(x_1) + 1}$, $f_2(x) = x_1^2 + x_1 x_2^2$, $\Delta_1 = 0.5\xi^2 x_1 x_2$, $\Delta_2 = \xi^2 \sin(x_1 x_2)$, $\sigma_1 = 1$, $\sigma_2 = 2$ and consequently $\sigma = \max\{\sigma_1, \sigma_2\} = 3$. Additionally, the input delay is modeled by $\tau = 0.02$. The reference trajectory is selected as $y_d = \sin(0.5t)$. The initial conditions are chosen as $x_i(0) = 0.5$, $\hat{\Theta}_i(0) = 0$, and $r(0) = \xi(0)$, for $i = 1, 2$. The design constants are set to $T_d = 2$, $\varsigma = 4/99$, $\tau_i = 0.5$, and $\delta_0 = 2$. The fuzzy logic systems employ Gaussian-type basis functions given by $\Phi_1 = \exp\left[-\frac{(x_1 - i)^2}{2}\right]$ and $\Phi_2 = \exp\left[-\frac{(x_1 - i)^2}{2} - \frac{(x_2 - j)^2}{2}\right]$, where the indices satisfy $i, j, k \in \{-5, -4, \dots, 4, 5\}$. Define the nonlinear functions as $\Psi_{1,1} = \frac{1}{4}x_1^2 \sin^2(x_1)$, $\Psi_{2,1} = \frac{1}{4}x_1^2 x_2^2$, and $\Psi_{1,2} = \Psi_{2,2} = \xi^2$. Under these selections, Assumption 1 is satisfied. To fulfill Assumption 2, choose the Lyapunov candidate $V(\xi) = \xi^2$ together with $\underline{\mathcal{L}}(\xi) = \frac{1}{2}\xi^2$ and $\bar{\mathcal{L}}(\xi) = \frac{3}{2}\xi^2$. In accordance with Lemma 2, select $\underline{g}_0 = \left(\frac{3}{\delta_0}\right)^{\varsigma/2} \frac{\pi}{\varsigma T_d}$, $\bar{g}_0 = \frac{j_0^{\frac{\varsigma}{2}} \pi}{\varsigma T_d}$, $\lambda(x_1^2) = \frac{5}{4}x_1^2$, $\bar{\omega} = \frac{14}{25}$. Consequently, the auxiliary dynamic signal r evolves according to $\dot{r} = -\underline{g}_0 r^{1+\varsigma/2} - \bar{g}_0 r^{1-\varsigma/2} + \frac{5}{4}x_1^4 + \frac{1}{25}$.

The first-step virtual controller α_1 is synthesized following the design in (36), while the real control signal u is generated in accordance with (69). The parameter estimation dynamics $\dot{\hat{\Theta}}_1$ and $\dot{\hat{\Theta}}_2$ are specified by the update laws given in (55).

The performance of the proposed adaptive control scheme is evaluated through the simulation results shown in Figures 2–7. Figure 2 compares the system output y with the desired reference signal y_r , from which it can be observed that the output accurately follows the reference trajectory, demonstrating satisfactory tracking behavior. The corresponding tracking error z_1 is shown in Figure 3, where it is seen to gradually decrease and remain confined within a small region near the origin, indicating successful achievement of the tracking objective. Figure 4 illustrates the time response of the state variable x_2 .

The results indicate that x_2 stays within acceptable bounds throughout the simulation interval, which supports the stability of the closed loop system. The evolution of the estimated parameters $\hat{\Theta}_1$ and $\hat{\Theta}_2$ is presented in Figure 5. Their bounded and well behaved trajectories confirm the capability of the adaptive laws to compensate for parametric uncertainties. The control input u together with the delayed signal $u(t - \tau)$ is shown in Figure 6. Both signals vary smoothly over time, demonstrating that the input delay does not induce oscillations or destabilizing effects. Finally, Figure 7 depicts the unmodeled dynamics ζ and r , which remain bounded during the entire simulation. Overall, these simulation outcomes verify that the proposed control framework ensures boundedness of all system signals while maintaining reliable and accurate tracking performance.

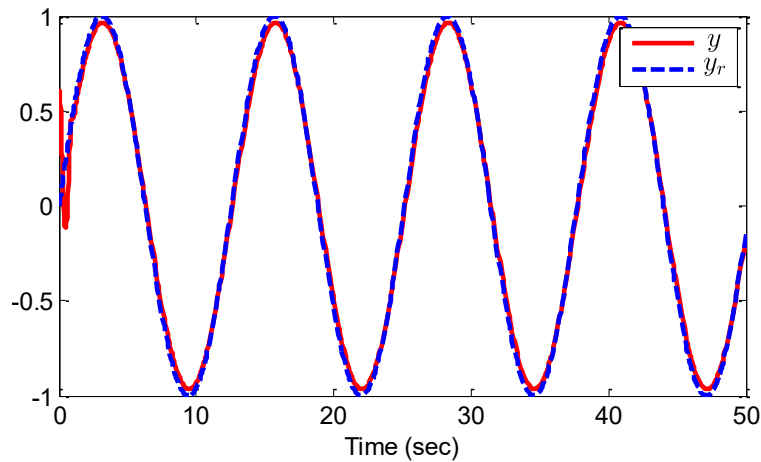


Figure 2. Trajectories of y and y_r .

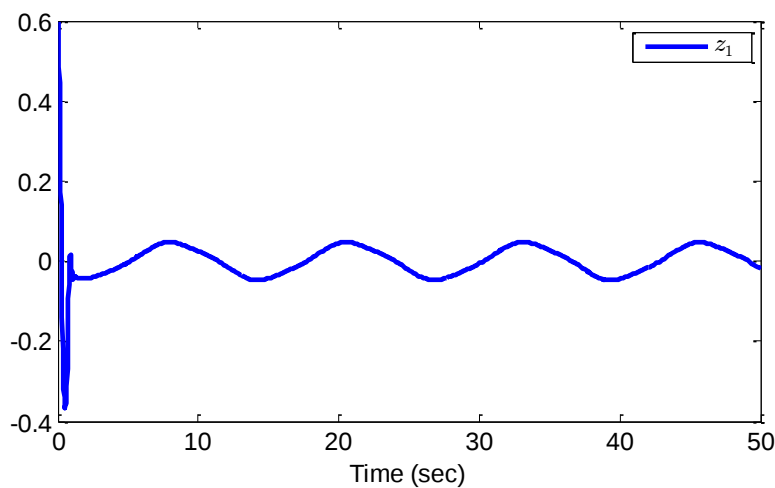


Figure 3. The trajectory of the tracking error z_1 .

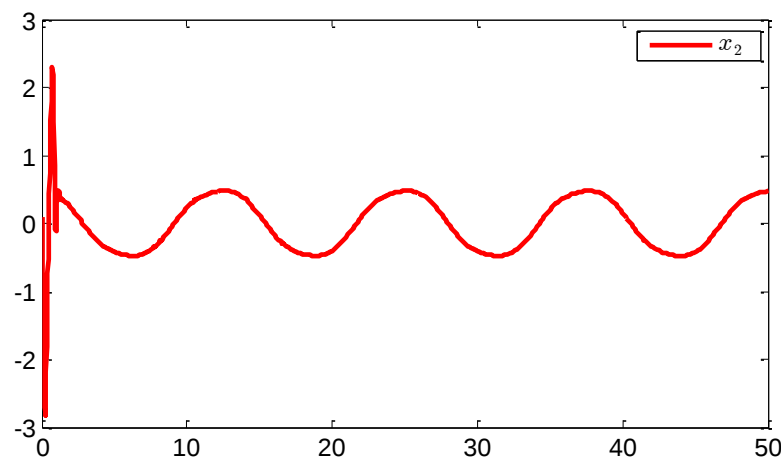


Figure 4. State variables x_2 .

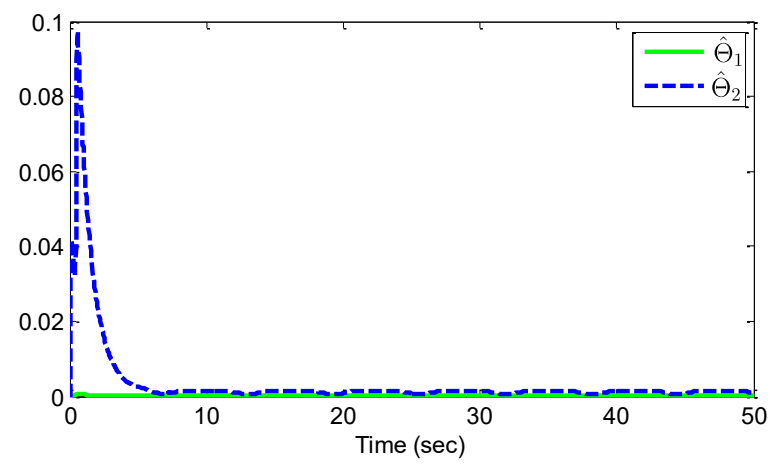


Figure 5. Adaptive laws $\hat{\theta}_1$ and $\hat{\theta}_2$.

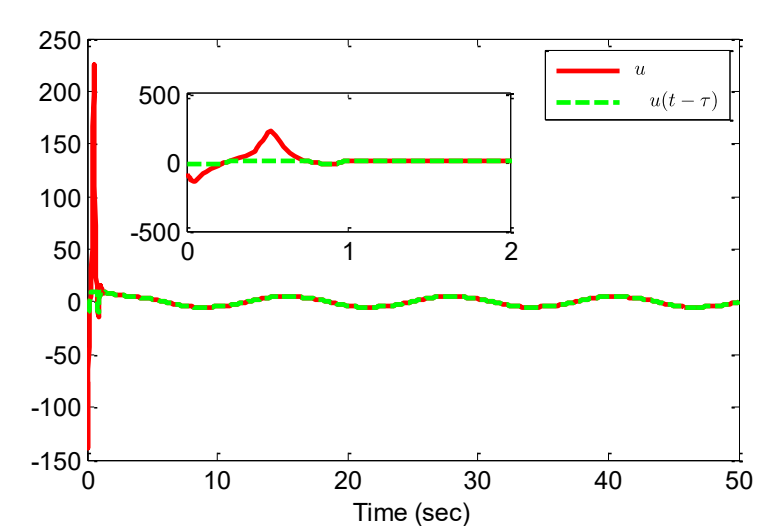


Figure 6. System input $u(t - \tau)$ and control input u .

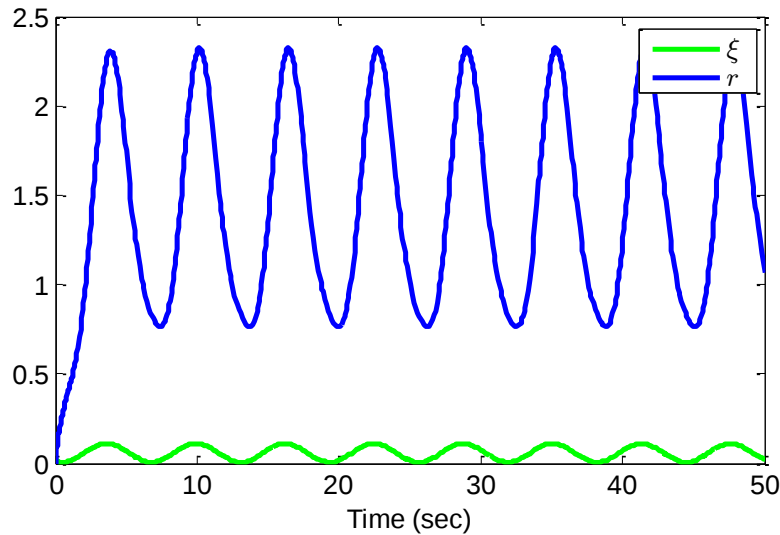


Figure 7. Trajectories of ξ and r .

Example 2: Consider a practical application involving a one-link robotic manipulator with motor dynamics, as investigated in [51]. The dynamic equations of the system are given by

$$\begin{cases} D\ddot{q} + B\dot{q} + N\sin(q) = I + I_d, \\ M\dot{I} + HI = -K_m\dot{q} + V, \end{cases} \quad (84)$$

where q , $\dot{q}$, and $\ddot{q}$ denote the link position, velocity, and acceleration, respectively. The variable I represents the generated motor torque, $I_d = \sin(\dot{q})\cos(I)$ denotes the disturbance torque, and V is the electromechanical control input. The system parameters are chosen as $D = 1 \text{ kg} \cdot \text{m}^2$, $B = 1 \text{ N} \cdot \text{m} \cdot \text{s}/\text{rad}$, $N = 10 \text{ N} \cdot \text{m}$, $M = 0.3 \text{ H}$, $H = 1.0 \Omega$, and $K_m = 2 \text{ N} \cdot \text{m}/\text{A}$. It is assumed that the system is subject to input delay and unmodeled dynamics. Define the state variables as $x_1 = q$, $x_2 = \dot{q}$, $x_3 = I/D$, and the delayed control input as $u(t - \tau) = V(t - \tau(t))/(DM)$.

Then, the system (84) can be transformed as

$$\begin{cases} \dot{\xi} = -2\xi + 0.25x_1^2, \\ \dot{x}_1 = x_2^{\sigma_1} + \Delta_1, \\ \dot{x}_2 = x_3^{\sigma_2} - \frac{N}{D}\sin(x_1) - \frac{B}{D}x_2 + \frac{1}{D}\sin(x_2)\cos(Dx_3) + \Delta_2, \\ \dot{x}_3 = -\frac{K_m}{MD}x_2 - \frac{H}{MD}x_3 + u^{\sigma_2}(t - \tau(t)) + \Delta_3, \\ y = x_1, \end{cases} \quad (85)$$

where $f_1(x) = 0$, $f_2(x) = -\frac{N}{D}\sin(x_1) - \frac{B}{D}x_2 + \frac{1}{D}\sin(x_2)\cos(Dx_3)$, $f_3(x) = -\frac{K_m}{MD}x_2 - \frac{H}{MD}x_3$, $\Delta_1 = \xi x_1 x_2 x_3$, $\Delta_2 = \xi x_2 \cos(x_1 x_2)$, $\Delta_3 = \xi x_3 \sin(x_1 x_2)$, $\sigma_1 = 1$, $\sigma_2 = 1$, $\sigma_3 = 3$, and consequently $\sigma = \max\{\sigma_1, \sigma_2, \sigma_3\} = 3$. Additionally, the input delay is modeled by $\tau = 0.02$.

The reference trajectory is selected as $y_r = 0.5(\sin(t))$. The initial states and parameter estimates are chosen as $x_i(0) = 0.5$, $\hat{\Theta}_i(0) = 0$, and $r(0) = \xi(0) = 0$, for $i = 1, 2, 3$. The design constants are set to $T_d = 2$, $\varsigma = 4/99$, $\tau_i = 0.5$, and $\delta_0 = 2$.

The fuzzy logic systems employ Gaussian-type basis functions given by $\Phi_1 = \exp\left[-\frac{(x_1-i)^2}{2}\right]$, $\Phi_2 = \exp\left[-\frac{(x_1-i)^2}{2} - \frac{(x_2-j)^2}{2}\right]$, $\Phi_3 = \exp\left[-\frac{(x_1-i)^2}{2} - \frac{(x_2-j)^2}{2} - \frac{(x_3-k)^2}{2}\right]$, where the indices satisfy $i, j, k \in \{-5, -4, \dots, 4, 5\}$.

Define the nonlinear functions as $\Psi_{1,1} = \frac{1}{4}x_1^2 \sin^2(x_1)$, $\Psi_{2,1} = \frac{1}{4}x_1^2 x_2^2$, $\Psi_{3,1} = \frac{1}{4}x_3^2 \cos^2(x_1 x_2)$, and $\Psi_{1,2} = \Psi_{2,2} = \Psi_{3,2} = \xi^2$. Under these selections, Assumption 1 is satisfied. To fulfill Assumption 2,

choose the Lyapunov candidate $V(\xi) = \xi^2$ together with $\underline{\mathcal{L}}(\xi) = \frac{1}{2}\xi^2$ and $\bar{\mathcal{L}}(\xi) = \frac{3}{2}\xi^2$. In accordance with Lemma 2, select $\underline{g}_0 = \left(\frac{3}{\delta_0}\right)^{\zeta/2} \frac{\pi}{\zeta T_d}$, $\bar{g}_0 = \frac{j_0^{\frac{\zeta}{2}} \pi}{\zeta T_d}$, $\lambda(x_1^2) = \frac{5}{4}x_1^2$, $\bar{\omega} = \frac{14}{25}$. Consequently, the auxiliary dynamic signal r evolves according to $\dot{r} = -\underline{g}_0 r^{1+\zeta/2} - \bar{g}_0 r^{1-\zeta/2} + \frac{5}{4}x_1^4 + \frac{1}{25}$. The first-step virtual controllers α_1 and α_2 are synthesized following the design in (54), while the real control signal u is generated in accordance with (69). The parameter estimation dynamics $\hat{\Theta}_1$, $\hat{\Theta}_2$ and $\hat{\Theta}_3$ are specified by the update laws given in (55). The performance of the proposed adaptive control method is evaluated through the simulation results depicted in Figures 8–13. Figure 8 presents the time responses of the system output y along with the desired reference signal y_r , showing close agreement and indicating that the controller achieves precise output tracking. The associated tracking error z_1 is shown in Figure 9, where it gradually decreases and remains within a small neighborhood of the origin, confirming the successful achievement of the tracking objective. The temporal behaviors of the state variables x_2 and x_3 are illustrated in Figure 10, demonstrating that both states remain bounded over the entire simulation period, which validates the stability of the closed-loop system under the proposed controller. Figure 11 depicts the evolution of the estimated adaptive parameters $\hat{\Theta}_1$, $\hat{\Theta}_2$, and $\hat{\Theta}_3$, whose bounded and smooth trajectories confirm the effectiveness of the adaptive laws in handling parametric uncertainties. Figure 12 compares the control input u with the delayed input $u(t - \tau)$, and the smooth variations of these signals indicate that input delays do not introduce instability or undesirable oscillations. Finally, Figure 13 shows the unmodeled dynamics ξ and r , which remain bounded throughout the simulation, further illustrating the robustness of the proposed approach. Collectively, these simulation results demonstrate that the developed control strategy ensures boundedness of all system signals while achieving accurate and reliable tracking performance.

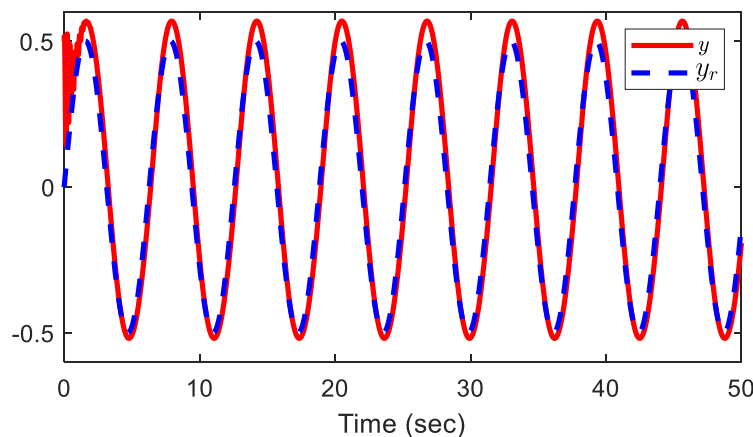


Figure 8. Trajectories of y and y_r .

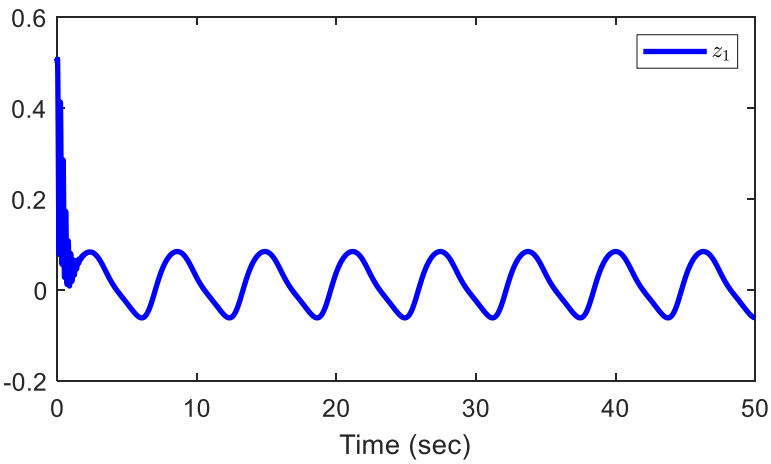


Figure 9. The trajectory of the tracking error z_1 .

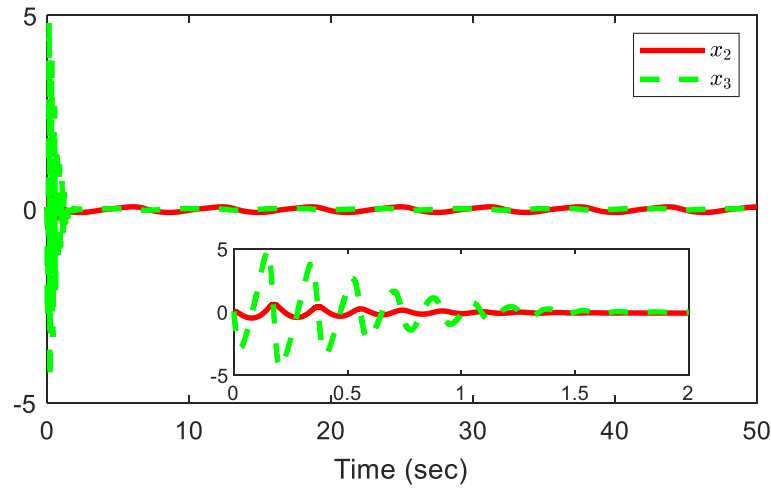


Figure 10. State variables x_2 and x_3 .

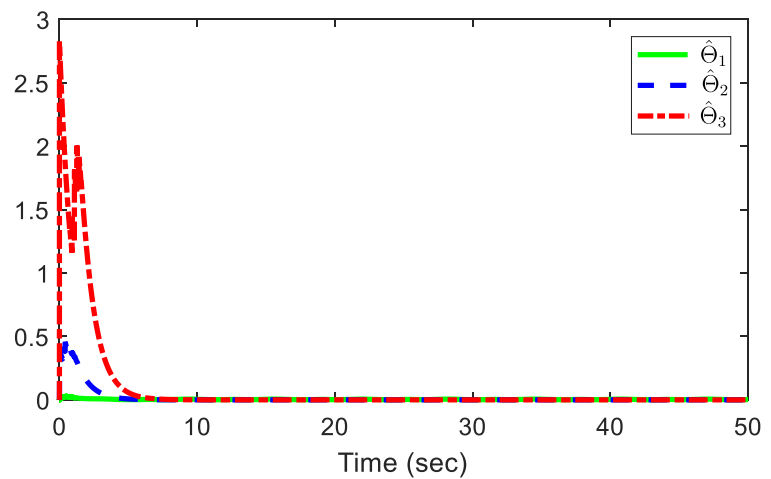


Figure 11. Adaptive laws $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$.

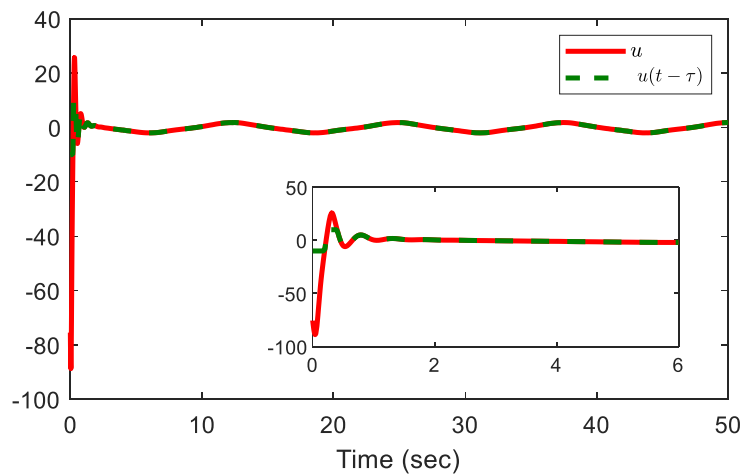


Figure 12. Control input u and system input $u(t - \tau)$.

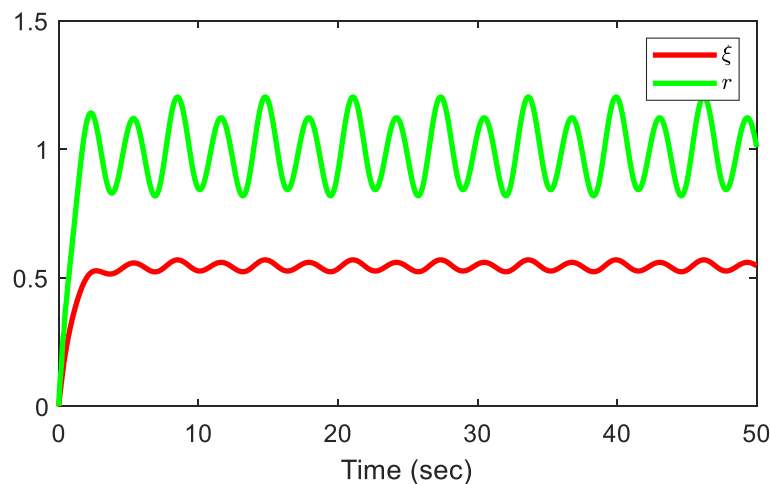


Figure 13. Trajectories of ξ and r .

5. Conclusions

This study has investigated the design of a predefined-time adaptive fuzzy control scheme for high-order nonlinear systems with nonstrict-feedback structures, considering the presence of unmodeled dynamics and input delays. To compensate for the effects of unmodeled dynamics, a predefined-time auxiliary dynamic signal was incorporated, while the impact of input delays was alleviated using a Padé approximation in conjunction with an intermediate variable. Fuzzy logic systems were utilized to approximate the unknown nonlinear functions, thereby reducing reliance on an accurate mathematical model of the system. By integrating the recursive backstepping technique with a power-type Lyapunov function framework, an adaptive fuzzy predefined-time tracking controller was developed. A comprehensive analytical study establishes that the designed feedback dynamics ensure practical predefined-time convergence of the closed-loop system, while suitable adjustment of the controller gains confines the tracking deviations within a small vicinity of the equilibrium point. The effectiveness and advantages of the proposed control strategy were further validated through simulation examples. Future research will focus on extending the proposed predefined-time adaptive fuzzy control framework to more complex system classes, such as switched stochastic nonlinear systems. In such systems, random disturbances, stochastic uncertainties, and switching behaviors coexist, posing significant challenges to stability analysis and controller design. Incorporating predefined-time performance guarantees under stochastic effects and switching signals,

while accounting for unmodeled dynamics and input delays, remains an open and meaningful research direction. Additionally, event-triggered mechanisms and network-induced constraints will be considered to improve communication efficiency and practical implementability in large-scale and distributed control systems.

Author Contributions: Mohamed Kharrat : Writing – original draft, Supervision, Paolo Mercorelli: Writing – review & editing. All authors have read and agreed to the published version of the manuscript.

Data Availability Statement: No data were used for the research described in the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Zhang, Y., Ning, X., Wang, Z., & Yu, D. (2021). High-order disturbance observer-based neural adaptive control for space unmanned systems with stochastic and high-dynamic uncertainties. *IEEE Access*, 9, 77028-77043.
2. Kharrat, M., & Mercorelli, P. (2025). Neural Network-Based Adaptive Finite-Time Control for Pure-Feedback Stochastic Nonlinear Systems with Full State Constraints, Actuator Faults, and Backlash-like Hysteresis. *Mathematics*, 14(1), 30.
3. Cui, Y., Duan, G., Liu, X., & Zheng, H. (2023). Adaptive fuzzy fault-tolerant control of high-order nonlinear systems: A fully actuated system approach. *International Journal of Fuzzy Systems*, 25(5), 1895-1906.
4. Sui, S., Chen, C. P., & Tong, S. (2021). Finite-time adaptive fuzzy prescribed performance control for high-order stochastic nonlinear systems. *IEEE Transactions on Fuzzy Systems*, 30(7), 2227-2240.
5. Deng, C., Wen, C., Wang, W., Li, X., & Yue, D. (2022). Distributed adaptive tracking control for high-order nonlinear multiagent systems over event-triggered communication. *IEEE Transactions on Automatic Control*, 68(2), 1176-1183.
6. Duan, G. (2021). High-order fully actuated system approaches: Part IV. Adaptive control and high-order backstepping. *International Journal of Systems Science*, 52(5), 972-989.
7. Hou, M., Shi, W., Fang, L., & Duan, G. (2023). Adaptive dynamic surface control of high-order strict feedback nonlinear systems with parameter estimations. *Science China Information Sciences*, 66(5), 159203.
8. Liu, S., Niu, B., Zong, G., Zhao, X., & Xu, N. (2023). Adaptive neural dynamic-memory event-triggered control of high-order random nonlinear systems with deferred output constraints. *IEEE Transactions on Automation Science and Engineering*, 21(3), 2779-2791.
9. Niu, Y., Yang, Y., Wang, H., Niu, B., & Shang, Z. (2024). Adaptive tracking control of high-order nonlinear systems with time-varying delays under asymmetric output constraints. *IEEE Transactions on Automation Science and Engineering*, 22, 2020-2030.
10. Kharrat, M., & Paolo, M. (2025). Predefined-Time Adaptive Command Filter Control for Nonstrict-Feedback Nonlinear Systems with Input Delay and Unmodeled Dynamics. *Mathematics*, 14(1), 14.
11. Shi, X., Xu, S., Jia, X., Chu, Y., & Zhang, Z. (2022). Adaptive neural control of state-constrained MIMO nonlinear systems with unmodeled dynamics. *Nonlinear Dynamics*, 108(4), 4005-4020.
12. Wang, H., Liu, P. X., Li, S., & Wang, D. (2017). Adaptive neural output-feedback control for a class of nonlower triangular nonlinear systems with unmodeled dynamics. *IEEE transactions on neural networks and learning systems*, 29(8), 3658-3668.
13. Jiang, Z. P., & Praly, L. (1998). Design of robust adaptive controllers for nonlinear systems with dynamic uncertainties. *Automatica*, 34(7), 825-840.
14. Sui, S., Chen, C. P., & Tong, S. (2020). Event-trigger-based finite-time fuzzy adaptive control for stochastic nonlinear system with unmodeled dynamics. *IEEE Transactions on Fuzzy Systems*, 29(7), 1914-1926.
15. Lyu, Z., Liu, Z., Xie, K., Chen, C. P., & Zhang, Y. (2019). Adaptive fuzzy output-feedback control for switched nonlinear systems with stable and unstable unmodeled dynamics. *IEEE Transactions on Fuzzy Systems*, 28(8), 1825-1839.
16. Shen, Q., Shi, P., Wang, S., & Shi, Y. (2019). Fuzzy adaptive control of a class of nonlinear systems with unmodeled dynamics. *International Journal of Adaptive Control and Signal Processing*, 33(4), 712-730.
17. Tong, S., & Li, Y. (2016). Adaptive fuzzy output feedback control for switched nonlinear systems with unmodeled dynamics. *IEEE transactions on cybernetics*, 47(2), 295-305.

18. Zhang, L., Wang, P., & Hua, C. (2023). Adaptive control of time-delay nonlinear HOFA systems with unmodeled dynamics and unknown dead-zone input. *International Journal of Robust and Nonlinear Control*, 33(4), 2615-2628.
19. Bi, W. (2023). Neural networks adaptive control for fractional-order non-linear system with unmodelled dynamics and actuator faults. *IET Control Theory & Applications*, 17(3), 259-269.
20. Sui, S., Chen, C. P., & Tong, S. (2020). Event-trigger-based finite-time fuzzy adaptive control for stochastic nonlinear system with unmodeled dynamics. *IEEE Transactions on Fuzzy Systems*, 29(7), 1914-1926.
21. Hua, Y., & Zhang, T. (2020). Adaptive control of pure-feedback nonlinear systems with full-state time-varying constraints and unmodeled dynamics. *International Journal of Adaptive Control and Signal Processing*, 34(2), 183-198.
22. Li, D. P., Liu, Y. J., Tong, S., Chen, C. P., & Li, D. J. (2018). Neural networks-based adaptive control for nonlinear state constrained systems with input delay. *IEEE transactions on cybernetics*, 49(4), 1249-1258.
23. Deng, W., Yao, J., & Ma, D. (2017). Adaptive control of input delayed uncertain nonlinear systems with time-varying output constraints. *IEEE Access*, 5, 15271-15282.
24. Zhao, J., Tong, S., & Li, Y. (2021). Observer-based fuzzy adaptive control for MIMO nonlinear systems with non-constant control gain and input delay. *IET Control Theory & Applications*, 15(11), 1488-1505.
25. Wang, Y., Zhang, J., Zhang, H., & Xie, X. (2020). Adaptive fuzzy output-constrained control for nonlinear stochastic systems with input delay and unknown control coefficients. *IEEE Transactions on Cybernetics*, 51(11), 5279-5290.
26. Han, Y. Q. (2021). Design of decentralized adaptive control approach for large-scale nonlinear systems subjected to input delays under prescribed performance. *Nonlinear Dynamics*, 106(1), 565-582.
27. Li, Z., Li, T., Feng, G., Zhao, R., & Shan, Q. (2018). Neural network-based adaptive control for pure-feedback stochastic nonlinear systems with time-varying delays and dead-zone input. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 50(12), 5317-5329.
28. Qi, X., Liu, W., & Lu, J. (2022). Observer-based finite-time adaptive prescribed performance control for nonlinear systems with input delay. *International Journal of Control, Automation and Systems*, 20(5), 1428-1438.
29. Han, Y. Q., He, W. J., Li, N., & Zhu, S. L. (2021). Adaptive tracking control of a class of nonlinear systems with input delay and dynamic uncertainties using multi-dimensional Taylor network. *International Journal of Control, Automation and Systems*, 19(12), 4078-4089.
30. Zhu, B., Karimi, H. R., Zhang, L., & Zhao, X. (2025). Neural network-based adaptive reinforcement learning for optimized backstepping tracking control of nonlinear systems with input delay. *Applied Intelligence*, 55(2), 129.
31. Zheng, X. Y. (2024). Adaptive neural control for non-strict feedback stochastic nonlinear systems with input delay. *Transactions of the Institute of Measurement and Control*, 46(1), 104-115.
32. Kharrat, M. (2025). Finite-Time Fuzzy Adaptive Control for Nonlinear Systems With Asymmetric Dead-Zone and Actuator Faults and via an Event-Triggered Mechanism. *International Journal of Adaptive Control and Signal Processing*.
33. Kharrat, M. (2025). Neural Network-Based Adaptive Finite-Time Command-Filter Control for Nonlinear Systems With Input Delay and Input Saturation. *International Journal of Adaptive Control and Signal Processing*, 39(1), 231-243.
34. Kharrat, M. (2025). Fixed-Time Adaptive Tracking Control for Nonlinear Systems With Unmodeled Dynamics and Input Dead-Zone and Saturation. *International Journal of Robust and Nonlinear Control*.
35. Kharrat, M. (2025). Adaptive fixed-time command-filtered funnel control for nonstrict-feedback nonlinear systems with input saturation. *Transactions of the Institute of Measurement and Control*, 01423312251361970.
36. Kharrat, M., & Alhazmi, H. (2025). Fixed-time adaptive control for nonstrict-feedback nonlinear systems with sensor faults and input saturation. *Journal of Low Frequency Noise, Vibration and Active Control*, 44(1), 565-587.
37. Kharrat, M., & Alhazmi, H. (2025). Fixed-Time Adaptive Control for Nonstrict-Feedback Nonlinear Systems with Input Delay and Unknown Backlash-Like Hysteresis. *Neural Processing Letters*, 57(3), 52.
38. Kharrat, M. (2025). Predefined-time adaptive fuzzy control for nonlinear cyber-physical systems under sensor and actuator attacks. *Journal of Vibration and Control*, 10775463251374117.
39. Zhao, L., Sui, S., & Chen, C. P. (2025). Adaptive Predefined-Time Control for High-Order Nonlinear Systems With Unmodeled Dynamics. *International Journal of Adaptive Control and Signal Processing*.

40. Sui, S., Chen, C. P., & Tong, S. (2024). Command filter-based predefined time adaptive control for nonlinear systems. *IEEE Transactions on Automatic Control*, 69(11), 7863-7870.
41. Wang, H., Tong, M., Zhao, X., Niu, B., & Yang, M. (2022). Predefined-time adaptive neural tracking control of switched nonlinear systems. *IEEE Transactions on Cybernetics*, 53(10), 6538-6548.
42. Zhang, Y., Chadli, M., & Xiang, Z. (2022). Predefined-time adaptive fuzzy control for a class of nonlinear systems with output hysteresis. *IEEE Transactions on Fuzzy Systems*, 31(8), 2522-2531.
43. Xu, H., Yu, D., & Liu, Y. J. (2023). Observer-based fuzzy adaptive predefined time control for uncertain nonlinear systems with full-state error constraints. *IEEE Transactions on Fuzzy Systems*, 32(3), 1370-1382.
44. Chen, P., Tan, J., Yao, Y., Zhang, X., & Yao, Y. (2025). Predefined-time adaptive fuzzy control for nonlinear systems with input saturation and deferred restriction. *Nonlinear Dynamics*, 113(5), 4727-4744.
45. Yang, Y., Sui, S., & Chen, C. P. (2025). Adaptive predefined time control for stochastic switched nonlinear systems with full-state error constraints and input quantization. *IEEE Transactions on Cybernetics*.
46. Wang, L., Xin, M., & Niu, J. (2025). Adaptive fuzzy predefined-time control for stochastic nonlinear systems against actuator and sensor faults. *IEEE Transactions on Instrumentation and Measurement*.
47. Sui, S., Zhao, L., & Chen, C. P. (2024). Adaptive fuzzy predefined-time tracking control design for nonstrict-feedback high-order nonlinear systems with input quantization. *IEEE Transactions on Fuzzy Systems*, 32(10), 5978-5990.
48. Ling, S., Wang, H., & Liu, P. X. (2020). Adaptive tracking control of high-order nonlinear systems under asymmetric output constraint. *Automatica*, 122, 109281.
49. Cui, D., & Xiang, Z. (2022). Nonsingular fixed-time fault-tolerant fuzzy control for switched uncertain nonlinear systems. *IEEE Transactions on Fuzzy Systems*, 31(1), 174-183.
50. Wu, Y., Xie, X. J., & Hou, Z. G. (2020). Adaptive fuzzy asymptotic tracking control of state-constrained high-order nonlinear time-delay systems and its applications. *IEEE Transactions on Cybernetics*, 52(3), 1671-1680.
51. Zhai, J., Wang, H., Tao, J., & He, Z. (2023). Observer-based adaptive fuzzy finite time control for non-strict feedback nonlinear systems with unmodeled dynamics and input delay. *Nonlinear Dynamics*, 111(2), 1417-1440.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.