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The Collatz Conjecture “The Non-Existence of Divergent Orbits Through an Analysis of the Codification of Linear Diophantine Dynamical Systems”

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The Collatz Conjecture "The Non-Existence of Divergent Orbits Through an Analysis of the Codification of Linear Diophantine Dynamical Systems."

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Abstract: We define the function $Col : \mathbb{N} \to \mathbb{N}$ as the Collatz function, given by $3n+1$ if n is odd and $\frac{n}{2}$ if n is even. The conjecture postulates that for any positive integer, at some point, its iteration will reach 1, or equivalently, every orbit will fall into the periodic cycle $\{4,2,1\}$. Two conditions would invalidate the conjecture: The existence of a divergent orbit or the presence of another cycle. We can study the dynamics of the orbits through the density of even terms in their orbit. If all points' accumulation density exceeds the value of $\frac{\ln(3)}{\ln(2)}$ then the orbit is bounded. The main result of this work is to show that there are no natural numbers such that the accumulation points of the pair density are less than $\frac{\ln(3)}{\ln(2)}$. In other words, there are no divergent orbits.

Keywords: collatz conjecture; dynamical system

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1. Background

A **metric space** is a set X equipped with a function $d : X \times X \to \mathbb{R}$, called a *metric*, that satisfies the following properties for all $x, y, z \in X$:

- **Non-negativity:** $d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$.
- **Symmetry:** $d(x,y) = d(y,x)$.
- **Triangle inequality:** $d(x,z) \leq d(x,y) + d(y,z)$.

The function $d(x,y)$ measures the "distance" between any two points x and y in the set X. The pair (X,d) is called a metric space.

Examples of Metrics:

- **Euclidean Metric (on $\mathbb{R}^n$):**
$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$
This metric defines the usual distance between two points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in Euclidean space $\mathbb{R}^n$.
- **Discrete Metric:**
$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$
In this metric, the distance between two distinct points is always 1, and the distance from any point to itself is 0.
- **Taxicab Metric (or Manhattan Metric, on $\mathbb{R}^n$):**
$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$
This metric measures the distance between two points x and y as the sum of the absolute differences of their coordinates. It corresponds to the distance a taxi would drive on a grid of city streets.
- **p-adic Metric (on $\mathbb{Q}$):**
$$d_p(x,y) = p^{-ord_p(x-y)}$$
Here, p is a fixed prime number, and $ord_p(x-y)$ denotes the p-adic valuation of $x - y$, which
$$ord_p(x) = \begin{cases} \text{the highest power } p \text{ which divides } x & \text{if} \quad x \in \mathbb{Z} \\ ord_p(a) - ord_p(b) & \text{if} \quad x = \frac{a}{b} \in \mathbb{Q} \end{cases}$$
This metric measures the distance between two rational numbers based on their divisibility by p. The p-adic metric induces a non-Archimedean topology, meaning that the "triangle inequality" is strengthened to $d_p(x,z) \leq \max\{d_p(x,y), d_p(y,z)\}$.

A **complete metric space** is a metric space in which every *Cauchy sequence* converges to a point within the space. Formally, a metric space (X,d) is called **complete** if, for every sequence $\{x_n\} \subset X$ that is Cauchy (i.e., for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $d(x_n, x_m) < \varepsilon$), there exists a point $x \in X$ such that:

$$\lim_{n \to \infty} x_n = x.$$

In other words, all Cauchy sequences in X must have a limit in X.
Examples:

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- **The Real Numbers $\mathbb{R}$ with the Euclidean Metric:** The set of real numbers $\mathbb{R}$ with the usual Euclidean metric $d(x,y) = |x-y|$ is a complete metric space. This is because every Cauchy sequence of real numbers converges to a real number.
- **The Rational Numbers $\mathbb{Q}$ with the Euclidean Metric:** The set of rational numbers $\mathbb{Q}$ with the Euclidean metric is not complete. For example, the sequence defined by $x_0 = 1$ and $x_{n+1} = \dfrac{x_n + \frac{2}{x_n}}{2}$ that approximate $\sqrt{2}$ is Cauchy in $\mathbb{Q}$ but does not converge to a rational number (since $\sqrt{2} \notin \mathbb{Q}$).
- **The p-adic Numbers $\mathbb{Q}_p$:** The set of p-adic numbers $\mathbb{Q}_p$, equipped with the p-adic metric $d_p(x,y) = \|x-y\|_2 = p^{-ord_p(x-y)}$, is a complete metric space. Every Cauchy sequence in $\mathbb{Q}_p$ converges to a p-adic number within $\mathbb{Q}_p$.

Limit Superior (lim sup) and Limit Inferior (lim inf) of a Sequence:

Given a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers and $X_n = \{x_n, x_{n+1}, \ldots\}$. Let us consider the following subsequences of $\{x_n\}_{n\in\mathbb{N}}$ given by

$$a_n = \inf X_n$$
$$b_n = \sup X_n$$

we have that the sequence $\{a_n\}$ is monotonically increasing and $\{b_n\}$ is monotonically decreasing, that is:

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots \leq b_n \leq \ldots \leq b_2 \leq b_1$$

Therefore there are limits

$$a = \lim_{n\to\infty} a_n = \sup_n a_n = \sup_n(\inf X_n)$$
$$b = \lim_{n\to\infty} b_n = \inf_n b_n = \inf_n(\sup X_n)$$

We will write $a = \liminf\limits_{n\to\infty} x_n$ and $b = \limsup\limits_{n\to\infty} x_n$ and we will call *Limit Inferior* and *Limit Superior* respectively.

Limit Superior (lim sup) and Limit Inferior (lim inf) of a Sequence of Sets: Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets in a space X. The limit superior and limit inferior of the sequence of sets are defined as follows:

1. **Set Sequence limit superior ($\limsup\limits_{n\to\infty} A_n$):**

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

2. **Set Sequence limit inferior ($\liminf\limits_{n\to\infty} A_n$):**

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m$$

3. **Limit of a Sequence of Sets** If $\limsup\limits_{n\to\infty} A_n = \liminf\limits_{n\to\infty} A_n$, then the sequence $\{A_n\}$ converges, and its limit is denoted as:

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$$

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Particular Case: Monotone Sequences of Sets Non-Decreasing Sequence ($A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$): If A_n is a non-decreasing sequence (i.e., $A_n \subseteq A_{n+1}$ for all n), then:

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

In this case, the limit of the sequence is simply the union of all the sets in the sequence. **Non-Increasing Sequence ($A_1 \supset A_2 \supset A_3 \supset \cdots$):** If A_n is a non-increasing sequence (i.e., $A_n \supset A_{n+1}$ for all n), then:

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

In this case, the limit of the sequence is simply the intersection of all the sets in the sequence.

A **discrete dynamical system** is a model of the evolution of a state over discrete time steps. Formally, it consists of a set X (called the *state space*) and a function $f : X \to X$ that describes how the state evolves from one time step to the next. The system is described by the equation:

$$x_{n+1} = f(x_n)$$

where $x_n \in X$ represents the state of the system at the n-th time step. The evolution of the system is typically studied by iterating the function, f starting from an initial state x_0. The sequence $\{x_n\}$, where $x_{n+1} = f(x_n)$, is called the *orbit* or *trajectory* of the initial state x_0.

Topologically Conjugate Dynamical Systems: Two discrete dynamical systems (X, f) and (Y, g) are said to be **topologically conjugate** if there exists a homeomorphism $h : X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

In other words, the systems (X, f) and (Y, g) are topologically conjugate if there is a bijective function $h : X \to Y$ such that:

- h is a **homeomorphism**, meaning h is continuous, bijective, and its inverse h^{-1} is also continuous.
- The following relation holds:
$$h \circ f = g \circ h.$$

This means that the dynamics of f on X and g on Y are the same up to a change of coordinates given by h. The systems (X, f) and (Y, g) have the same qualitative behavior, such as the structure of orbits and periodic points, despite potentially differing in their specific representations.

Properties of Topological Conjugation with Respect to Orbits and Periodic Points:

- **Preservation of Orbits:** If $f : X \to X$ and $g : Y \to Y$ are two topologically conjugate dynamical systems, with a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$, then h preserves the orbits of points. Specifically, for any point $x \in X$, the orbit of x under f is mapped to the orbit of $h(x)$ under g by h. Mathematically, this means:

$$h(f^n(x)) = g^n(h(x)) \text{ for all } n \geq 0.$$

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- **Preservation of Periodic Points:** If $x \in X$ is a periodic point of f with period p, then $h(x) \in Y$ is a periodic point of g with the same period p. Specifically, if $f^p(x) = x$, then:

$$g^p(h(x)) = h(f^p(x)) = h(x).$$

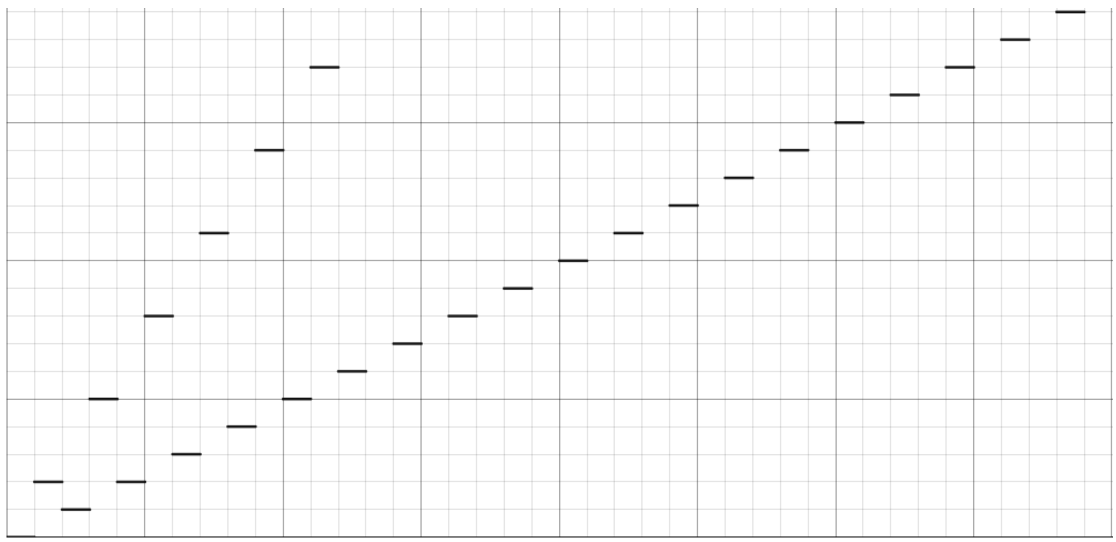
Conversely, if $y \in Y$ is a periodic point of g with period p, then $h^{-1}(y) \in X$ is a periodic point of f with the same period p.

2. Collatz's Conjecture

The Collatz conjecture, also known as the $3n + 1$ conjecture, is an unsolved problem in number theory proposed by the German mathematician Lothar Collatz in 1937. Despite its seemingly simple formulation, it has challenged mathematicians for decades due to the difficulty of proving its validity or finding a counterexample. The formal formulation of Collatz's Conjecture is as follows:

Let $Col : \mathbb{N} \to \mathbb{N}$, defined by:

$$Col(n) = \begin{cases} 3n+1 & \text{if } n \text{ odd} \\ \frac{n}{2} & \text{if } n \text{ even,} \end{cases}$$



then, for all $n \in \mathbb{N}$ exist $k \in \mathbb{N}$ such that:

$$Col^k(n) = 1.$$

An equivalent formulation of the conjecture argues that starting from any positive integer, one will eventually reach cycles $\{4, 2, 1\}$. Two cases that could invalidate the Conjecture are that there is a cycle different from $\{4, 2, 1\}$ or that there is some divergent orbit. To date no evidence has been found for any of these options, however no demonstration has been presented that completely rules out these cases. In 2019 Terense Tao [15] presented a demonstration that almost every orbit falls in the cycle $\{4, 2, 1\}$.

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Example 1.

$$
\begin{aligned}
27 &\to 82 \to 41 \to 124 \to 62 \to 31 \to 94 \to 47 \to 142 \to 71 \\
&\to 214 \to 107 \to 322 \to 161 \to 484 \to 242 \to 121 \to 364 \to 182 \\
&\to 91 \to 274 \to 137 \to 412 \to 206 \to 103 \to 310 \to 155 \to 466 \\
&\to 233 \to 700 \to 350 \to 175 \to 526 \to 263 \to 790 \to 395 \to 1186 \\
&\to 593 \to 1780 \to 890 \to 445 \to 1336 \to 668 \to 334 \to 167 \to 502 \\
&\to 251 \to 754 \to 377 \to 1132 \to 566 \to 283 \to 850 \to 425 \to 1276 \\
&\to 638 \to 319 \to 958 \to 479 \to 1438 \to 719 \to 2158 \to 1079 \to 3238 \\
&\to 1619 \to 4858 \to 2429 \to 7288 \to 3644 \to 1822 \to 911 \to 2734 \\
&\to 1367 \to 4102 \to 2051 \to 6154 \to 3077 \to 9232 \to 4616 \to 2308 \\
&\to 1154 \to 577 \to 1732 \to 866 \to 433 \to 1300 \to 650 \to 325 \to 976 \\
&\to 488 \to 244 \to 122 \to 61 \to 184 \to 92 \to 46 \to 23 \to 70 \\
&\to 35 \to 106 \to 53 \to 160 \to 80 \to 40 \to 20 \to 10 \to 5 \\
&\to 16 \to 8 \to 4 \to 2 \to 1
\end{aligned}
$$

Example 2.

$$
\begin{aligned}
871 &\to 2614 \to 1307 \to 3922 \to 1961 \to 5884 \to 2942 \to 1471 \to 4414 \to 2207 \\
&\to 6622 \to 3311 \to 9934 \to 4967 \to 14902 \to 7451 \to 22354 \to 11177 \to 33532 \\
&\to 16766 \to 8383 \to 25150 \to 12575 \to 37726 \to 18863 \to 56590 \to 28295 \to 84886 \\
&\to 42443 \to 127330 \to 63665 \to 190996 \to 95498 \to 47749 \to 143248 \to 71624 \to 35812 \\
&\to 17906 \to 8953 \to 26860 \to 13430 \to 6715 \to 20146 \to 10073 \to 30220 \to 15110 \\
&\to 7555 \to 22666 \to 11333 \to 34000 \to 17000 \to 8500 \to 4250 \to 2125 \to 6376 \\
&\to 3188 \to 1594 \to 797 \to 2392 \to 1196 \to 598 \to 299 \to 898 \to 449 \to 1348 \\
&\to 674 \to 337 \to 1012 \to 506 \to 253 \to 760 \to 380 \to 190 \to 95 \to 286 \\
&\to 143 \to 430 \to 215 \to 646 \to 323 \to 970 \to 485 \to 1456 \to 728 \to 364 \\
&\to 182 \to 91 \to 274 \to 137 \to 412 \to 206 \to 103 \to 310 \to 155 \to 466 \\
&\to 233 \to 700 \to 350 \to 175 \to 526 \to 263 \to 790 \to 395 \to 1186 \to 593 \\
&\to 1780 \to 890 \to 445 \to 1336 \to 668 \to 334 \to 167 \to 502 \to 251 \to 754 \\
&\to 377 \to 1132 \to 566 \to 283 \to 850 \to 425 \to 1276 \to 638 \to 319 \to 958 \\
&\to 479 \to 1438 \to 719 \to 2158 \to 1079 \to 3238 \to 1619 \to 4858 \to 2429 \to 7288 \\
&\to 3644 \to 1822 \to 911 \to 2734 \to 1367 \to 4102 \to 2051 \to 6154 \to 3077 \to 9232 \\
&\to 4616 \to 2308 \to 1154 \to 577 \to 1732 \to 866 \to 433 \to 1300 \to 650 \to 325 \\
&\to 976 \to 488 \to 244 \to 122 \to 61 \to 184 \to 92 \to 46 \to 23 \to 70 \\
&\to 35 \to 106 \to 53 \to 160 \to 80 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \\
&\to 4 \to 2 \to 1
\end{aligned}
$$

2.1. Main Idea of This Work

To each natural number, we can associate a unique binary sequence ξ that represents the parity of iterations through the Collatz function. The sequence is defined as 0 if the number is even and 10 if the number is odd. We define the density function $\frac{a_{k+1}}{k}$, where a_k is the number of 0's up to the k-th 1. These sequences can be classified into various sets depending on the behavior of the density function. However, we focus on three particular sets:

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- G_0: the set of sequences ξ such that $\liminf_{k\to\infty} \frac{a_{k+1}}{k} > \frac{\ln(3)}{\ln(2)}$,
- G_∞: the set of sequences ξ such that $\limsup_{k\to\infty} \frac{a_{k+1}}{k} < \frac{\ln(3)}{\ln(2)}$,
- G_1: the set of sequences ξ such that $\lim_{k\to\infty} \frac{a_{k+1}}{k} = \frac{\ln(3)}{\ln(2)}$.

We have the following results on these sets

1. There is no $n \in \mathbb{N}$ such that its encoding belongs to $G_\infty \cup G_1$.
2. If the coding of $n \in \mathbb{N}$ belongs to G_0, then its orbit is bounded.
3. If $n \in \mathbb{N}$ is a periodic point, then its coding belongs to G_0.

We will use these results to demonstrate the main result of this work **Divergent Orbits Theorem**:

Theorem 10: *There are no divergent orbits for the Collatz function on natural numbers.*

The idea of the proof is as follows: Suppose there exists an n whose orbit is divergent. Then necessarily, its encoding cannot belong to G_0. This means there must exist a subsequence of the density function such that its limit is less than $\frac{\ln(3)}{\ln(2)}$. Moreover, the density function of ξ cannot have accumulation points greater than $\frac{\ln(3)}{\ln(2)}$, because otherwise, there would exist a sub-orbit of n that is bounded. This is a consequence of the Corollary 7 where we obtain an upper bound for the sub-orbit of n. Since the set of possible values of the subsequence is finite, the subsequence must be periodic, causing the entire sequence to collapse into a periodic orbit. This would imply that the encoding belongs to G_0, which would mean the orbit is bounded a contradiction. Thus, all accumulation points of the density function must be less than $\frac{\ln(3)}{\ln(2)}$. In other words, $\limsup \frac{a_{k+1}}{k} \leq \frac{\ln(3)}{\ln(2)}$. Hence, the encoding of n belongs to $G_\infty \cup G_1$.

We now provide a brief demonstration of points 1, 2, and 3.

We use the following symbology to refer to an arbitrary composition of functions:

$$\bigodot_{i=1}^{n} f_i(x) = f_n \circ f_{n-1} \circ \ldots \circ f_2 \circ f_1.$$

Let $\theta, \psi : \mathbb{R} \to \mathbb{R}$ be real functions defined by

$$\theta(x) = \frac{x}{2} \text{ and } \psi(x) = \frac{3x+1}{2}.$$

Define the set $\langle \theta, \psi \rangle$ as:

$$S : \mathbb{R} \to \mathbb{R}; \quad S(x) = \bigodot_{i=1}^{k} s_i(x), \quad s_i(x) = \theta(x) \text{ or } \psi(x) \text{ for } i \geq 1,\ s_0 = \mathrm{id}.$$

We call k the length of S. This composition is unique for each element in $\langle \theta, \psi \rangle$ (Proposition 3). Let $S \in \langle \theta, \psi \rangle$ we define the integer set of S as

$$\mathbb{E}(S) = \{n \in \mathbb{Z}, S(n) \in \mathbb{Z}\}$$

This set is monotonically decreasing with respect to the composition of functions (Proposition 1). Let $S, H \in \langle \theta, \psi \rangle$ then

$$\mathbb{E}(H \circ S) \subset \mathbb{E}(S)$$

For any $S \in \langle \theta, \psi \rangle$ defines the minimum positive integer value of S

$$\rho_0(S) = \min\{\mathbb{E}(S) \cap \mathbb{N}\}$$

Let $\{S_k\}_{k \in \mathbb{N}} \subset \langle \theta, \psi \rangle$ be a sequence of functions given by $S_k(x) = \bigodot_{j=1}^{k} s_j(x)$. Since the entire set is a mononally decreasing set with respect to the composition of functions in $\langle \theta, \psi \rangle$. We have that the minimum positive integer value is a monotonically increasing function, so

$$\lim_{k \to \infty} \rho_0(S_k) = \infty \text{ or } \lim_{k \to \infty} \rho_0(S_k) = \rho_0(S_{k_0}) \text{ for any } k_0 \in \mathbb{N}$$

We will say that the sequence $S_k(x) = \bigodot_{j=1}^{k} s_j(x)$ is positively stable if $\lim_{k \to \infty} \rho_0(S_k) = \rho_0(S_{k_0})$ for any $k_0 \in \mathbb{N}$. Let $S \in \langle \theta, \psi \rangle$ with length k. We define the following application $Cod : \langle \theta, \psi \rangle \to \{0, 10\}^k$ defined by

$$Cod\left(\bigodot_{j=1}^{k} s_j(x) \right) = \{\xi_j\}_{j=1}^{k} \text{ with } s_j \in \{\theta, \psi\}$$

with

$$\xi_j = \begin{cases} 0 & \text{if} \quad s_j(x) \text{ is } \theta(x) \\ 10 & \text{if} \quad s_j(x) \text{ is } \psi(x) \end{cases}$$

***Theorem* 8:** *Let $\{S_k\}_{k=1}^{\infty} \subset \langle \theta, \psi \rangle$ with $S_k(x) = \bigodot_{j=1}^{k} s_j(x)$ such that $Cod\left(\bigodot_{j=1}^{\infty} s_j(x) \right) \in G_\infty$, then $S_k(x)$ is positively unstable.*

Now there is a relationship between the stability of the sequence $S_k(x)$ and the existence of numbers with a given coding

***Proposition* 11:** *Let $p \in \mathbb{Z}$ and $S(x) \in \langle \theta, \psi \rangle$ of length k, then*

$$p \in \mathbb{E}(S) \textit{ if and only if } Cod^k(p) = Cod(S)$$

Then suppose that there exists a natural number n whose coding is in G_∞ and let be $\{S_k\}_{k \in \mathbb{N}} \subset \langle \theta, \psi \rangle$ such that $Cod(S_k) = Cod^k(n)$, then by Proposition 11 we have $n \in \mathbb{E}(S_k)$ for all $k \in \mathbb{N}$, exist $k_0 \in \mathbb{N}$ such that $\rho_0(S_k) = n$ for all $k \geq k_0$, then $\{S_k\}_{k \in \mathbb{N}}$ is positively stable, which contradicts the Theorem 8. Therefore not exist $n \in \mathbb{N}$ such that $Cod(n) \in G_\infty$. When we extend the Collatz function on rational numbers Definition 7 we have to given an coding $\xi \in \Sigma_2^*$, if there exists a rational whose coding is ξ, then it is unique. On the other hand, by the Theorem 7, we have that if this rational exists, it must be negative. Therefore $n \in \mathbb{N}$ cannot exist such that its coding is in G_∞

Now we are going to show that there is no $n \in \mathbb{N}$ such that its coding is in G_1. To prove this we are going to extend the Collatz function on the dyadic integers $\mathbb{Z}_2$, initiated by Lagarias [9] (1985), defines (Definition 19) the following extension of the Collatz function on $\mathbb{Z}_2$: Let $Col : \mathbb{Z}_2 \to \mathbb{Z}_2$ be given by

$$Col(\beta) = \begin{cases} \dfrac{3\beta + 1}{2} & \text{if } \beta \mod 2 = 1, \\ \dfrac{\beta}{2} & \text{if } \beta \mod 2 = 0. \end{cases}$$

It has been shown (see Mathews and Watts[10](1984) and Müller[11](1991)) that the extended Col is surjective, not injective, infinitely many times differentiable, not analytic, measure-preserving with

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respect to the Haar measure, and strongly mixing. Similar results concerning iterates of Col may be found in Lagarias[9](1985), Müller[11](1991),[12](1994) and Terras[14](1976). Defining the shift map $\sigma_{-1} : \mathbb{Z}_2 \to \mathbb{Z}_2$ as

$$\sigma_{-1}(x) = \begin{cases} \dfrac{x-1}{2} & \text{if } x \equiv 1 \pmod 2, \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod 2, \end{cases}$$

Lagarias[9](1985) proved that Col is conjugate to σ_{-1} via the parity vector map $\Phi^{-1} : \mathbb{Z}_2 \to \mathbb{Z}_2$ defined by

$$\Phi^{-1}(x) = \sum_{k=0}^{\infty} 2^k \Big(T^k(x) \mod 2 \Big),$$

that is, $\Phi \circ T \circ \Phi^{-1} = \sigma_{-1}$. Bernstein[3](1994) gives an explicit formula for the inverse conjugacy Φ, namely

$$\Phi(2^{d_0} + 2^{d_1} + 2^{d_2} + \ldots) = -\frac{1}{3}2^{d_0} - \frac{1}{9}2^{d_1} - \frac{1}{27}2^{d_2} - \ldots$$

See [6] page 57. In the notations that we will use in this work, we will denote the Bernstein function Φ by $-\pi^1$ and its inverse by Cod. We will use Σ_2^* instead of $\mathbb{Z}_2$ as the domain of the Bernstein function. As a consequence, we have that the periodic points of the Collatz function form a dense set. In particular, the periodic points of G_0 and G_∞ are dense and admit rational representation Proposition 34. We will use these results to show that, in fact, the elements of G_1 are positive and negative unstable (Theorem 6). Then we have that there is no natural number whose encodings are in $G_1 \cap G_\infty$.

The second point we have:

***Theorem* 9:** *Let $n \in \mathbb{N}$ such that $Cod(n) \in G_0$ then the orbit of n is bounded.*

In the Definition 20 the functions π^1 and π^2 defined on Σ_2^* are introduced. Let $n \in \mathbb{N}$ be such that if the coding is in G_0, then its orbit can be approximated through the function π^2 for relatively large values. In the Proposition 35, we have that when the coding is in G_0 then the function π^2 is upper bounded, then so is the orbit of n.

Finally in point 3, suppose that $n \in \mathbb{N}$ is a periodic point, then its coding is a periodic sequence in Σ_2^* and by Proposition 11, we have that the sequence of functions S_k on $\langle \theta, \psi \rangle$ must be of the form S^k for some $S \in \langle \theta, \psi \rangle$, that is, n corresponds to the fixed point of $S(x)$. The fixed points are of the form $\dfrac{N}{2^a - 3^b}$, where this expression we obtain that the fixed point is positive in G_0 and negative in G_∞

2.2. Notations and Conventions

In this work, we are going to denote the set of positive integers as $\mathbb{N}$, the set of non-negative integers as $\mathbb{N}_0$ and the greatest common divisor of a and b as (a, b) and the least common multiple of a and b as $[a, b]$.
We use the following symbology to refer to an arbitrary composition of functions:

$$\bigodot_{i=1}^{n} f_i(x) = f_n \circ f_{n-1} \circ \ldots \circ f_2 \circ f_1.$$

3. Set Generate by θ and ψ^q

In this section, we delve into functions generated by the composition of two real linear functions, θ and ψ^q, focusing on their properties over integers. We define the set $\langle \theta, \psi^q \rangle$, representing compositions of these functions, and examine their orbits and associated sets of integers. Before delving into their properties, we introduce the crucial concept of the integer set of a function. Denoted as $\mathbb{E}(S)$, this set represents the integers generated by the orbit of the function S. We emphasize the one-to-one

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correspondence between functions of the same length and the partition of integers into sets based on this length. These results provide a solid foundation for a detailed understanding of the properties of these functions and their application in the study of iterative functions over rational numbers.

3.1. Summary of Propositions in the Section

1. **Definition 1:** Introduces the set $\langle \theta, \psi^q \rangle$, generated by two real linear functions θ and ψ^q.
2. **Definition 2 :** Defines the integer set of a function $S \in \langle \theta, \psi^q \rangle$ as $\mathcal{O}_S(x) = \{x, s_1(x), \ldots, S(x)\}$, where s_i are functions in the composition of S.
3. **Definition 3 :** Defines the entire set of a function.
4. **Lemma 1** Monotony of Integer Set Lemma.
5. **Proposition 1 :** Establishes a relation of monotony in the entire sets concerning the composition of functions.
6. **Lemma 2** Establishes a characterization of the integer sets.
7. **Proposition 2 :** Establishes a one-to-one correspondence between functions of the same length and integer sets of the same length, and Affirms that the integer sets of functions of the same length are disjoint.
8. **Theorem 1:** Ensures that the integer sets are the disjoint union of the integer sets of functions in $\langle \theta, \psi^q \rangle$ with the same length.
9. **Proposition 3:** Guarantees the existence of a unique sequence of elements for a function $S \in \langle \theta, \psi^q \rangle$.

3.2. Set Generate by θ and ψ^q

The Generated Spaces, denoted as $\langle \theta, \psi^q \rangle$. These spaces arise from the iterative composition of functions, where the individual contributions of θ and ψ^q combine to form an enriched dynamic structure.

Definition 1 (Set Generated by θ and ψ^q)**.** *Let $q \in \mathbb{Z}$ and $\theta, \psi^q : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\theta(x) = \dfrac{x}{2}$ and $\psi^q(x) = \dfrac{3x+q}{2}$, then we define the set $\langle \theta, \psi^q \rangle$ as:*

$$S : \mathbb{R} \longrightarrow \mathbb{R}; \quad S(x) = \bigodot_{i=1}^{n} s_i(x), s_i(x) = \theta(x), \psi^q(x) \text{ for } i \geq 1, s_0 = id$$

We will call the number n length of S.

Now let's define the S-Orbits. These orbits are ordered sequences that reveal how each element evolves under the iterative action of the functions θ and ψ^q.

Definition 2 (S-Orbit and Integer S-Orbit)**.** *Let $S(x) = \bigodot_{i=1}^{n} s_i(x) \in \langle \theta, \psi^q \rangle$ and $x \in \mathbb{R}$. We define the S-orbit of x_0 as the set:*

$$\mathcal{O}_S(x_0) = \left\{ x_0, s_1(x_0), \ldots, \bigodot_{i=1}^{k} s_i(x_0), \ldots, S(x_0) \right\}$$

Let $p \in \mathbb{N}$. We will say that an S-orbit of p is integer when

$$\bigodot_{i=1}^{k} s_i(p) \in \mathbb{N} \text{ for all } k \leq n$$

Now let's define The integer sets of a function, denoted by $\mathbb{E}(S)$, represent the integer values that a specific function takes on its domain. Examining $\mathbb{E}(S)$ allows for the identification of patterns and

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regularities in the interaction of the function with integers, which is essential for understanding the structure of spaces generated by such functions.

Definition 3 (Integer Set of a Function). *Let $S : \mathbb{R} \longrightarrow \mathbb{R}$ a function, we called integer set of f or the integers of f the set:*

$$\mathbb{E}(S) = \{n \in \mathbb{Z}; S(n) \in \mathbb{Z}\}.$$

In the following Proposition we are going to see that integer sets have a monotonic behavior concerning the composition of linear functions, this property will be fundamental to studying integer S-orbits.

Lemma 1 (Monotony of Integer Set Lemma). *Let $A, B, a, b \in \mathbb{Z}$ such that $(A, 2) = (B, 2) = 1$ and $\alpha, \beta \in \mathbb{N}$. Let $S(x) = \dfrac{Ax + a}{2^\alpha}$ and $H(x) = \dfrac{Bx + b}{2^\beta}$, then $\mathbb{E}(H \circ S) \subset \mathbb{E}(S)$.*

Proof. In fact, we have $H \circ S(x) = \dfrac{ABx + Ba + 2^\alpha b}{2^{\alpha+\beta}}$ Since $(AB, 2) = 1$ there are solutions.

Let $x_0 \in \mathbb{E}(H \circ S)$ then by definition $\dfrac{ABx_0 + Ba + 2^\alpha b}{2^{\alpha+\beta}} \in \mathbb{Z}$, then we have

$$\frac{ABx_0 + Ba + 2^\alpha b}{2^{\alpha+\beta}} = \frac{B}{2^\beta}\left(\frac{Ax_0 + a}{2^\alpha}\right) + \frac{b}{2^\beta} \in \mathbb{Z} \Rightarrow B\left(\frac{Ax_0 + a}{2^\alpha}\right) + b \in \mathbb{Z}$$

as $(B, 2) = 1$ then we have

$$\frac{Ax_0 + a}{2^\alpha} \in \mathbb{Z}$$

□

Proposition 1 (Monotony of Integer Set). *Let $\bigodot_{i=1}^{n} s_i(x) \in \langle \theta, \psi^q \rangle$ with $s_i \in \{\theta, \psi^q\}$ and $L, M \in \mathbb{N}$. Then if $n \geq L \geq M$ we have:*

$$\mathbb{E}\left(\bigodot_{i=1}^{L} s_i(x)\right) \subset \mathbb{E}\left(\bigodot_{i=1}^{M} s_i(x)\right)$$

Proof. As the functions generated by $\langle \theta, \psi^q \rangle$ are linear of the form $\dfrac{3^b x + N}{2^a}$ with $a, b.N \in \mathbb{N}$. The result follows inductively from Lemma 1. □

Example 3. *Let $S_n(x) = \theta^{2n}\psi(x) \in \langle \theta, \psi \rangle$. We will calculate the integer set of S_n*

$$2^{2n}y - 3x = 1$$

we have that $y = 1$ and $\dfrac{2^n - 1}{3} \in \mathbb{N}$ are solutions of the Diophantine Equation, then the integer set is:

$$\mathbb{E}(\theta^{2n}\psi) = \frac{2^n - 1}{3} + 2^{2n}\mathbb{Z}$$

Let $m \geq n$ then

$$\mathbb{E}(\theta^{2m}\psi) \subset \mathbb{E}(\theta^{2n}\psi)$$

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indeed

$$\frac{3\left(\frac{2^m-1}{3}\right)+1}{2^n}=2^{m-n}\in\mathbb{N}$$

The following lemma states that an $S-$orbit is integer if and only if its last value is an integer.

Lemma 2 (Containment in Integer Sets). *Let $s_i\in\{\theta,\psi^q\}$, then $u\in\mathbb{E}\left(\bigodot_{i=1}^{j}s_i(x)\right)$ if and only if $\bigodot_{i=1}^{r}s_i(u)\in\mathbb{E}\left(\bigodot_{i=r+1}^{j}s_i(x)\right)$ with $r\le j$.*

Proof. Let $u\in\mathbb{E}\left(\bigodot_{i=1}^{j}s_i(x)\right)$ then by proposition 1 we have $u\in\mathbb{E}\left(\bigodot_{i=1}^{r}s_i(x)\right)$ with $r<j$, then $\bigodot_{i=1}^{r}s_i(u)\in\mathbb{Z}$. On the other hand, we have $\bigodot_{i=r+1}^{j}s_i\left(\bigodot_{i=1}^{r}s_i(u)\right)\in\mathbb{Z}$ then $\bigodot_{i=1}^{r}s_i(u)\in\mathbb{E}\left(\bigodot_{i=r+1}^{j}s_i(x)\right)$.
If $\bigodot_{i=1}^{r}s_i(x)\in\mathbb{E}\left(\bigodot_{i=r+1}^{j}s_i(x)\right)$ with $r<j$ then $\bigodot_{i=1}^{r}s_i(u)\in\mathbb{Z}$ so

$$\bigodot_{i=r+1}^{j}s_i\left(\bigodot_{i=1}^{r}s_i(u)\right)=\bigodot_{i=1}^{j}s_i(u)\in\mathbb{Z}$$

then $u\in\mathbb{E}\left(\bigodot_{i=1}^{j}s_i(x)\right)$. □

In the following proposition, We will demonstrate that the integer sets associated with functions of the same length are disjoint. That is, if two integer sets share at least one element, then the functions must be the same.

Proposition 2 (One-to-One Correspondence and Disjointedness). *Let $S,H\in\langle\theta,\psi^q\rangle$ of length k, if $H\neq S$ if and only if $\mathbb{E}(H)\cap\mathbb{E}(S)=\varnothing$.*

Proof. Let $S(x)=\bigodot_{i=1}^{j}s_i(x)$ and $H(x)=\bigodot_{i=1}^{j}h_i(x)$ with $s_i,h_i\in\{\theta(x),\psi(x)\}$ and let k_0 be the largest index such that $s_i(x)=h_i(x)$ for all $i<k_0$
If $k_0=1$ This means that they have different first terms. Then $\mathbb{E}(S)\subset\mathbb{E}(\theta)=2\mathbb{Z}$ and $\mathbb{E}(H)\subset\mathbb{E}(\psi^q)=2\mathbb{Z}+1$ or, $\mathbb{E}(H)\subset\mathbb{E}(\theta)=2\mathbb{Z}$ and $\mathbb{E}(S)\subset\mathbb{E}(\psi^q)=2\mathbb{Z}+1$ in either case we have $\mathbb{E}(H)\cap\mathbb{E}(S)=\varnothing$.
Suppose that exist $u\in\mathbb{E}(H)\cap\mathbb{E}(S)$ by proposition 1 we have

$$u\in\mathbb{E}\left(\bigodot_{i=1}^{r}s_i(x)\right)\cap\mathbb{E}\left(\bigodot_{i=1}^{r}h_i(x)\right)\text{ with } r\le j.$$

Taken $r=k_0$

$$u\in\mathbb{E}\left(\bigodot_{i=1}^{k_0}s_i(x)\right)\cap\mathbb{E}\left(\bigodot_{i=1}^{k_0}h_i(x)\right).$$

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and by lemma 2

$$\bigodot_{i=1}^{k_0-1} s_i(u) = \bigodot_{i=1}^{k_0-1} h_i(u) \in \mathbb{E}\big(s_{k_0}(x)\big) \cap \mathbb{E}\big(s_{k_0}(x)\big) = 2\mathbb{Z} \cap (2\mathbb{Z}+1) = \emptyset$$

which is a contradiction, On the other hand if $\mathbb{E}(H) \cap \mathbb{E}(S) = \emptyset$ then $S \neq H$ otherwise we would have

$$\mathbb{E}(S) = \mathbb{E}(H) = \emptyset$$

however, neither set can be empty $\square$

As a consequence of the above proposition we have

Theorem 1 (Partition of Integers). *Let $\{S_j\}_{j\in\mathbb{N}} \subset \langle \theta, \psi^q \rangle$ with length k, then*

$$\mathbb{Z} = \bigsqcup_{S \text{ with length } k} \mathbb{E}(S)$$

i.e., the sets of integers are equal to the disjoint union of the integer sets of functions S of length k

Proof. it is evident that

$$\bigsqcup_{S \text{ with length } k} \mathbb{E}(S) \subset \mathbb{Z}$$

To prove the other contention we consider the $q-$Collatz function defined by $Col_q : \mathbb{Z} \to \mathbb{Z}$ given by

$$Col_q(u) = \begin{cases} \dfrac{3u+q}{2} & \text{if } u \text{ odd} \\ \dfrac{u}{2} & \text{if } u \text{ even,} \end{cases}$$

let's take an integer u and calculate its k-th orbit, this orbit can be written as compositions of functions in $\langle \theta, \psi^q \rangle$, let's call the resulting function S since all the values of the orbit are integers, we have by the lemma 2 we can conclude that u is in the entire set of the function S.

$$u \in \mathbb{E}(S) \subset \bigsqcup_{S \text{ with length } k} \mathbb{E}(S).$$

$\square$

So far we know that if two functions have the same length, then their integer sets are disjoint, this means that if we take elements of each integer set, the S-orbits of these integers are disjoint sets. Now, if we remove the condition that the lengths are the same, could it be that there is some number such that, given two different functions without being a part of another, it generates the same S-orbit? The answer is in the following proposition.

Proposition 3 (Uniqueness of Basis Representation). *Let $S \in \langle \theta, \psi^q \rangle$. Then there exists a unique sequence of k elements $\{s_j\}_{j=1}^{k}$ with $s_j \in \{\theta, \psi^q\}$ such that*

$$S(x) = \bigodot_{j=1}^{k} s_j(x)$$

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Proof. Since $S \in \langle \theta, \psi^q \rangle$, then there exists a sequence of k_0 elements $\{s_j\}_{j=1}^{k_0}$ with $s_j \in \{\theta, \psi^q\}$ such that $S(x) = \bigodot_{j=1}^{k_0} s_j(x)$. Suppose for absurdity, that there is another sequence but of l_0 elements $\{h_j\}_{j=1}^{l_0}$ such that $S(x) = \bigodot_{j=1}^{l_0} h_j(x)$. Let $u \in \mathbb{E}(S)$, we have the following cases

1. If $k_0 > l_0$. We have

$$S(x) = \bigodot_{j=1}^{k_0} s_j(x) = \bigodot_{j=1}^{l_0} h_j(x)$$

$$u \in \mathbb{E}\left(\bigodot_{j=1}^{k_0} s_j(x)\right) = \mathbb{E}\left(\bigodot_{j=1}^{l_0} h_j(x)\right)$$

by Proposition 1 we have $u \in \mathbb{E}(s_1) \cap \mathbb{E}(h_1)$, since $s_1, h_1 \in \{\theta, \psi^q\}$ and $\mathbb{E}(\theta) \cap \mathbb{E}(\psi^q) = \emptyset$ then $s_1 = h_1$. Since s_1 and h_1 are invertible functions, we have

$$\bigodot_{j=2}^{k_0} s_j(x) = \bigodot_{j=2}^{l_0} h_j(x).$$

Following the same idea up to k_0, we have

$$\bigodot_{j=k_0}^{l_0} s_j(x) = x$$

The latter is impossible since the slope of the resulting line is of the form $\dfrac{3^b}{2^a}$ with $a, b \in \mathbb{N}$. The case $k_0 > l_0$ is completely analogous, therefore the case where k_0 and l_0 are different is not possible.

2. If $k_0 = l_0$. Since the sequences are different, there must exist some $k < k_0$ such that

$$\bigodot_{j=1}^{k} s_j(x) \neq \bigodot_{j=1}^{k} h_j(x)$$

then by Proposition 2 we have

$$\mathbb{E}\left(\bigodot_{j=1}^{k} s_j(x)\right) \cap \mathbb{E}\left(\bigodot_{j=1}^{k} h_j(x)\right) = \emptyset$$

However, this is a contradiction to the Proposition 1, because $u \in \mathbb{E}\left(\bigodot_{j=1}^{k} s_j(x)\right) \cap \mathbb{E}\left(\bigodot_{j=1}^{k} h_j(x)\right)$ for all $k \leq k_0$. Then both sequences must be identical.

□

4. Stability and Instability of Integer Set

In this section, we delve into the stability and instability of sequences associated with integer sets. We begin by defining functions ρ_0 and ρ_1 that map real functions to integers. We introduce the concepts of positive and negative stability for sequences $\{S_j(x)\}_{j=1}^{\infty}$. The monotony of ρ_0 and ρ_1 is

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established through Proposition 1, demonstrating the non-decreasing of ρ_0 and the non-increasing of ρ_1 for a given sequence $S_j(x)$. Further, the Proposition formally defines positive and negative stability, incorporating limits and intersections of sets. The ensuing Stability Limit Theorem (2) establishes the asymptotic behavior of the integer set of an iterative sequence.

4.1. Summary of Propositions in the Section

1. **Definition 4 :** Definition of functions ρ_0 and ρ_1.
2. **Proposition 4:** monotony of the functions ρ_0 and ρ_1.
3. **Definition 5:** Definition of positively (negatively) stable (unstable) sequences.
4. **Theorem 2:** Establishes the asymptotic behavior of the integer set when we have a positively (negatively) stable (unstable) sequence.

4.2. Stability and Instability of Integer Set

We initiate this section by introducing functions that associate each integer set with its minimum positive integer value and maximum negative integer value. These values are determined by the solutions closest to zero for the variable x in the Diophantine equation $2^a y - 3^b x = N$. This equation is representative of the Diophantine equation linked to an element within the space generated by ψ and θ.

Definition 4 (ρ_0 and ρ_1 functions.). *Define the function $\rho_0, \rho_1 : \{f : \mathbb{R} \longrightarrow \mathbb{R} : f \text{ function } \} \to \mathbb{Z}$ by*

$$\rho_0(f) = \min\{\mathbb{E}(f) \cap \mathbb{N}\}$$

and

$$\rho_1(f) = \max\{\mathbb{E}(f) \cap -\mathbb{N}\}$$

As a consequence of the Proposition 1. We have that the functions ρ_0 and ρ_1 are monotone.

Proposition 4 (monotony of ρ_0 and ρ_1). *Let $\{S_j(x)\}_{j\in\mathbb{N}} = \left\{ \bigodot_{i=1}^{j} s_i(x) \right\}_{j\in\mathbb{N}} \subset \langle \theta, \psi^q \rangle$ then $\rho_0(S_j(x))$ is a non-decreasing function, and $\rho_1(S_j(x))$ it is a non-increasing function.*

Proof. By the proposition 1 we have that

$$\rho_0(S_{j+1}), \rho_1(S_{j+1}) \in \mathbb{E}(S_{j+1}) \subset \mathbb{E}(S_j)$$

then $\rho_0(S_j) \leq \rho_0(S_{j+1})$ and $\rho_1(S_j) \geq \rho_1(S_{j+1})$. □

From the result of the proposition above, we are going to make a classification of the sequences $S_j = \bigodot_{i=1}^{j} s_i(x)$ according to the behavior of the functions ρ_0 and ρ_1.

Definition 5 (Stability of Sequences). *Let $\{S_j(x)\}_{j=1}^{\infty}$ sequence on $\langle \theta, \psi^q \rangle$ given by $S_j = \bigodot_{i=1}^{j} s_i(x)$. We will say that $\{S_j(x)\}_{j=1}^{\infty}$ is positively stable if $\lim_{j\to\infty} \rho_0\left(\bigodot_{i=1}^{j} S_k(j)\right) < \infty$ otherwise we will say that it is positively unstable. On the other hand, we will say that $\{S_j(x)\}_{j=1}^{\infty}$ is negatively stable if $\lim_{j\to\infty} \rho_1\left(\bigodot_{i=1}^{j} S_k(j)\right) > -\infty$, otherwise, we will say that it is negatively unstable.*

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Now we will give the central theorem of this section, which establishes the asymptotic behavior of the integer sets of S_j from the stability or stability of this.

Theorem 2 (Stability Limit Theorem). *Let* $\{S_j(x)\}_{j=1}^{\infty} \subset \langle \theta, \psi^q \rangle$ *and*

$$\mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \mathbb{E}\left(\bigodot_{j=1}^{k} S_j(x)\right) \cap \mathbb{N}$$

and

$$\mathbb{E}^-\left(\bigodot_{j=1}^{k} S_j(x)\right) = \mathbb{E}\left(\bigodot_{j=1}^{k} S_j(x)\right) \cap -\mathbb{N}.$$

We have:

1. *if* $\{S_j(x)\}_{j=1}^{\infty}$ *is positively stable then*

$$\lim_{k\to\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \bigcap_{k=1}^{\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \left\{\lim_{k\to\infty} \rho_0\left(\bigodot_{j=1}^{k} S_j(x)\right)\right\}$$

2. *if* $\{S_j(x)\}_{j=1}^{\infty}$ *is positively unstable, then*

$$\lim_{k\to\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \bigcap_{k=1}^{\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \emptyset$$

analogously

1. *if* $\{S_j(x)\}_{j=1}^{\infty}$ *is negatively stable then*

$$\lim_{k\to\infty} \mathbb{E}^-\left(\bigodot_{j=1}^{k} S_j(x)\right) = \bigcap_{k=1}^{\infty} \mathbb{E}^-\left(\bigodot_{j=1}^{k} S_j(x)\right) = \left\{\lim_{k\to\infty} \rho_1\left(\bigodot_{j=1}^{k} S_j(x)\right)\right\}$$

2. *if* $\{S_j(x)\}_{j=1}^{\infty}$ *is positively unstable, then*

$$\lim_{k\to\infty} \mathbb{E}^-\left(\bigodot_{j=1}^{k} S_j(x)\right) = \bigcap_{k=1}^{\infty} \mathbb{E}^-\left(\bigodot_{j=1}^{k} S_j(x)\right) = \emptyset$$

Proof. : Let $\rho_0^k = \rho_0\left(\bigodot_{j=1}^{k} S_j(x)\right)$ and $\bigodot_{j=1}^{k} S_j(x) = \dfrac{3^{B_k}x + N}{2^{A_k}}$ with A_k =numbers from θ to $\bigodot_{j=1}^{k} S_j(x)$ and B_k =numbers from ψ^q to $\bigodot_{j=1}^{k} S_j(x)$. We have

$$\mathbb{E}\left(\bigodot_{j=1}^{k} S_j(x)\right) = \rho_0^k + 2^{A_k}\mathbb{N}.$$

supposed that $\{S_j(x)\}_{j=1}^{\infty}$ is stable, we will first prove that $\lim_{k\to\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right)$ is non-empty. By Proposition 1 the sequence of sets $\mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right)$ is a decreasing sequence of sets i.e. that the next set

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is a subset of the previous one, then the limit set corresponds to the intersection of all the sets of the sequence.

$$\lim_{k\to\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \bigcap_{k\in\mathbb{N}} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right)$$

Now since ρ_0^k is a function of the natural ones in the natural ones and is convergent, it implies that this function reaches its limit in a finite amount of steps

$$\rho_0^k = cts \text{ for all } k > K$$

this implies

$$\lim_{k\to\infty} \rho_0^k \in \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) \text{ for all } k \in \mathbb{N}$$

then the limit set is non-empty.

Now we will prove the limit set contains a single element. Suppose there exists another element u_0 that is contained in all positive integer sets, then there exists a non-negative integer t such that

$$u_0 = \rho_0^k + 2^{A_k} t$$

without loss of generality, we can assume that ρ_0^k is constant. Solving the equation in terms of t, we have

$$t = \frac{u_0 - \rho_0^k}{2^{A_k}}$$

This solution is a fraction less than 1 for k large enough., which contradicts the fact that t is an integer.

Now let us take the unstable case. Suppose there exists an element u_0 in the limiting set i.e. an element that is contained in all non-negative integer sets, then there exists t a non-negative integer such that

$$u_0 = \rho_0^k + 2^{A_k} t \geq \rho_0^k$$

as ρ_0^k diverges and u_0 constant, then there exists a K such that ρ_0^k is greater than u_0, then u_0 cannot belong to any integer set with $k > K$, which is a contradiction. Analogously for the other case. □

Example 4. *$S_n(x) = \theta^{2n}\psi(x)$ is positively and negatively unstable. Indeed, by example 3, we have $\mathbb{E}(\theta^{2n}\psi(x)) = \dfrac{2^n-1}{3} + 2^{2n}\mathbb{Z}$ where $\rho_0(\theta^{2n}\psi(x)) = \dfrac{2^n-1}{3} \to \infty$ as $n \to \infty$ then $\lim\limits_{n\to\infty} \mathbb{E}^+(\theta^{2n}\psi(x)) = \emptyset$. On the other hand $\rho_1(\theta^{2n}\psi(x)) = \dfrac{2^n-1}{3} - 2^{2n} \to -\infty$.*

Example 5. *$S_n(x) = (\theta\psi(x))^n$ is positively unstable and negatively stable. Let's calculate the integer set $\mathbb{E}((\theta\psi(x))^n)$, let's observe that*

$$\theta\psi(-1) = \frac{3(-1)+1}{2} = -1$$

Then $\mathbb{E}(\theta\psi(x))^n = -1 + 2^n\mathbb{Z}$ then we have $\rho_1((\theta\psi(x))^n) = -1$ and $\rho_0((\theta\psi(x))^n) = 2^n - 1 \to \infty$ as $n \to \infty$

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5. Coding of the Orbits

In this section, we will delve into the study of the coding of the orbits of the Collatz function. The main results of this section are the invariance of the coding between Cod_q and Cod on the fractions with denominator q and the one-to-one identification of each element of $\mathbb{Q}_{odd}$ with its coding.

5.1. Summary of Propositions in the Section

1. **Definition 6:** Coding maps and Σ_2^* the space of sequences 0 and 10.
2. **Proposition 5:** General form of the elements generated by ψ and θ.
3. **Proposition 6:** Fist Cod invariance: $Cod(S) = Cod_q(S_q)$.
4. **Definition 9:** Definition of $Col_q : \mathbb{Z} \to \mathbb{Z}$.
5. **Definition 7:** Extension of Collatz function on $\mathbb{Q} = \mathbb{Q}_{odd} \cup \mathbb{Q}_{even}$.
6. **Proposition 7**: Extension of Collatz function on $\mathbb{Q}$ is well-defined.
7. **Proposition 8:** $Col(\mathbb{Q}_{odd}) \subset \mathbb{Q}_{odd}$.
8. **Definition 8:** Extension of the Collatz function on $\mathbb{Q}_{odd}$.
9. **Proposition 9:** $Col - Col_q$ equivalence : if $\frac{p}{q} \in \mathbb{Q}_{odd}$ then $Col^k\left(\frac{p}{q}\right) = \frac{1}{q}Col_q^k(p)$.
10. **Proposition 10:** Second Cod invariance: $Cod^k\left(\frac{p}{q}\right) = Cod_q^k(p)$.
11. **Proposition 11:** $p \in \mathbb{E}(S)$ if and only if $Cod_q^k(p) = Cod_q(S)$.
12. **Proposition 12** $p \in \mathbb{E}(S)$ if and only if $Cod^k\left(\frac{p}{q}\right) = Cod_q(S)$.
13. **Proposition 13:** $Cod^k\left(\frac{p}{q}\right) = Cod^k\left(\frac{p}{q} + \frac{2^{A_k}T}{q}\right)$.
14. **Definition 10:** The Coding set $Cod^k(\xi)$.
15. **Proposition 14:** Monotony of the Coding set $Cod^{k+1}(\xi) \subset Cod^k(\xi)$.
16. **Proposition 15:** Generating property: if $\frac{p}{q}, \frac{t}{r} \in Cod^k(\xi)$ then $\frac{p}{q} = \frac{t}{r} + \frac{2^{A_k}T}{qr}$.
17. **Theorem 3:** Uniqueness of the full coding $\mathbb{Q}_{odd}$.

5.2. Coding of the Orbits

It is a common practice in dynamical systems to encode orbits based on specific criteria. In our case, we will encode the orbits of the Collatz function according to the parity of its elements, assigning the value 1 when they are odd and 0 when they are even. Since our primary focus is on the Collatz function over $\mathbb{Q}_{odd}$, we will modify the initial coding by assigning 10 when it is odd, as opposed to just 1. We will denote the space where these codings reside as Σ_2^* since it is a subset of the sequence space consisting of 0s and 1s, denoted in dynamics as Σ_2. Formally, we express this as

Definition 6 (Coding of the Orbits). *Let $S \in \langle \theta, \psi^q \rangle$ with length k. We define the following application $Cod_q : \langle \theta, \psi^q \rangle \to \{0, 10\}^k$ defined by*

$$Cod_q\left(\bigodot_{j=1}^{k} s_j(x)\right) = \{\xi_j\}_{j=1}^{k} \text{ with } s_j \in \{\theta, \psi^q\}$$

with

$$\xi_j = \begin{cases} 0 & \text{if} \quad s_j(x) \text{ is } \theta(x) \\ 10 & \text{if} \quad s_j(x) \text{ is } \psi^q(x) \end{cases}$$

To rigorously examine the properties of the coding, it is essential to establish a precise form for the elements generated by ψ and θ.

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Proposition 5 (General form of S)**.** *Let $S \in \langle \theta, \psi^q \rangle$ and Let $\{\theta_j\}_{j=1}^{k+1} \in \mathbb{N}_0 \times \mathbb{N} \times \ldots \times \mathbb{N}$ and $Cod_q(S) = 0^{\theta_1}10^{\theta_2}\ldots 0^{\theta_k}10^{\theta_{k+1}}$ and let $a_j = \sum_{i=1}^{j} \theta_i$ and $N : \Sigma_2^* \to \mathbb{N}$ defined as*

$$N(Cod_q(S)) = \begin{cases} 3^{k-1}2^{a_1} + 3^{k-2}2^{a_2} + \ldots + 2^{a_k} & if \quad k > 0 \\ 0 & if \quad k = 0 \end{cases}$$

then

$$S(x) = \frac{3^k x + qN(Cod_q(S))}{2^{a_{k+1}}}$$

Proof. We will prove by induction on k. For $a_{k+1} = 1$ we have

1. $Cod_q(\psi^q(x)) = 10$, then, $k = 1, a_1 = 0, a_2 = 1$ and $N = 3^0 2^0 = 1$ then $S(x) = \dfrac{3x+q}{2} = \psi^q(x)$.
2. $Cod_q(\theta(x)) = 0$, then, $k = 0, a_1 = 1$ and $N = 0$ then $S(x) = \dfrac{3^0 x + 0q}{2} = \dfrac{x}{2} = \theta(x)$.

Suppose the statement is true up to k, let $S \circ H \in \langle \theta, \psi^q \rangle$ of length a_{k+1} with H of length a_k.

Claim 1: $Cod_q(\psi^q \circ H) = Cod_q(H)10$. We have:

1. $a_{k+2}(Cod_q(H)10) = a_{k+1} + 1$ and $a_j(Cod_q(H)10) = a_j(Cod_q(H))$ for $j \le k+1$.
2. $N(Cod_q(H)10) = 3^{k+1}2^{a_1} + 3^k 2^{a_2} + \ldots + 3^1 2^{a_k} + 2^{a_{k+1}}$

On the other hand, we have:

$$\begin{aligned} \psi^q\left(\frac{3^k x + qN(Cod_q(H))}{2^{a_{k+1}}}\right) &= \frac{3^{k+1}x + q3N(Cod_q(H)) + q2^{a_{k+1}}}{2^{a_{k+1}+1}} \\ &= \frac{3^{k+1}x + q3(3^{k-1}2^{a_1} + 3^{k-2}2^{a_2} + \ldots + 2^{a_k}) + q2^{a_{k+1}}}{2^{a_{k+1}+1}} \\ &= \frac{3^{k+1}x + q(3^k 2^{a_1} + 3^{k-1}2^{a_2} + \ldots + 3^1 2^{a_k} + 2^{a_{k+1}})}{2^{a_{k+1}+1}} \\ &= \frac{3^{k+1}x + qN(Cod_q(H)10)}{2^{a_{k+2}}} \end{aligned}$$

where we observe that the values coincide with those calculated.

Claim 2: $Cod_q(\theta \circ H) = Cod_q(H)0$. We have

1. $a_{k+2}(Cod_q(H)0) = a_k(Cod_q(H)) + 1$.
2. $N(Cod_q(H)0) = 3^{k-1}2^{a_1} + 3^{k-2}2^{a_2} + \ldots + 2^{a_k} = N(Cod_q(H))$.

On the other hand, we have

$$\theta\left(\frac{3^k x + qN(Cod_q(H))}{2^{a_{k+1}}}\right) = \frac{3^k x + qN(Cod_q(H)0)}{2^{a_{k+2}}}$$

where we observe that the values coincide with those calculated, then the statement is true. □

Now we will see the first property of the coding

Proposition 6 (First Cod invariance)**.** *Let $S \in \langle \theta, \psi \rangle$ given by $S(x) = \dfrac{3^b x + N}{2^a}$, we defined $S_q \in \langle \theta, \psi^q \rangle$ given by $S_q(x) = \dfrac{3^b x + qN}{2^a}$. Then $Cod(S) = Cod_q(S_q)$.*

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Proof. Let $q \in \mathbb{Z}$. To prove that they have the same coding, we have to prove that they have the same decomposition in principle, except that where there is ψ we have a ψ^q. let us observe that q has commutative properties with θ and ψ.

1. $q\theta(x) = \dfrac{qx}{2} = \theta(qx)$.
2. $q\psi(x) = q\left(\dfrac{3x+1}{2}\right) = \dfrac{3(qx)+q}{2} = \psi^q(qx)$

As $S \in \langle \theta, \psi \rangle$ then there exists $s_j \in \{\theta, \psi\}$ such that

$$S(x) = \bigodot_{j=1}^{a} s_j(x).$$

For convenience we will denote $s_j^q \in \{\theta, \psi^q\}$. Then we have

$$S_q(x) = \frac{3^b x + qN}{2^a} = q\left(\frac{3^b\left(\frac{x}{q}\right) + N}{2^a}\right) = q\bigodot_{j=1}^{a} s_j\left(\frac{x}{q}\right) = \bigodot_{j=1}^{a} s_j^q\left(q\frac{x}{q}\right) = \bigodot_{j=1}^{a} s_j^q(x)$$

We have that if s_j is θ then s_j^q is still θ and if s_j is ψ then s_j^q corresponds to ψ^q. By Proposition 3 we have that the coding of S_q has to be the same as that of S. $\square$

Let us contemplate a generalization of the Collatz function applied to integers. In this variant, rather than adding 1, the function adds $q \in \mathbb{Z}$, where q is an odd integer. Subsequently, we will establish the compatibility of this generalization with the extension of the Collatz function to $\mathbb{Q}_{odd}$.

5.3. Extension of the Collatz Function on $\mathbb{Q}$

As the concept of parity is a concept defined for integers, the Collatz function can be naturally extended to the set of integers. This concept is not trivially extended to the set of rational numbers, as there is no unique representation, We are going to consider a modification of extension on the rationals proposed by Lagaria in [9], Lagaria defined the Collatz function for fractions $\frac{p}{q}$ such that $(q,2) = (q,3) = 1$. We will distinguish two subsets of $\mathbb{Q}$. The set of rationals with odd denominators, denoted by $\mathbb{Q}_{odd}$, and the set of rationals with even denominators such that the numerator and denominator are co-prime, denoted by $\mathbb{Q}_{even}$. We will say that a rational number in $\mathbb{Q}_{\text{odd}}$ is odd if the numerator is odd, and it is even if its numerator is even. In the case of $\mathbb{Q}_{\text{even}}$, since the denominator is already even and due to coprimality, all elements are odd. We are going to consider the following extension of the Collatz function.

Definition 7 (Extension of the Collatz function). *Let's consider the following sets*

$$\mathbb{Q}_{odd} = \left\{ \frac{p}{q} \in \mathbb{Q}, \text{ such that } q \text{ is odd} \right\}$$

and

$$\mathbb{Q}_{even} = \left\{ \frac{p}{q} \in \mathbb{Q}, \text{ such that } q \text{ is even and } (p,q) = 1 \right\}.$$

We defined the Collatz's function by $Col : \mathbb{Q}_{odd} \cup \mathbb{Q}_{even} \to \mathbb{Q}_{odd} \cup \mathbb{Q}_{even}$ *by*

$$Col\left(\frac{p}{q}\right) = \begin{cases} \dfrac{3p+q}{q} & \text{if } p \text{ odd} \\ \dfrac{p}{2q} & \text{if } p \text{ even,} \end{cases}$$

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We are going to show that the extension of the Collatz function that we defined is well-defined on $\mathbb{Q}_{odd}$.

Proposition 7 (Well-Defined)**.** *The Collatz's function on $\mathbb{Q}_{odd}$ is well-defined.*

Proof. We are going to show that Col is well-defined over $\mathbb{Q}_{odd}$. Let $p, q \in \mathbb{Z}$ with $q \neq 0$ and $(q, 2) = (p, q) = 1$. Let λ an odd number, then

$$Col\left(\frac{\lambda p}{\lambda q}\right) = \begin{cases} \dfrac{3\lambda p + \lambda q}{\lambda q} & \text{if } \lambda p \text{ odd} \\ \dfrac{\lambda p}{2\lambda q} & \text{if } \lambda p \text{ even,} \end{cases} = \begin{cases} \dfrac{3p+q}{q} & \text{if } p \text{ odd} \\ \dfrac{p}{2q} & \text{if } p \text{ even,} \end{cases} = Col\left(\frac{p}{q}\right)$$

□

Let's observe that when we apply the Collatz function to $\frac{p}{q} \in \mathbb{Q}_{\text{odd}}$ with odd number, we always obtain a fraction with an even numerator, and when applied to $\mathbb{Q}_{\text{even}}$, we always obtain an odd number. This will be very important since in section 5, we are going to define how to coding the orbits, assigning 1 if it is odd and 0 if it is even, in the case of $\mathbb{Q}_{\text{even}}$, we will have that all its elements have the same encoding which is $1111\ldots$ unlike $\mathbb{Q}_{\text{odd}}$, where the codings will be generated by 10 and 0. For this reason we are going to work mainly on $\mathbb{Q}_{\text{odd}}$, let's simplify the Collatz function a bit, as $Col : \mathbb{Q}_{\text{odd}} \to \mathbb{Q}_{\text{odd}} \cup \mathbb{Q}_{\text{even}}$ given by

$$Col\left(\frac{p}{q}\right) = \begin{cases} \dfrac{3p+q}{2q} & \text{if } p \text{ odd} \\ \dfrac{p}{2q} & \text{if } p \text{ even,} \end{cases}$$

Proposition 8 (Invariance of $\mathbb{Q}_{odd}$)**.** *The Collatz function defined above satisfies that $Col(\mathbb{Q}_{odd}) \subset \mathbb{Q}_{odd}$*

Proof. We will show that Col does not change the parity of the numerator.

1. if p is odd, we have $3p + q = 2r$ with $r \in \mathbb{Z}$, then

$$\frac{3p+q}{2q} = \frac{2r}{2q} = \frac{r}{q}$$

   Since q is odd, we have independent of the simplification $\frac{r}{q} \in \mathbb{Q}_{odd}$.

2. if $p = 2r$ with $r \in \mathbb{Z}$, we have

$$\frac{p}{2q} = \frac{2r}{2q} = \frac{r}{q}$$

   Since q is odd, we have independent of the simplification, we have $\frac{r}{q} \in \mathbb{Q}_{odd}$.

□

Considering the proposition above, we are going to define the Collatz function on $\mathbb{Q}_{odd}$ as

Definition 8 (Extension of the Collatz function on $\mathbb{Q}_{odd}$)**.** *We define the Collatz Function on $\mathbb{Q}_{odd}$ by $Col : \mathbb{Q}_{odd} \to \mathbb{Q}_{odd}$*

$$Col\left(\frac{p}{q}\right) = \begin{cases} \dfrac{3p+q}{2q} & \text{if } p \text{ odd} \\ \dfrac{p}{2q} & \text{if } p \text{ even,} \end{cases}$$

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Example 6. *Let* $\frac{5}{7} \in \mathbb{Q}_{odd}$ *and* $\frac{5}{2} \in \mathbb{Q}_{even}$ *we have:*

$$\mathcal{O}\left(\frac{5}{7}\right) : \frac{5}{7} \to \frac{11}{7} \to \frac{20}{7} \to \frac{10}{7} \to \frac{5}{7}$$

and

$$\mathcal{O}\left(\frac{5}{2}\right) : \frac{5}{2} \to \frac{17}{2} \to \frac{53}{2} \to \frac{161}{2} \to \frac{485}{2} \to \ldots \to \frac{3^k 5 + (3^k - 1)}{2} \to \ldots \to \infty$$

We can observe that the extension of the conjecture on the set of rationals is false, since we have found a fraction with a divergent otbit, The first objective of the work is to show that there are no divergent orbits in $\mathbb{Q}_{odd}$*..*

We define the following generalization of the Collatz function.

Definition 9 (The Col_q map)**.** *Let* $q \in \mathbb{Z}$*, we define the* $q-$*Collatz function defined by* $Col_q : \mathbb{Z} \to \mathbb{Z}$ *given by*

$$Col_q(n) = \begin{cases} \dfrac{3n+q}{2} & \text{if } u \text{ odd} \\ \dfrac{n}{2} & \text{if } u \text{ even,} \end{cases}$$

Now, we will demonstrate the compatibility of this generalization

Proposition 9 ($Col - Col_q$ equivalence)**.** *Let* $\frac{p}{q} \in \mathbb{Q}_{odd}$*. Then for all integer numbers* $k \geq 0$ *we have*

$$Col^k\left(\frac{p}{q}\right) = \frac{1}{q} Col_q^k(p)$$

Proof. We let's observe that

$$Col\left(\frac{p}{q}\right) = \begin{cases} \dfrac{\frac{p}{q}+1}{2} & \text{if } p = 1 \mod 2 \\ \dfrac{p}{2q} & \text{if } p = 0 \mod 2 \end{cases} = \frac{1}{q}\begin{cases} \dfrac{3p+q}{2} & \text{if } p = 1 \mod 2 \\ \dfrac{p}{2} & \text{if } p = 0 \mod 2 \end{cases} = \frac{1}{q} Col_q(p)$$

Suppose first that $(q,3) = 1$. This fraction is irreducible. Indeed, we have that $(3p+q,q) = (3p,q) = 1$. Then the parity of the fraction depends only on the numerator since there is no possibility of simplification that changes the parity of the numerator, and we can continue with the iteration for all k since the irreducibility of the iterations only depends on the initial fraction is irreducible. Then we have

$$Col^k\left(\frac{p}{q}\right) = \frac{1}{q} Col_q^k(p)$$

Now to suppose that $(q,3) \neq 1$, for this case, the resulting fraction is not irreducible. However, as we are going to prove below, this does not change the parity of the orbits, so the formula would continue to be valid for this case. Suppose that, $q = 3^l v$ with $(v,2) = (v,p) = 1$ and let $Col_q^k(p) = \frac{3^b p + 3^l v N}{2^a}$. We will divide this proof into two parts.

Case one $b \leq l$**:** We are going to prove the statement by induction. To $k = 1$

$$\frac{1}{3^l v} Col_{3^l v}(p) = \frac{1}{3^l v}\left(\frac{3p + 3^l v}{2}\right) = \frac{3\left(\frac{p}{3^l v}\right) + 1}{2} = Col\left(\frac{p}{3^l v}\right)$$

Now suppose that the statement is true for k, observe before continuing that the expressions $\left(\frac{3^b p+3^l vN}{2^a}\right)$ and $\left(\frac{p+3^{l-b}vN}{2^a}\right)$ have the same parity. Indeed,

$$\frac{3^b p+3^l vN}{2^a}=3^b\left(\frac{p+3^{l-b}vN}{2^a}\right)$$

if the expression on the left-hand side is even, if and only if $\left(\frac{p+3^{l-b}vN}{2^a}\right)$ it is even. On the other hand, if the left side is odd, $\left(\frac{p+3^{l-b}vN}{2^a}\right)$ must be odd and if $\left(\frac{p+3^{l-b}vN}{2^a}\right)$ is odd, since the product of odd is odd, the left side is odd, so the expressions have the same parity.

1. if $\left(\frac{3^b p+3^l vN}{2^a}\right)$ it is odd. Expanding the left-hand side of the proposition,

$$\begin{aligned}\frac{1}{3^l v}Col^{k+1}(p)=\frac{1}{3^l v}\psi\left(\frac{3^b p+3^l vN}{2^a}\right)&=\frac{1}{3^l v}\left(\frac{3^{b+1}p+3^{l+1}vN+2^a 3^l v}{2^{a+1}}\right)\\&=\frac{1}{3^{l-(b+1)}v}\left(\frac{p+3^{l+1-(b+1)}vN+2^a 3^{l-(b+1)}v}{2^{a+1}}\right)\\&=\frac{1}{3^{l-(b+1)}v}\left(\frac{p+3^{l-b}vN+2^a 3^{l-(b+1)}v}{2^{a+1}}\right)\end{aligned}$$

developing the right-hand side of the proposition,

$$\begin{aligned}Col^{k+1}\left(\frac{p}{3^l v}\right)=Col\left(Col^k\left(\frac{p}{3^l v}\right)\right)&=Col\left(\frac{1}{3^l v}\left(\frac{3^b p+3^l vN}{2^a}\right)\right)\\&=Col\left(\frac{1}{3^{l-b}v}\left(\frac{p+3^{l-b}vN}{2^a}\right)\right)\\&=\frac{1}{2}\left(\frac{3}{3^{l-b}v}\left(\frac{p+3^{l-b}vN}{2^a}\right)+1\right)\\&=\frac{1}{2}\left(\frac{1}{3^{l-(b+1)}v}\left(\frac{p+3^{l-b}vN+2^a 3^{l-(b+1)}v}{2^a}\right)\right)\\&=\frac{1}{3^{l-(b+1)}v}\left(\frac{p+3^{l-b}vN+2^a 3^{l-(b+1)}v}{2^{a+1}}\right)\end{aligned}$$

We conclude in this case that both parts are equal

2. if $\left(\frac{3^b p+3^l vN}{2^a}\right)$ it is even. Expanding the left-hand side of the proposition,

$$\begin{aligned}\frac{1}{3^l v}Col^{k+1}(p)=\frac{1}{3^l v}\theta\left(\frac{3^b p+3^l vN}{2^a}\right)&=\frac{1}{3^l v}\left(\frac{3^b p+3^l vN}{2^{a+1}}\right)\\&=\frac{1}{3^{l-p}v}\left(\frac{p+3^{l-b}vN}{2^{a+1}}\right)\end{aligned}$$

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developing the right-hand side of the proposition,

$$\begin{aligned}
Col^{k+1}\left(\frac{p}{3^l v}\right) = Col\left(Col^k\left(\frac{p}{3^l v}\right)\right) &= Col\left(\frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^a}\right)\right) \\
&= Col\left(\frac{1}{3^{l-b} v}\left(\frac{p + 3^{l-b} vN}{2^a}\right)\right) \\
&= \frac{1}{3^{l-b} v}\left(\frac{p + 3^{l-b} vN}{2^{a+1}}\right)
\end{aligned}$$

We conclude in this case that both parts are equal. Since in both cases it gave equality, we conclude that the proposition is true.

Case two $b \geq l$**:** We are going to prove the statement by induction. To $k = 1$

$$\frac{1}{3v} Col_{3v}(p) = \frac{1}{3v}\left(\frac{3p + 3v}{2}\right) = \frac{3\left(\frac{p}{3}\right) + 1}{2} = Col\left(\frac{p}{3v}\right)$$

Now suppose that the statement is true for k, observe before continuing that the expressions $\left(\frac{3^b p + 3^l vN}{2^a}\right)$ and $\left(\frac{3^{b-l} p + vN}{2^a}\right)$ have the same parity. Indeed,

$$\frac{3^b p + 3^l vN}{2^a} = 3^l\left(\frac{3^{b-l} p + vN}{2^a}\right)$$

if the expression on the left-hand side is even, if and only if $\left(\frac{3^{b-l} p + vN}{2^a}\right)$ it is even. On the other hand, if the left side is odd, $\left(\frac{3^{b-l} p + vN}{2^a}\right)$ must be odd and if $\left(\frac{3^{b-l} p + vN}{2^a}\right)$ is odd since the product of odd is odd, the left side is odd, so the expressions have the same parity.

1. if $\left(\frac{3^b p + 3^l vN}{2^a}\right)$ it is odd. Expanding the left-hand side of the proposition,

$$\begin{aligned}
\frac{1}{3^l v} Col^{k+1}(p) = \frac{1}{3^l v}\psi\left(\frac{3^b p + 3^l vN}{2^a}\right) &= \frac{1}{3^l v}\left(\frac{3^{b+1} p + 3^{l+1} vN + 2^a 3^l v}{2^{a+1}}\right) \\
&= \frac{3^{b+1-l} p + 3vN + 2^a v}{2^{a+1} v}
\end{aligned}$$

developing the right-hand side of the proposition,

$$\begin{aligned}
Col^{k+1}\left(\frac{p}{3^l v}\right) = Col\left(Col^k\left(\frac{p}{3^l v}\right)\right) &= Col\left(\frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^a}\right)\right) \\
&= Col\left(\frac{3^{b-l} p + vN}{2^a v}\right) \\
&= \frac{1}{2}\left(3\left(\frac{3^{b-l} p + vN}{2^a v}\right) + 1\right) \\
&= \frac{3^{b+1-l} p + 3vN + 2^a v}{2^{a+1} v}
\end{aligned}$$

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We conclude in this case that both parts are equal.

2. if $\left(\dfrac{3^b p + 3^l vN}{2^a}\right)$ it is even. Expanding the left-hand side of the proposition,

$$\begin{aligned}\frac{1}{3^l v} Col^{k+1}(p) = \frac{1}{3^l v}\theta\left(\frac{3^b p + 3^l vN}{2^a}\right) &= \frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^{a+1}}\right)\\ &= \frac{3^{b-l}p + vN}{2^{a+1}v}\end{aligned}$$

developing the right-hand side of the proposition,

$$\begin{aligned}Col^{k+1}\left(\frac{p}{3^l v}\right) = Col\left(Col^k\left(\frac{p}{3^l v}\right)\right) &= Col\left(\frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^a}\right)\right)\\ &= Col\left(\frac{3^{b-l}p + vN}{2^a v}\right)\\ &= \frac{3^{b-l}p + vN}{2^{a+1}v}\end{aligned}$$

We conclude in this case that both parts are equal. Since in both cases it gave equality, we conclude that the proposition is true.

□

We will define a coding function for the Collatz q-functions and demonstrate that they produce the same coding as the fractions with denominator q.

We are going to consider the set of sequences 0 and 10 that we will denote by Σ_2^* and we formally define it as

$$\Sigma_2^* = \left\{\{\xi_j\}_{j=1}^{\infty}, \xi_j \in \{0, 10\}\right\}$$

this set can be seen as a subset of the set of sequences 0 and 1 where after the entry 1 enters 0. Let's consider the following application: $Cod^k : \mathbb{Q}_{odd} \to \Sigma_2^*$ defined by

$$Cod^k\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{k} 0^{\infty}$$

with

$$\xi_j = \begin{cases} 10 & \text{if} \quad Col^{j-1}\left(\dfrac{p}{q}\right) \text{ is odd} \\ 0 & \text{if} \quad Col^{j-1}\left(\dfrac{p}{q}\right) \text{ is even} \end{cases}$$

and

Proposition 10 (Second Cod invariance)**.** *Let* $\dfrac{p}{q} \in \mathbb{Q}$ *an irreducible fraction with* $(q, 2) = 1$ *and* $Cod^k :$ $\mathbb{Z} \to \Sigma_2^*$ *defined by*

$$Cod_q^k(p) = \{\xi_j\}_{j=1}^{k} 0^{\infty}, \textit{ with } \xi_i \in \{10, 0\}$$

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with

$$\xi_j = \begin{cases} 10 & \text{if} \quad Col_q^{j-1}(p) = 1 \mod 2 \\ 0 & \text{if} \quad Col_q^{j-1}(p) = 0 \mod 2 \end{cases}$$

then we have

$$Cod^k\left(\frac{p}{q}\right) = Cod_q^k(p)$$

Proof. By proposition 9 we have

$$Col^k\left(\frac{p}{q}\right) = \frac{1}{q}Col_q^k(p)$$

Since q it is odd, then, we have coding of $Col^k\left(\frac{p}{q}\right)$ and $Col_q^k(p)$ must be the same. □

We will now establish the initial connection between sets of integers and coding. Specifically, we will demonstrate that all elements within the integer set S share the same coding.

Proposition 11 (First characterization of $\mathbb{E}(S)$)**.** *Let $p \in \mathbb{Z}$ and $S(x) \in \langle \theta, \psi^q \rangle$ of length k with $(q, 2) = 1$, then*

$$p \in \mathbb{E}(S) \textit{ if and only if } Cod_q^k(p) = Cod_q(S)$$

Proof. Let $p \in \mathbb{E}(S)$ and $S(x) = \bigodot_{i=1}^{k} s_i(x) \in \langle \theta, \psi^q \rangle$ then by definition $S(p) \in \mathbb{Z}$ by Proposition 1 we have $\bigodot_{i=1}^{l} s_i(p) \in \mathbb{Z}$ with $l \leq k$, then $Cod(S) = Cod_q^k(p)$.

Suppose that $Cod_q^k(p) = Cod_q(S)$ then $\{p, s_1(p), s_2 \circ s_1(p), \ldots, \bigodot_{i=1}^{k-1} s_i(p), \bigodot_{i=1}^{k} s_i(p) = S(p)\} \in \mathbb{Z}^{k+1}$, then $p \in \mathbb{E}(S)$. □

We show below the second connection between the integer sets and the encoding. Specifically, we demonstrate that all values p within the integer set S_q indeed have the same coding as the corresponding fraction $\frac{p}{q}$.

Proposition 12 (second characterization of $\mathbb{E}(S)$)**.** *Let $p \in \mathbb{Z}$ and $S(x) \in \langle \theta, \psi^q \rangle$ of length k with $(q, 2) = 1$, then we have:*

$$p \in \mathbb{E}(S) \textit{ if and only if } Cod^k\left(\frac{p}{q}\right) = Cod_q(S)$$

Proof. Let $S(x) \in \langle \theta, \psi^q \rangle$ of length k such that $Cod^k(p) = Cod_q(S)$, for the proposition 9, we have

$$qCol^k\left(\frac{p}{q}\right) = Col_q^k(p) = S(p)$$

then $Cod^k\left(\frac{n}{q}\right) = Cod_q(S)$ finally by the proposition 11, we have $p \in \mathbb{E}(S)$ if and only if $Cod^k\left(\frac{p}{q}\right) = Cod_q(S)$. □

The following proposition demonstrates that for a given rational number, we can generate a family of rationals that share the same encoding. This suggests that there exist many rationals with the same k-th encoding

Example 7. *Let's consider the coding $\xi = 10010100$, we want to find rational numbers $\frac{p}{q}$ such that $Cod^5\left(\frac{p}{q}\right) = \xi$, we have that the function $S(x) \in \langle \theta, \psi \rangle$ with coding ξ is $S(x) = \dfrac{27x+29}{32}$*

1. *$p = 1$, we have to calculate some solution of the entire set of $S_7(x) = \dfrac{27x+29}{32}$. We have $x_0 = 25$, then $\mathcal{O}_5(25) = \{25, 38, 19, 29, 44, 22\}$ then $Cod^5\left(\dfrac{15}{7}\right) = \xi$.*
2. *$p = 7$, we have to calculate some solution of the entire set of $S_7(x) = \dfrac{27x+29\cdot 7}{32}$. We have $x_0 = 15$, then $\mathcal{O}_5\left(\dfrac{15}{7}\right) = \left\{\dfrac{15}{7}, \dfrac{26}{7}, \dfrac{13}{7}, \dfrac{23}{7}, \dfrac{38}{7}, \dfrac{19}{7}\right\}$ then $Cod^5\left(\dfrac{15}{7}\right) = \xi$.*
3. *$p = 27$, we have to calculate some solution of the entire set of $S_{27}(x) = \dfrac{27x+29\cdot 27}{32}$. We have that $x_0 = 3$, then $\mathcal{O}_5\left(\dfrac{3}{27}\right) = \left\{\dfrac{3}{27}, \dfrac{2}{3}, \dfrac{1}{3}, 1, 2, 1\right\}$ then we have that $Cod^5\left(\dfrac{3}{27}\right) = \xi$.*

Proposition 13 (Invariance property of Coding of rational)**.** *Let $\frac{p}{q} \in \mathbb{Q}$ an irreducible fraction with $(q, 2) = 1$, A_k =numbers from 0 to $Cod^k\left(\frac{p}{q}\right)$ and $T \in \mathbb{Z}$ then*

$$Cod^k\left(\frac{p}{q}\right) = Cod^k\left(\frac{p}{q} + \frac{2^{A_k}T}{q}\right)$$

.

Proof. Let $S(x) \in \langle \theta, \psi^q \rangle$ such that $Cod(S) = Cod^k_q(p)$ then $S(p) = Col^k_q(p) \in \mathbb{Z}$ this implies

$$\mathbb{E}(S) = p + 2^{A_k}T \Rightarrow Cod^k_q(p + 2^{A_k}T) = Cod^k_q(p)$$

then

$$Col^k\left(\frac{p+2^{A_k}T}{q}\right) = Col^k\left(\frac{p}{q} + \frac{2^{A_k}T}{q}\right) = Cod^k\left(\frac{p}{q}\right)$$

□

As we have seen so far, we can characterize the entire set S from its encoding. Exploiting this property, we generalize the entire set S to encompass all fractions sharing the same encoding. We will call the $k-$Coding set.

Definition 10 (The $k-$Coding set)**.** *Let $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$, we define the k-th coding set of $\{\xi_j\}_{j=1}^{\infty}$*

$$Cod^k\{\xi_j\}_{j=1}^{\infty} = \left\{\frac{p}{q} \in \mathbb{Q}_{odd}, Cod^k\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{k} 0^{\infty}\right\}$$

The encoding set also exhibits the property of monotony, similar to the integer set of S.

Proposition 14 (Monotony of the Coding set)**.** *Let $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$ then*

$$Cod^{k+1}\{\xi_j\}_{j=1}^{\infty} \subset Cod^k\{\xi_j\}_{j=1}^{\infty}.$$

Proof. Let $\frac{p}{q} \in Cod^{k+1}\{\xi_j\}_{j=1}^{\infty}$ by definition $Cod^{k+1}\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{k+1}0^{\infty}$ then trivially we have $Cod^{k}\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{k}0^{\infty}$, then $\frac{p}{q} \in Cod^{k}\{\xi_j\}_{j=1}^{\infty}$. □

Definition 11 (The Coding set)**.** *Let $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$, we define the k-th coding set of $\{\xi_j\}_{j=1}^{\infty}$ the coding set of $\{\xi_j\}_{j=1}^{\infty}$*

$$Cod\{\xi_j\}_{j=1}^{\infty} := \bigcap_{k\in\mathbb{N}} Cod^{k}\{\xi_j\}_{j=1}^{\infty}.$$

Similarly, the behavior of the solutions of Diophantine equations, in which knowing a particular solution allows us to determine other solutions, is reflected in the coding set. This connection is illustrated in the following proposition.

Proposition 15 (Generating property)**.** *Let $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$, A_k =numbers from 0 to $\{\xi_j\}_{j=1}^{k}$ and $\frac{p}{q}, \frac{t}{r} \in Cod^{k}\{\xi_j\}_{j=1}^{\infty}$ then exist $T \in \mathbb{Z}$ such that*

$$\frac{p}{q} = \frac{t}{r} + \frac{2^{A_k}T}{qr}$$

Proof. Let $\frac{p}{q}, \frac{t}{r} \in Cod^{k}\{\xi_j\}_{j=1}^{\infty}$ and $S_k(x) = \frac{3^b x + N}{2^{A_k}} \in \langle\theta, \psi\rangle$ such that $Cod(S_k) = \{\xi_j\}_{j=1}^{k}$, now consider $S_k^q \in \langle\theta, \psi^q\rangle$ and $S_k^r \in \langle\theta, \psi^r\rangle$ such that $Cod_q^k(S_k^q) = Cod_r^k(S_k^r) = \{\xi_j\}_{j=1}^{k}$ by proposition 12 we have $S_k^q(p), S_k^r(t) \in \mathbb{Z}$ the latter is equivalent

$$\frac{3^b p + qN}{2^{A_k}}, \frac{3^b t + rN}{2^{A_k}} \in \mathbb{Z}$$

We are going to prove that pr and qt are elements of $\mathbb{E}(S_k^{qr})$ with $s^r(x) = \frac{3^b x + qrN}{2^{A_k}}$. Indeed,

$$\frac{3^b pr + qrN}{2^{A_k}} = r\left(\frac{3^b p + qN}{2^{A_k}}\right) \in \mathbb{Z}$$

and

$$\frac{3^b qt + qrN}{2^{A_k}} = q\left(\frac{3^b t + rN}{2^{A_k}}\right) \in \mathbb{Z}$$

then

$$pr = qt + 2^{A_k}T \text{ with } T \in \mathbb{Z} \Rightarrow \frac{p}{q} = \frac{t}{r} + \frac{2^{A_k}T}{qr} \text{ with } T \in \mathbb{Z}$$

□

Now, we will present the main theorem of this section, establishing that the encoding of a rational number is unique.

Theorem 3 (Uniqueness of the full coding on $\mathbb{Q}_{odd}$)**.** *Let $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$. If it exists $\frac{p}{q} \in \mathbb{Q}_{odd}$ such that $Cod\left(\frac{p}{q}\right) := \lim_{k\to\infty} Cod^{k}\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{\infty}$ then it is unique.*

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Proof. Let A_k =numbers from 0 to $\{\xi_j\}_{j=1}^{k}$. Suppose there is another element, $\frac{t}{r} \in \mathbb{Q}_{odd}$ such than $Cod\left(\frac{t}{r}\right) = \lim_{k\to\infty} Cod^k\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{\infty}$ by proposition 15 exist $\{T_k\}_{k\in\mathbb{N}} \in \mathbb{Z}$ such that

$$\frac{t}{r} = \frac{p}{q} + \frac{2^{A_k}T}{rq} \text{ with } T \in \mathbb{Z} \text{ and } k \in \mathbb{N}$$

Since $\frac{p}{q} \neq \frac{t}{r}$ then $T_k \in \mathbb{Z}$ for all $k \in \mathbb{N}$. So

$$rq\left(\frac{t}{r} - \frac{p}{q}\right) = 2^{A_k}T_k \text{ then } T_k = \frac{rq\left(\frac{t}{r} - \frac{p}{q}\right)}{A_k} \to 0 \text{ as } k \to \infty$$

which is a contradiction. □

6. The Sigma Function

In this section, we immerse ourselves in the study of Diophantine equations of the form $2^k y - ax = n$, where a, k, n are integers. Solving these equations in the domain of integers x and y is a problem in number theory. Usually, these types of Diophantine equations are solved using Euclid's algorithm or some similar technique, even by trial and error. However, these techniques begin to have a high degree of complexity for very large values. This mainly complicates when we want to study the behavior of the minimum positive values since in this case, we are interested in asymptotic solutions. We introduce the sigma function, symbolized as $\sigma_a^k(n)$ to address this challenge. This function, whose detailed analysis will constitute the core of our research, plays a fundamental role in the quest for specific solutions to the aforementioned Diophantine equations. Particularly noteworthy is the sigma function's remarkable property of delivering solutions that are closest to zero in the context of these equations.

6.1. Summary of Propositions in the Section

1. **Definition 12:** Definition of the sigma function.
2. **Theorem 4:** Establish that $\sigma_a^k(n)$ and $\frac{1}{a}(2^k\sigma_a^k(n) - n)$ are solutions of the Diophantine equation $2^k y - ax = n$. Additionally, $\frac{1}{a}(2^k\sigma_a^k(n) - n)$ is the minimum non-negative integer value.
3. **Corollary 1:** Establishes that the minimum value grows based on the number of times the sigma function takes odd values.
4. **Corollary 2:** $\sigma_a^k(n) - \sigma_{-a}^k(n) = a$
5. **Proposition 16:** Establishes inequalities that estimate the values of the sigma function
6. **Proposition 17:** It establishes the periods for the periodic points.
7. **Proposition 18:** Establish algebraic properties of additivity, dependent on the parity of the addends
8. **Corollary 3** Establish algebraic properties' linearity modulo a.
9. **Proposition 19** Establish that the sigma function is homogeneous modulo a.
10. **Definition 13:** Extension of the sigma function on $\mathbb{Q}_{odd}$
11. **Definition 14:** Characteristic Function
12. **Lemma 3:** Establishes an invariance in the coding of the orbits of the sigma function.
13. **Proposition 21:** Establishes homogeneity properties of the extension of the sigma function.
14. **Proposition 22** Algebraic properties of the Extension of the Sigma function.
15. **Definition 15:** Definition of dyadic numbers.
16. **Proposition 23:** Characterization of the dyadic representation of rational numbers.
17. **Definition 16:** Definition of Cod-Sigma function.
18. **Lemma 4:** Invariant coding lemma for Cod-Sigma function.

19. **Proposition 24:** Change of basis of the Cod-Sigma function.
20. **Proposition 25:** Let $r \in \mathbb{N}$ and $\gamma = 3^{b-1}$ and let $k \in \mathbb{N}$ such that $r < 2^{2\gamma k}$ and $\delta_j \in \{0,1\}$ such that $r\left\{\dfrac{2^{2\gamma k}-1}{3\gamma}\right\} = \sum_{j=1}^{2\gamma k-1} \delta_j 2^j$ then $Cod\sigma_{3\gamma}^{2\gamma k}(r) = \delta_{2\gamma k-1}, \ldots, \delta_0$.
21. **Corollary 5:** Let $r_1, r_2 \in \mathbb{N}$. Then $Cod\sigma_{3^b}^{a}(r_1+r_2) = Cod\sigma_{3^b}^{a}(r_1) + Cod\sigma_{3^b}^{a}(r_2) \mod 2^a$.
22. **Proposition 26** $Cod\sigma_{3^b} : \mathbb{N} \to \mathbb{Z}_2$ is linear.
23. **Lemma 5:** Rational equivalence of the Cod-Sigma function.
24. **Definition 17:** we will say that $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$ has a null tail of index J if $J > 0$ the smallest index such that $j > J$ we have $\xi_j = 0$.
25. **Proposition 27:** $-\pi^1(\xi) \in \mathbb{Z}_2$ and $Cod\sigma(\pi^1(\xi)) = -\pi^1(\xi) \in \mathbb{Z}_2$.
26. **Lemma 6:** $\rho(S_k) = Cod\sigma_{3^k}^{A_k}(N_k)$.
27. **Proposition 28** $\rho_0(S_k) = Cod\sigma\Big(\pi^1(\xi)\Big) \mod 2^{a_k}$.

6.2. The Sigma Function

We are going to define the sigma function. This function is very similar to the Collatz function except that in this function, we do not multiply by 3.

Definition 12 (The Sigma function)**.** *Let $x, a \in \mathbb{Z}$ such that $(a,2) = 1$. We define the sigma function $\sigma_a : \mathbb{Z} \to \mathbb{Z}$*

$$\sigma_a(x) = \begin{cases} \dfrac{x+a}{2} & \text{if } x = 1 \mod 2 \\ \\ \dfrac{x}{2} & \text{if } x = 0 \mod 2 \end{cases}$$

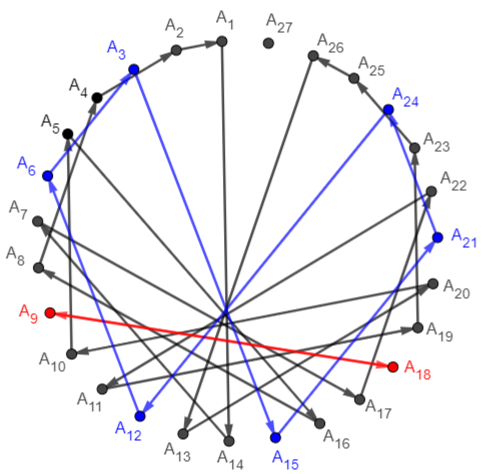

Figure 1. Orbits of σ_{27} for $0 < x \leq 27$

In the following theorem, we explore solutions to the Diophantine equation $2^k y - ax = n$, where a, k, and n are integers. This equation arises frequently in number theory, particularly in the study of Diophantine equations. We'll demonstrate that the sigma function provides particular solutions for y, shedding light on the behavior of solutions in both positive and negative domains. Additionally, we'll establish formulas for the smallest non-negative solution ρ_0 and the largest non-positive solution ρ_1 for the variable x, offering valuable insights into the structure of solutions to this equation.

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Theorem 4 (Theorem on Diophantine Solutions). *Let $a, k \in \mathbb{N}$ with $(a, 2) = 1$ and $n \in \mathbb{Z}$. Consider the Diophantine Equation $2^k y - ax = n$. Then a particular solution for y is given by*

$$\sigma_a^k(n) \text{ and } \sigma_{-a}^k(n).$$

where $\sigma_a^k(n) = \underbrace{\sigma_a \circ \ldots \circ \sigma_a(n)}_{k-times}$ and $\sigma_{-a}^k(n) = \underbrace{\sigma_{-a} \circ \ldots \circ \sigma_{-a}(n)}_{k-times}$. Furthermore. Let ρ_0 be the smallest non-negative solution for x, then

$$\rho_0 = \frac{1}{a}(2^k \sigma_a^k(n) - n)$$

let ρ_1 be the largest non-positive solution for x if then

$$\rho_1 = \frac{1}{a}(2^k \sigma_{-a}^k(n) - n)$$

Proof. We can write the sigma function as

$$\sigma_{\pm a}(n) = \frac{n \pm a\delta(n)}{2} \text{ where } \delta(n) = \begin{cases} 1 & \text{if } n = 1 \mod 2 \\ 0 & \text{if } n = 0 \mod 2 \end{cases}$$

Since the sigma function is defined on the set of integers in the integers, we have that its k-th composition is also an integer value: Let $\delta_j = \delta(\sigma_a^j(n))$ then

$$\sigma_a^k(n) = \frac{\dfrac{\dfrac{n+\delta_0}{2}}{\ddots} + a\delta_k}{2} = \frac{n + aL}{2^k} \in \mathbb{Z} \text{ where } L = \sum_{j=1}^{k-1} \delta_j 2^j$$

and Let $\varepsilon_j = \delta(\sigma_{-a}^j(n))$ then

$$\sigma_{-a}^k(n) = \frac{\dfrac{\dfrac{n-a\varepsilon_0}{2}}{\ddots} - a\varepsilon_{k-1}}{2} = \frac{n - aU}{2^k} \in \mathbb{Z} \text{ where } U = \sum_{j=1}^{k-1} \varepsilon_j 2^j$$

replacing the k-th iteration sigma function σ_a in the equation $2^k y - ax = n$ and solving for x_0, we have

$$x_0 = \frac{1}{a}(2^k \sigma_a^k(n) - n) = L \in \mathbb{Z}$$

and replacing the k-th iteration sigma function σ_{-a} in the equation $2^k y - ax = n$ and solving for x_0, we have

$$x_0 = \frac{1}{a}(2^k \sigma_{-a}^k(n) - n) = -U \in \mathbb{Z}$$

For the positive case, we have that $0 \leq L = \sum_{j=1}^{k-1} \delta_j 2^j \leq 2^k - 1$, then due to the uniqueness of solutions in $[0, 2^k) \cap \mathbb{Z}$, L corresponds to the non-negative minimum value and for the negative case we have $0 \geq -U - \sum_{j=1}^{k-1} \varepsilon_j 2^j \geq -(2^k - 1)$, again due to uniqueness of solutions in $(2^k, 0] \cap \mathbb{Z}$, we have that $-U$ is the maximum non-positive solution. $\square$

Example 8. *Let us consider the following Diophantine equation* $16y - 7x = 45$ *then*

$$\sigma_7^4(45) = 5 \text{ and } \frac{1}{7}(16 \cdot 5 - 45) = 5$$

are solutions of the equation.

We will demonstrate that this minimum value increases every time $\sigma_a^k(n)$ is an odd number. This result is crucial for understanding how the parity of the sigma function influences the structure of non-negative solutions of the associated Diophantine equation.

Corollary 1 (Monotony relation)**.** *Let* $a \in \mathbb{N}$ *such that* $(a, 2) = 1$ *and* $S_a(x) = \dfrac{ax + n}{2^k}$ *and* $\rho_0(S_a)$ *the minimum non-negative value of* $S_a(x)$*. Then* $\rho_0(S_a)$ *increases every time* $\sigma_a^k(n)$ *is an odd number. In particular* $\rho_0(S_a) = \sum_{j=1}^{k-1} \delta_j 2^j$ *with* $\delta_j = 0$ *if* $\sigma_a^{j-1}(n)$ *is even and* $\delta_j(n) = 1$ *if* σ_a^{j-1} *is odd.*

Proof. Let $S_a(x) = \dfrac{ax + n}{2^k}$ and $\delta_j = \delta(\sigma_a^j(n))$ then by Theorem 4 we have

$$\sigma_a^k(n) = \frac{1}{2^k}\left\{ n + a \sum_{j=1}^{k-1} \delta_j 2^j \right\}$$

then $S_a^{-1}(\sigma_a^k(n)) = \rho_0(S_a) = \sum_{j=1}^{k-1} \delta_j 2^j$. So, we have that every time $\delta_j = 1$, the minimum positive integer value increases, and this only happens when $\sigma_a^{j-1}(n)$ is odd.
□

In the following corollary, we explore the relationship between the sigma functions $\sigma_a^k(n)$ and $\sigma_{-a}^k(n)$ in the context of the Diophantine equation $2^k y - ax = n$.

Corollary 2 (Relation between $\sigma_a^k(n)$ and $\sigma_{-a}^k(n)$)**.** *Let* $a, k \in \mathbb{N}$ *and* $n \in \mathbb{Z}$*. Consider the Diophantine Equation,* $2^k y - ax = n$*, then*

$$\sigma_a^k(n) - \sigma_{-a}^k(n) = a$$

Proof. By definition, we have that ρ_0 is the nearest non-negative solution to 0, and ρ_1 is the nearest non-positive solution to 0, which means that ρ_0 and ρ_1 are consecutive solutions. Therefore, $\rho_0 - 2^k = \rho_1$. then we have

$$2^k = \rho_0(S) - \rho_1(S) = \frac{1}{a}\left\{2^k \sigma_a^k(n) - n\right\} - \frac{1}{a}\left\{2^k \sigma_{-a}^k(n) - n\right\} = \frac{1}{a} 2^k \sigma_a^k(n) - \frac{1}{a} 2^k \sigma_{-a}^k(n)$$

Therefore $\sigma_a^k(n) - \sigma_{-a}^k(n) = a$ □

In the following proposition, we examine the inequalities and estimations for the sigma function $\sigma_a^k(n)$ and $\sigma_{-a}^k(n)$, where n is an integer. We show that the sigma function lies in the interval $\left[\dfrac{n}{2^k}, \dfrac{n}{2^k} + a\right)$ for $\sigma_a^k(n)$, and in the interval $\left[\dfrac{n}{2^k} - a, n\right)$ for $\sigma_{-a}^k(n)$. These inequalities are fundamental to understand the range of values the sigma function can take in the context of the considered Diophantine equations.

Proposition 16 (Inequality and estimation of the sigma function)**.** *Let* $a, k \in \mathbb{N}$ *and* $n \in \mathbb{Z}$*. Then,*

$$\frac{n}{2^k} \le \sigma_a^k(n) < \left(\frac{n}{2^k} + a\right) \text{ and } \frac{n}{2^k} - a < \sigma_{-a}^k(n) \le \frac{n}{2^k}$$

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Proof. For $\sigma_a^k(n)$ we have two possible extreme paths, either we always get even or we always get odd, for the first case we would always have division by 2

$$\frac{n}{2^k} \leq \sigma_a^k(n)$$

for the second we would have

$$\sigma_a^k(n) \leq \frac{n + a(1 + 2 + \ldots + 2^{k-1})}{2^k} = \frac{n}{2^k} + a\left(\frac{1 + 2 + \ldots + 2^{k-1}}{2^k}\right) = \frac{n}{2^k} + a\left(\frac{2^k - 1}{2^k}\right) < \frac{n}{2^k} + a$$

For $\sigma_a^k(n)$, regardless of the cases, we always get a less stringent value to the initial value. If it is always even, we will have that it is always divided by 2, now in the case that it is always odd we have

$$\sigma_{-a}^k(n) \geq \frac{n - a(1 + 2 + \ldots + 2^{k-1})}{2^k} = \frac{n}{2^k} - a\left(\frac{1 + 2 + \ldots + 2^{k-1}}{2^k}\right) > \frac{n}{2^k} - a$$

and clearly, we have

$$\frac{n}{2^k} > \frac{n}{2^k} - a$$

□

6.3. Periodicity of the Sigma Function

Proposition 17 (Periodicity of periodic orbits). *Let $a \in \mathbb{N}$ The sigma function σ_a has the following properties,*

1. *The only fixed points are a and 0.*
2. *If, $n < a$ then, its orbit by is periodic with period given by*

$$\lambda\left(\frac{a}{(a,u)}\right) = \begin{cases} \varphi\left(\frac{a}{(a,u)}\right) & \text{if } \frac{a}{(a,u)} \text{ is prime} \\ [\varphi(p_1^{\alpha_1}), \ldots, \varphi(p_r^{\alpha_r})] & \text{if } \frac{a}{(a,u)} = \prod_{i=1}^{r} p_i^{\alpha_i} \text{ with } p_i \text{ prime and } \alpha_i \in \mathbb{N} \end{cases}$$

where φ is the Euler's totient function.
In particular, if $a = 3^r$ and $(n,3) = 1$ then the periodic is $3^{r-1}2$. Let $\gamma = 3^{r-1}$ then

$$\sigma_{3\gamma}^{2\gamma}(n) = n$$

In particular, all points terminate in some periodic orbit (including periodic points) between 0 and a.

Proof. we have

1. Let $\sigma_a(u) = u$, if u is odd, then $\frac{u+a}{2} = u$ which implies $u = a$. If u is even, we have, $\frac{u}{2} = u$ which implies $u = 0$.
2. Let $u \in \mathbb{N}$ such that $u < a$ and $\sigma_a^k(u) = u$, so

$$\frac{u + aL}{2^k} = u \text{ for any } L = \sum_{j=1}^{k-1} \delta_j 2^j \text{ where } \delta_j \in \{0, 1\}$$

Then

$$u + aL = 2^k u \Rightarrow aL = (2^k - 1)u \Rightarrow (2^k - 1)u = 0 \mod a$$

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suppose that $(u, a) = 1$, this implies that u is an invertible $\mod a$ then, the equation is equivalent

$$2^k = 1 \mod a$$

The minimum value of k is given by the Carmichael function given by

$$\lambda(a) = \begin{cases} \varphi(a) & \text{if } a \text{ is prime} \\ \\ [\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})] & \text{if } a = \prod_{i=1}^{r} p_i^{\alpha_i} \text{ with } p_i \text{ prime and } \alpha_i \in \mathbb{N} \end{cases}$$

Let $k_0 = \lambda(a)$, then

$$(2^{k_0} - 1)u = 0 \mod a \Rightarrow (2^{k_0} - 1)u = L_0 a \text{ for any } L_0 \in \mathbb{N}$$

$$(2^{k_0} - 1) = L_0 \frac{a}{u}$$

as $\frac{a}{u} > 1$ then $(2^{k_0} - 1) > L_0$, which is the necessary and sufficient condition for L_0 to admit decomposition in base 2 up to the power $k_0 - 1$ which implies that there exist $\delta_j \in \{0, 1\}$ such that $L_0 = \sum_{j=1}^{k_0-1} \delta_j 2^j$.

Now suppose that $(u, a) = d > 1$, then we divide $aL = (2^k - 1)u$ by, d

$$\frac{a}{d} L = (2^k - 1)\frac{u}{d} \text{ where } \left(\frac{a}{d}, \frac{a}{d}\right) = 1$$

then the development is completely analogous to the first case.

In particular, when $a = 3^r$ and u are co-prime with 3, then the period of the orbit of u corresponds to the Euler's totient function, ϕ which in this case is $3^{r-1}2$.

□

Let us observe that for the equation $\sigma_a^k(u) = u$ to have a solution it is necessary and sufficient that $n < a$ since the function is monotonically decreasing for $n > a$.

Example 9. *For, $a = 7$ we have $\mathcal{O}(5) = \{5, 6, 3\}$ and $\mathcal{O}(4) = \{4, 2, 1\}$.*
For $a = 15$ we have $\mathcal{O}(1) = \{1, 8, 4, 2\}$, $\mathcal{O}(3) = \{3, 9, 12, 6\}$, $\mathcal{O}(5) = \{5, 10\}$ and $\mathcal{O}(7) = \{7, 11, 13, 14\}$

6.4. Linearity of the Sigma Function Modulo A

In this section, we address the linearity of the sigma function modulo a. Proposition 18 establishes the addition rules for the sigma function under different parity conditions of the involved numbers. We will see in the corollary 3 that the function sigma modulo a is an automorphism of $\mathbb{Z}/a\mathbb{Z}$ Furthermore, Proposition 19 establishing a relationship between $\sigma_a^k(m)$ and $\sigma_a^k(1)$.

Proposition 18 (Algebraic properties of the Sigma function)**.** *Let $a \in \mathbb{Z}$ and $\sigma_a : \mathbb{Z} \to \mathbb{Z}$, then we have*

1. *if n, m are even numbers, then $\sigma_a(n + m) = \sigma_a(n) + \sigma_a(m)$.*

2. *if n is an even number and m is an odd number, then $\sigma_a(n + m) = \sigma_a(n) + \sigma_a(m)$.*

3. *if n, m are odd numbers, then $\sigma_a(n + m) = \sigma_a(n) + \sigma_a(m) - a$.*

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Proof. Let $m.n \in \mathbb{Z}$, then we have

1. If m, n are even, we have

$$\sigma_a(m+n) = \frac{m+n}{2} = \frac{m}{2} + \frac{n}{2} = \sigma_a(m) + \sigma_a(n)$$

2. If m is even and n is odd, we have

$$\sigma_a(m+n) = \frac{m+n+a}{2} = \frac{m}{2} + \frac{n+a}{2} = \sigma_a(m) + \sigma_a(n)$$

3. If m, n are odd, we have

$$\sigma_a(m+n) = \frac{m+n}{2} = \frac{m}{2} + \frac{n}{2} + a - a = \frac{m+a}{2} + \frac{n+a}{2} - a = \sigma_a(m) + \sigma_a(n) - a.$$

□

This corollary states that the sigma function, seen as a function on the set $\mathbb{Z}/a\mathbb{Z}$ and taking values in $\mathbb{Z}/a\mathbb{Z}$, acts as a group additive automorphism. In other words, it preserves the group structure under modular addition in $\mathbb{Z}/a\mathbb{Z}$

Corollary 3 (Linearity modulo a)**.** *We consider the function sigma as a function of $\mathbb{Z}/a\mathbb{Z}$ in $\mathbb{Z}/a\mathbb{Z}$, then it is a group additive automorphism. i.e.*

$$\sigma_a^k(m+n) = \sigma_a^k(m) + \sigma_a^k(n) \mod a$$

Proof. From the previous proposition we have that the sigma function is linearly distributed except for a term $-a$ that appears when both addends are odd, this term is congruent to $0 \mod a$ □

This proposition establishes the concept of homogeneity modulo a for the sigma function. It relates the value of $\sigma_a^k(m)$ to $m\sigma_a^k(1)$ under modular arithmetic. This relationship highlights a consistent behavior of the sigma function concerning scaling by m, providing valuable insights into its algebraic properties.

Proposition 19 (Homogeneity mod a)**.** *Let $a, k, m \in \mathbb{N}$ such that $(a, 2) = 1$, then we have*

$$\sigma_a^k(m) = m\sigma_a^k(1) \mod a$$

Proof. Let $a, b \in \mathbb{N}$ such that $(a, 2) = 1$ and consider the following Diophantine Equation $2^k y - ax = m$. Since $(2^k, a) = 1$ we have that, this equation is equivalent to $2^k Y - aX = 1$. The Theorem 4 we have $\sigma_a^k(1)$ a particular solution of Y, then $m\sigma_a^k(1)$ is a solution for y of $2^k y - ax = m$, then

$$m\sigma_a^k(1) = \sigma_a^k(m) \mod a$$

□

Proposition 20 (Fundamental equation)**.** *Let $b, n \in \mathbb{N}$ such that $n < 3^b$ and $(n, 3^b) = 1$, $\gamma = 3^b$ then*

$$\sigma_{3\gamma}^{a}(n) + \sigma_{3\gamma}^{a+\gamma}(n) = 3\gamma$$

Proof. Let n co-prime and less than 3^b. Let's consider the following equation:

$$n + \sigma_{3^b}^k(n) = 3^b$$

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Let $T_k(n) = \sum_{j=1}^{k-1} \delta_j 2^j$, then

$$\frac{n + 3^b T_k}{2^k} = 3^b - n \Rightarrow n = -2^k n \mod 3^b$$
$$\Rightarrow 2^k = -1 \mod 3^b$$
$$\Rightarrow k = 3^{b-1}$$

Now we will prove that $T_{3^{b-1}} \leq 2^{3^{b-1}} - 1$, we have

$$T_{3^{b-1}} = \frac{2^{3^{b-1}}(3^b - n) - n}{3^b} = 2^{3^{b-1}} - n\left(\frac{2^{3^{b-1}} + 1}{3^b}\right)$$

on the other hand

$$n\left(\frac{2^{3^{b-1}} + 1}{3^b}\right) \geq \left(\frac{2^{3^{b-1}} + 1}{3^b}\right)$$

Let $f : \mathbb{R} \to \mathbb{R}$ given $f(x) = \frac{2^{3^{x-1}} + 1}{3^x}$ we have

1. $f(1) = \frac{2^{3^{1-1}} + 1}{3^1} = 1$ and $f(2) = \frac{2^{3^{2-1}} + 1}{3^2} = 1$,
2. $f'(x) = \frac{1}{3} 2^{3^{x-1}} \ln(2)\ln(3) - 3^{-x}(2^{3^{x-1}} + 1)\ln(3)$ and $f'(x) > 0$ if $x > 1{,}5572$. That is to say that the function from this point on is monotonically increasing, therefore, from 2 we have that the function is always greater than 1.

then

$$\frac{2^{3^{b-1}} + 1}{3^b} \geq 1 \text{ for all } b \in \mathbb{N} \Rightarrow T_k \leq 2^{3^{b-1}} - 1 \text{ for all } b \in \mathbb{N}$$

Then we have $n + \sigma_{3^b}^{3^{b-1}}(n) = 3^b$ for all n co-prime and less than 3^b, in particular we take $\sigma_{3^b}^a(n)$

$$\sigma_{3^b}^a(n) + \sigma_{3^b}^{3^{b-1}}(\sigma^a(n)) = 3^b$$

□

Corollary 4. *Let $b, n \in \mathbb{N}$ such that $n < 3^b$ and $(n, 3^b) = 1$, $\gamma = 3^b$ then*

$$\sigma_{3\gamma}^{\gamma}(n) = 3\gamma - n$$

Proof. By Proposition 17 and 20

$$\sigma_{3\gamma}^{\gamma}(n) + \sigma_{3\gamma}^{2\gamma}(n) = 3\gamma \Rightarrow \sigma_{3\gamma}^{\gamma}(n) = 3\gamma - n$$

□

6.5. *Extension on the $\mathbb{Q}_{odd}$ of the Sigma Function*

We can extend the domain of the sigma function to the set of rationals, in the following way,

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Definition 13 ($\mathbb{Q}_{odd}$-extension of the sigma function)**.** *Let* $\frac{u}{v}, \frac{x}{y} \in \mathbb{Q}_{odd}$. *We define the sigma function* $\sigma_{\frac{u}{v}} : \mathbb{Q}_{odd} \to \mathbb{Q}_{odd}$

$$\sigma_{\frac{u}{v}}\left(\frac{x}{y}\right) = \begin{cases} \frac{1}{2}\left(\frac{x}{y} + \frac{u}{v}\right) & \textit{if } x = 1 \mod 2 \\ \frac{x}{2y} & \textit{if } x = 0 \mod 2 \end{cases}$$

We are going to provide a numerical interpretation of the extension of the sigma function. Thus far, we understand that the sigma function provides us with the non-negative solution to the Diophantine equation $2^k y - ax = n$ through the equation $\frac{ax+n}{2^k} = \sigma_a^k(n)$. We can utilize the latter equation to extend the sigma function to the set of fractions with odd denominators, employing the following equation on $r \in \mathbb{Z}$:

$$\frac{yur + vx}{2^k} = vy\sigma^k_{\frac{u}{v}}\left(\frac{x}{y}\right)$$

or equivalently

$$\frac{\frac{u}{v}r + \frac{x}{y}}{2^k} = \sigma^k_{\frac{u}{v}}\left(\frac{x}{y}\right)$$

That is to say, the extension of the sigma function gives the fraction that solves the equation

$$\frac{u}{v}r + \frac{x}{y} = 2^k t \text{ with } r \in \mathbb{Z} \text{ and } t, \frac{u}{v}, \frac{x}{y} \in \mathbb{Q}_{odd}$$

6.6. Properties of the Extension of the Sigma Function

The introduction of the sigma function extended to odd rationals is crucial for understanding its behavior in a broader domain. This extension, defined on the set $\mathbb{Q}_{odd}$, allows us to explore the algebraic and arithmetic properties of the sigma function in a more general context. In this section, we delve into this extension and explore its implications, focusing on how the sigma function modifies its behavior when applied to fractions with odd denominators. Additionally, we present an important lemma that establishes an invariant relationship between the characteristic function δ and the sigma function, providing a deeper understanding of how the sigma function preserves certain properties under different transformations.

Definition 14 (Characteristic Function)**.** *We define the characteristic function* $\delta : \mathbb{Q}_{odd} \to \{0, 1\}$ *given by*

$$\delta\left(\frac{p}{q}\right) = \begin{cases} 1 & \textit{if } p = 1 \mod 2 \\ 0 & \textit{if } p = 0 \mod 2 \end{cases}$$

The Invariant Coding Lemma, stated in Lemma 3, establishes a fundamental relationship between the characteristic function δ and the sigma function under certain conditions. Specifically, it asserts that for co-prime integers u and v, with u being odd, the characteristic function δ remains invariant under iterations of the sigma function. This means that the parity of the output of $\sigma_u^j(v)$ is the same as the parity of $\sigma_1^j\left(\frac{v}{u}\right)$ for all non-negative integers j. Furthermore, if v is odd, the lemma demonstrates that the parity of $\sigma_u^j(v)$ is identical to the parity of $\sigma^j_{\frac{u}{v}}(1)$ for all non-negative integers j.

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Lemma 3 (Invariant Characteristic Function Lemma)**.** *Let $u, v \in \mathbb{Z}$ with u not null, such that $(u, v) = 1$ and $(u, 2) = 1$ then*

1.
$$\delta\left(\sigma_u^j(v)\right) = \delta\left(\sigma_1^j\left(\frac{v}{u}\right)\right) \text{ for } j \geq 0.$$

2. *if v is odd then*
$$\delta(\sigma_u^j(v)) = \delta\left(\sigma_{\frac{u}{v}}^j(1)\right) \text{ for } j \geq 0.$$

Proof. We have

1. Let $\sigma_u^k(v) = \dfrac{v + uT_k^u(v)}{2^a}$ where $T_k^u(v) = \sum_{j=1}^{k-1} \delta(\sigma_u^j(v))2^j$ and $\sigma_1^j\left(\frac{v}{u}\right) = \dfrac{\frac{v}{u} + T_k^1\left(\frac{v}{u}\right)}{2^a}$ where $T_k^1\left(\frac{v}{u}\right) = \sum_{j=1}^{k-1} \delta\left(\sigma_1^j\left(\frac{v}{u}\right)\right)2^j$. We will prove by induction that
$$\delta\left(\sigma_u^j(v)\right) = \delta\left(\sigma_1^j\left(\frac{v}{u}\right)\right)$$
For $j = 0$, Since if v is odd (or even) then $\frac{v}{u}$ is odd (or even) then
$$\delta\left(\sigma_u^0(v)\right) = \delta(v) = \delta\left(\frac{v}{u}\right) = \delta\left(\sigma_1^0\left(\frac{v}{u}\right)\right).$$
Suppose $\delta\left(\sigma_u^j(v)\right) = \delta\left(\sigma_1^j\left(\frac{v}{u}\right)\right)$ for $j \leq k$, then $T_k^u(v) = T_k^1\left(\frac{v}{u}\right)$, then we have
$$\sigma_u^k(v) = \frac{v + uT_k^u(v)}{2^a} = \frac{v + uT_k^1\left(\frac{v}{u}\right)}{2^a} = u\left(\frac{\frac{v}{u} + T_k^1\left(\frac{v}{u}\right)}{2^a}\right) = u\sigma_1^k\left(\frac{v}{u}\right)$$
Since u is odd, we have that $\sigma_u^k(v)$ and $\sigma_1^k\left(\frac{v}{u}\right)$ have the same parity, then
$$\delta\left(\sigma_u^{k+1}(v)\right) = \delta\left(\sigma_1^{k+1}\left(\frac{v}{u}\right)\right)$$

2. Let $\sigma_u^k(v) = \dfrac{v + uT_k^u(v)}{2^a}$ where $T_k^u(v) = \sum_{j=1}^{k-1} \delta(\sigma_u^j(v))2^j$ and $\sigma_{\frac{u}{v}}^j(1) = \dfrac{1 + T_k^{\frac{u}{v}}(1)}{2^a}$ where $T_k^{\frac{u}{v}}(1) = \sum_{j=1}^{k-1} \delta\left(\sigma_{\frac{u}{v}}^j(1)\right)2^j$. We will prove by induction that
$$\delta\left(\sigma_u^j(v)\right) = \delta\left(\sigma_{\frac{u}{v}}^j(1)\right)$$
For $j = 0$, Since v is odd
$$\delta\left(\sigma_u^0(v)\right) = \delta(v) = \delta(1) = \delta\left(\sigma_{\frac{u}{v}}^0(1)\right).$$

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Suppose $\delta\left(\sigma_u^j(v)\right) = \delta\left(\sigma_{\frac{u}{v}}^j(1)\right)$ for $j \le k$, then $T_k^u(v) = T_k^{\frac{u}{v}}(1)$, then we have

$$\sigma_u^k(v) = \frac{v + uT_k^u(v)}{2^a} = \frac{v + uT_k^{\frac{u}{v}}(1)}{2^a} = v\left(\frac{1 + \frac{u}{v}T_k^{\frac{u}{v}}(1)}{2^a}\right) = v\sigma_{\frac{u}{v}}^k(1)$$

Since v is odd, we have that $\sigma_u^k(v)$ and $\sigma_{\frac{u}{v}}^k(1)$ have the same parity, then

$$\delta\left(\sigma_u^{k+1}(v)\right) = \delta\left(\sigma_{\frac{u}{v}}^{k+1}(1)\right)$$

□

In the following proposition, we demonstrate homogeneity properties that leave the coding of the orbits of the sigma function invariant.

Proposition 21 (homogeneity). *Let $u, v \in \mathbb{Z}$ with u not null, such that $(u,v) = 1$ and $(u,2) = 1$ then we have*

1. $\sigma_u^k(v) = u\sigma_1^k\left(\frac{v}{u}\right)$.
2. $\sigma_u^k(v) = v\sigma_{\frac{u}{v}}^k(1)$.

Proof. We have

1. Let $\sigma_u^k(v) = \dfrac{v + uT_k^u(v)}{2^a}$ where $T_k^u(v) = \sum_{j=1}^{k-1} \delta(\sigma_u^j(v))2^j$ and $\sigma_1^j\left(\frac{v}{u}\right) = \dfrac{\frac{v}{u} + T_k^1\left(\frac{v}{u}\right)}{2^a}$ where $T_k^1\left(\frac{v}{u}\right) = \sum_{j=1}^{k-1} \delta\left(\sigma_1^j\left(\frac{v}{u}\right)\right)2^j$. Then we have

$$\sigma_u^k(v) = \frac{v + uT_k^u(v)}{2^a} = \frac{v + uT_k^1\left(\frac{v}{u}\right)}{2^a} = u\left(\frac{\frac{v}{u} + T_k^1\left(\frac{v}{u}\right)}{2^a}\right) = u\sigma_1^k\left(\frac{v}{u}\right)$$

2. Let $\sigma_u^k(v) = \dfrac{v + uT_k^u(v)}{2^a}$ where $T_k^u(v) = \sum_{j=1}^{k-1} \delta(\sigma_u^j(v))2^j$ and $\sigma_{\frac{u}{v}}^j(1) = \dfrac{1 + T_k^{\frac{u}{v}}(1)}{2^a}$ where $T_k^{\frac{u}{v}}(1) = \sum_{j=1}^{k-1} \delta\left(\sigma_{\frac{u}{v}}^j(1)\right)2^j$. Then we have

$$\sigma_u^k(v) = \frac{v + uT_k^u(v)}{2^a} = \frac{v + uT_k^{\frac{u}{v}}(1)}{2^a} = v\left(\frac{1 + T_k^{\frac{u}{v}}(1)}{2^a}\right) = v\sigma_{\frac{u}{v}}^k(1)$$

□

Proposition 22 (Algebraic properties of the Extension of the Sigma function). *Let $a \in \mathbb{Q}_{odd}$ with odd numerator and $\sigma_a : \mathbb{Q}_{odd} \to \mathbb{Q}_{odd}$, satisfies the following identities*

1. *if $\frac{n}{q}, \frac{m}{p}$ are even fractions, then $\sigma_a\left(\frac{n}{q} + \frac{m}{p}\right) = \sigma_a\left(\frac{n}{q}\right) + \sigma_a\left(\frac{m}{p}\right)$.*
2. *if $\frac{n}{q}$ is an even fraction and $\frac{m}{p}$ is an odd fraction, then $\sigma_a\left(\frac{n}{q} + \frac{m}{p}\right) = \sigma_a\left(\frac{n}{q}\right) + \sigma_a\left(\frac{m}{p}\right)$.*
3. *if $\frac{n}{q}, \frac{m}{p}$ are odd fractions, then $\sigma_a\left(\frac{n}{q} + \frac{m}{p}\right) = \sigma_a\left(\frac{n}{q}\right) + \sigma_a\left(\frac{m}{p}\right) - a$.*

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Proof. Let's proved first for $a = 1$. Let $\frac{n}{q}, \frac{m}{p} \in \mathbb{Q}_{odd}$ and let $\beta : \mathbb{Z} \times \mathbb{Z} \to \{0, 1\}$ given by $\beta(n, m) = 0$ if n or m is even fraction and $\beta(n, m) = 1$ if n and m are odd fraction.

$$\begin{aligned}
pq\sigma\left(\frac{n}{q} + \frac{m}{p}\right) &= pq\sigma\left(\frac{n}{q} + \frac{m}{p}\right) \\
&= pq\sigma\left(\frac{np + mq}{pq}\right) \\
&= \sigma_{pq}\left(pq\frac{np + mq}{pq}\right) \\
&= \sigma_{pq}(np + mq) \\
&= \sigma_{pq}(np) + \sigma_{pq}(mq) - pq\beta(n, m) \\
&= pq\sigma\left(\frac{np}{pq}\right) + pq\sigma\left(\frac{mq}{pq}\right) - pq\beta(n, m) \\
&= pq\sigma\left(\frac{n}{q}\right) + pq\sigma\left(\frac{m}{p}\right) - pq\beta(n, m)
\end{aligned}$$

Dividing everything by pq, we have

$$\sigma\left(\frac{n}{q} + \frac{m}{p}\right) = \sigma\left(\frac{n}{q}\right) + \sigma\left(\frac{m}{p}\right) - \beta(n, m)$$

Now let a an odd fraction with odd numerator, then we have $\frac{1}{a}$ is odd fraction, then multiplying by $\frac{1}{a}$ does not change the parity of $\frac{n}{q}$ or $\frac{m}{q}$. Then we have

$$\begin{aligned}
\sigma\left(\frac{1}{a}\frac{n}{q} + \frac{1}{a}\frac{m}{p}\right) &= \sigma\left(\frac{1}{a}\frac{n}{q}\right) + \sigma\left(\frac{1}{a}\frac{m}{p}\right) - \beta(n, m) \text{ multiplying by } a \\
a\sigma\left(\frac{1}{a}\frac{n}{q} + \frac{1}{a}\frac{m}{p}\right) &= a\sigma\left(\frac{1}{a}\frac{n}{q}\right) + a\sigma\left(\frac{1}{a}\frac{m}{p}\right) - a\beta(n, m) \\
\sigma_a\left(\frac{n}{q} + \frac{m}{p}\right) &= \sigma_a\left(\frac{n}{q}\right) + \sigma_a\left(\frac{m}{p}\right) - a\beta(n, m)
\end{aligned}$$

□

6.7. *Coding of Sigma Function*

6.7.1. The $P-$Adic Numbers

Kurt Hensel, a German mathematician from the late 1800s, is credited with creating the p-adic numbers. Their creation was in response to the requirement to expand the rational numbers $\mathbb{Q}$ in order to address algebraic and number theory issues that are difficult to resolve with just real numbers.

The notion of the p-adic valuation gives rise to the introduction of p-adic numbers, which offer an alternative to the standard metric based on absolute difference for measuring the "size" or "distance" between numbers. The largest power of p that divides n for a prime number p is the p-adic valuation of an integer n, represented by $ord_p(n)$. The following formula extends this valuation to rational numbers:

$$ord_p\left(\frac{a}{b}\right) = ord_p(a) - ord_p(b)$$

where $\frac{a}{b}$ is a fraction of integers a and b, with $b \neq 0$. The associated p-adic metric is defined for any pair of rational numbers x and y as:

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$$\|x-y\|_p = p^{-ord_p(x-y)}$$

With regard to this metric, this metric offers a way to finish the set of rational numbers $\mathbb{Q}$ and produce the p-adic field $\mathbb{Q}_p$. Every Cauchy sequence in $\mathbb{Q}$ can have a limit within $\mathbb{Q}_p$ thanks to the completion of $\mathbb{Q}$ with regard to the p-adic metric, completing the space and permitting a more flexible algebraic representation.

Every number can be uniquely described in terms of an infinite series consisting of positive powers of p and a finite number of negative powers in the p-adic field $\mathbb{Q}_p$. In addition to generalizing the rational numbers, this algebraic structure offers strong tools for solving issues in number theory, algebra, and other mathematical fields.

Definition 15. *Let $p \in \mathbb{N}$ be any prime number. Define a norm $\|.\|_p$ on $\mathbb{Q}$ as follows*

$$\|x\|_p = \begin{cases} p^{-ord_p(x)} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

where

$$ord_p(x) = \begin{cases} \text{the highest power } p \text{ which divides } x & if \quad x \in \mathbb{Z} \\ ord_p(a) - ord_p(b) & if \quad x = \frac{a}{b} \in \mathbb{Q} \end{cases}$$

Let $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ through the norm $\|.\|_p$.

We have the following properties

1. if $a, b \in \mathbb{N}$ then $a = b \mod p^k$ if and only if $\|a-b\|_p \leq p^{-k}$.
2. Let $\beta \in \mathbb{Q}_p$ then this is uniquely represented by convergent series (with norm $\|.\|_p$) as

$$\beta = \sum_{n=-k}^{\infty} \delta_j 2^j = (\ldots\delta_j\ldots\delta_2\delta_1\delta_0.\delta_{-1}\ldots\delta_{-k})_p$$

3. The $p-$adic expansion allows us to perform arithmetical operations in $\mathbb{Q}_p$ in way very similar to that in $\mathbb{R}$. Moreover, we will see that the operations in $\mathbb{Q}_p$ are, in fact, easier to perform than $\mathbb{R}$. Let $\alpha = \sum_{j=-k}^{\infty} \varepsilon_j p^j$ and $\beta = \sum_{j=-k}^{\infty} \delta_j p^j$

$$\alpha \pm \beta = \sum_{j=-k}^{\infty} (\varepsilon_j \pm \delta_j) p^j$$

$$\alpha \cdot \beta = \left(\sum_{j=-k}^{\infty} \varepsilon_j p^j\right)\left(\sum_{j=-k}^{\infty} \delta_j p^j\right) = \sum_{r=-k}^{\infty}\left(\sum_{i+j=r} \varepsilon_j\delta_j\right)p^r$$

.

4. A $p-$adic number $\beta \in \mathbb{Q}_p$ is said to be a $p-$adic integer if its canonical expansion contains only non-negative power of p. The set of $p-$adic integers is denoted by $\mathbb{Z}_p$, so

$$\mathbb{Z}_p = \left\{\sum_{i=1}^{\infty} \delta_i p^i, \text{ with } \delta_j = 0, \ldots, p-1\right\} = \{\beta \in \mathbb{Q}_p; \|\beta\|_p \leq 1\}$$

This set has the property of being a complete metric subspace (Proposition 2.10 page 59 of [7]).

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One of the main characteristics of $p-$adics numbers is

Proposition 23. *The canonical $p-$adic expansion $\alpha = \sum_{n=-k}^{\infty} \varepsilon_j 2^j$ represents a rational number if and only if is eventually periodic to the left.*

6.7.2. Coding of Sigma Function

Now we are going to define the coding of the sigma function for $\mathbb{Q}_{odd}$.

Definition 16. *Let $\frac{p}{q} \in \mathbb{Q}_{odd}$ and $g \in \mathbb{Z}$ odd number. We define the Coding of $\frac{p}{q}$ by σ as $Cod\sigma_g\left(\frac{p}{q}\right) = \prod_{j\in\mathbb{N}_0} \delta_j \in \mathbb{Z}_2$ given by*

$$\delta_j = \begin{cases} 1 & \text{if} \quad \sigma_g^j\left(\frac{p}{q}\right) \text{ is odd} \\ \\ 0 & \text{if} \quad \sigma_g^j\left(\frac{p}{q}\right) \text{ is even} \end{cases}$$

and the finite $k-$coding as $Cod\sigma_g^k\left(\frac{p}{q}\right) = \ldots 0000\delta_k \ldots \delta_0$

The following lemma is a reformulation of lemma 3 for $Cod\sigma$.

Lemma 4 (Invariant coding lemma). *Let $u, v \in \mathbb{Z}$ with u not null, such that $(u, v) = 1$ and $(u, 2) = 1$ then*

1.

$$Cod\sigma_u^j(v) = Cod\sigma^j\left(\frac{v}{u}\right) \text{ for } j \geq 0.$$

2. *if v is odd then*

$$Cod\sigma_u^j(v) = Cod\sigma_{\frac{u}{v}}^j(1) \text{ for } j \geq 0.$$

Proof. Reformulation of the Lemma 3 □

Proposition 24 (Change of basis of the Cod-Sigma function). *Let $N_1 = \sum_{j=1}^{n_1} \delta_j 2^j$ and $N_2 = \sum_{j=1}^{n_2} \varepsilon_j 2^j$ with $\delta_j, \varepsilon_j \in \{0, 1\}$, and $a \in \mathbb{N}$ such that $N_1, N_2 \leq 2^a - 1$. Let $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ given by $\mathcal{H}(x) = \frac{x + N_1 N_2}{2^a}$. Then if $x_0 \in \mathbb{E}(\mathcal{H})$ and N_2 is an odd number we have*

$$Cod\sigma_{N_1}^a(x_0) = \underbrace{0, \ldots, 0, \varepsilon_{n_2}, \ldots .\varepsilon_0,}_{a}$$

or if N_1 is an odd number we have

$$Cod\sigma_{N_2}^a(x_0) = \underbrace{0, \ldots, 0, \delta_{n_1}, \ldots, \delta_0}_{a}$$

Additionally. if N_1, N_2 are odd numbers we have

$$\sigma_{N_1}^a(x_0) = \sigma_{N_2}^a(x_0)$$

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Proof. Let's prove by induction that if N_2 is odd, then

$$Cod\sigma^a_{N_2}(x_0) = \underbrace{0,\ldots,0,\delta_{n_1},\ldots,\delta_0}_{a}.$$

Completing with $\delta_j = 0$ if necessary, we have $N_1 = \sum_{j=1}^{n_1} \delta_j 2^j = \sum_{j=1}^{a-1} \delta_j 2^j$, then

$$\frac{x + N_1 N_2}{2^a} = \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{x+\delta_0 N_2}{2} + \delta_1 N_2}{2} + \delta_2 N_2}{2}}{\ddots} + \delta_{a-1} N_2}{2} = \bigodot_{j=1}^{a}\left(\frac{x+\delta_j N_2}{2}\right)$$

Since we trivially have that $(1,2) = 1$ and by hypothesis we have that $x_0 \in \mathbb{E}(\mathcal{H})$, by the Lemma 1 we have

$$x_0 \in \mathbb{E}\left(\bigodot_{j=1}^{K}\left(\frac{x+\delta_j N_2}{2}\right)\right) \text{ for all } K \leq a$$

In particular, for $K = 0$, we have $x_0 \in \mathbb{E}\left(\frac{x+\delta_0 N_2}{2}\right)$ and N_2 is odd, we have $\mathbb{E}(\theta) \cap \mathbb{E}(\lambda^{N_2}) = \emptyset$ then $\sigma_{N_2}(x_0) = \frac{x_0+\delta_0 N_2}{2}$.

For $K = 1$, we have $x_0 \in \mathbb{E}\left(\frac{x+\delta_1 N_2}{2} \circ \frac{x+\delta_0 N_2}{2}\right)$, then $\sigma_{N_2}(x_0) \in \mathbb{E}\left(\frac{x_0+\delta_1 N_2}{2}\right)$, then $\sigma^2_{N_2}(x_0) = \frac{x_0 + N_2(\delta_0 + \delta_1 2)}{2^2}$. Suppose this continues until for $K = r$ i.e.

$$\sigma^r_{N_2}(x_0) = \frac{x_0 + N_2 \sum_{j=1}^{r-1} \delta_j 2^j}{2^r} \in \mathbb{Z}$$

We have that $x_0 \in \mathbb{E}\left(\bigodot_{j=1}^{r+1}\left(\frac{x+\delta_j N_2}{2}\right)\right)$ then by definitions $\bigodot_{j=1}^{r+1}\left(\frac{x_0+\delta_j N_2}{2}\right) \in \mathbb{Z}$ and by inductive hypothesis $\sigma^r_{N_2}(x_0) = \bigodot_{j=1}^{r}\left(\frac{x_0+\delta_j N_2}{2}\right) \in \mathbb{Z}$ then $\sigma^r_{N_2}(x_0) \in \mathbb{E}\left(\frac{x+\delta_r N_2}{2}\right)$. Therefore, $\sigma^{r+1}_{N_2}(x_0) = \frac{x_0 + N_2 \sum_{j=1}^{r} \delta_j 2^j}{2^{r+1}}$. Then we have to

$$\sigma^a_{N_2}(x_0) = \frac{x_0 + N_2 \sum_{j=1}^{a-1} \delta_j 2^j}{2^a} = \frac{x_0 + N_2 N_1}{2^a}.$$

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Similarly, Completing with $\delta_j = 0$ if necessary we have. Let $N_2 = \sum_{j=1}^{n_2} \varepsilon_j 2^j = \sum_{j=1}^{a-1} \varepsilon_j 2^j$, then

$$\sigma^a_{N_1}(x_0) = \frac{x_0 + N_1 \sum_{j=1}^{a-1} \varepsilon_j 2^j}{2^a} = \frac{x_0 + N_1 N_2}{2^a}$$

Now. If N_1, N_2 are odd numbers, we have

$$\sigma^a_{N_1}(x_0) = \sigma^a_{N_2}(x_0)$$

In particular, we have

$$\sigma^a_{N_2}(x_0) = \frac{\dfrac{\dfrac{\dfrac{\dfrac{x+\delta_0 N_2}{2} + \delta_1 N_2}{2} + \delta_2 N_2}{2}}{\ddots} + \delta_{a-1} N_2}{2}$$

$$= \frac{\dfrac{\dfrac{\dfrac{\dfrac{x+\varepsilon_0 N_1}{2} + \varepsilon_1 N_1}{2} + \varepsilon_2 N_1}{2}}{\ddots} + \varepsilon_{a-1} N_1}{2} = \sigma^a_{N_1}(x_0)$$

then we have that the coding of $\sigma^a_{N_2}(x_0)$ is $\delta_{a-1} \ldots \delta_1 \delta_0$ and of the $\sigma^a_{N_1}(x_0)$ is $\varepsilon_{a-1} \ldots \varepsilon_1 \varepsilon_0$. $\square$

Example 10. *Let $\mathcal{H}(x) = \dfrac{x+15}{2^5}$, we have $x_0 = 2^5 - 15 = 17 \in \mathbb{E}(\mathcal{H})$, then*

1. *$\sigma_3(17) = 10, \sigma_3(10) = 5, \sigma_3(5) = 4, \sigma_3(4) = 2, \sigma_3(2) = 1$ then $\sigma_3^5(17) = 1$. Then we have $Cod\sigma_3(17) = 00101$ taking the coding coefficients, to base 2 we have*

$$1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3 + 0 \cdot 2^4 = 5.$$

2. *$\sigma_5(17) = 11, \sigma_5(11) = 8, \sigma_5(8) = 4, \sigma_5(4) = 2, \sigma_5(2) = 1$, then $\sigma_5^5(17) = 1$. Then we have $Cod\sigma_5(17) = 00011$ taking the coding coefficients, to base 2 we have*

$$1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 + 0 \cdot 2^4 = 3.$$

We observe that the obits are equal from the third iteration, which corresponds to the maximum power of two, where all subsequent coefficients are null. We can also observe that from the third term, the values that appear in the orbits are even, they, unfortunately, cannot continue forever, since as we have seen, the orbit of the sigma functions falls into a periodic orbit with the same number of even and odd numbers, so at some point this orbit must fall into an odd one, which means all the initial values must change.

Proposition 25. *Let $r \in \mathbb{N}$ and $\gamma = 3^{b-1}$ and let $k \in \mathbb{N}$ such that $r < 2^{2\gamma k}$ and $\delta_j \in \{0, 1\}$ such that $r\left\{\dfrac{2^{2\gamma k} - 1}{3\gamma}\right\} = \sum_{j=1}^{2\gamma k - 1} \delta_j 2^j$ then $Cod\sigma^{2\gamma k}_{3\gamma}(r) = \delta_{2\gamma k - 1}, \ldots, \delta_0$.*

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Proof. Let $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ given by $\mathcal{H}(x) = \dfrac{3\gamma x + r}{2^{2\gamma k}}$, then we want to find the minimum positive value of $\mathcal{H}$, then solving the following equation.

$$\frac{3\gamma x + r}{2^{2\gamma k}} = \sigma_{3\gamma}^{2\gamma k}(r)$$

By Proposition 17 we have $\sigma_{3\gamma}^{2\gamma k}(r) = r$, then

$$\frac{3\gamma x + r}{2^{2\gamma k}} = r \text{ then } x = \frac{2^{2\gamma k} r - r}{3\gamma} = r\left\{\frac{2^{2\gamma k} - 1}{3\gamma}\right\}$$

Let $\delta_j \in \{0,1\}$ such that $r\left\{\dfrac{2^{2\gamma k} - 1}{3^{3\gamma}}\right\} = \sum_{j=1}^{2\gamma k - 1} \delta_j 2^j$. By Proposition 24 we have

$$Cod\sigma_{3\gamma}^{2\gamma k}(r) = \delta_{2\gamma k-1}, \ldots, \delta_0$$

□

Corollary 5. *Let $r_1, r_2 \in \mathbb{N}$. Then*

$$Cod\sigma_{3^b}^{a}(r_1 + r_2) = Cod\sigma_{3^b}^{a}(r_1) + Cod\sigma_{3^b}^{a}(r_2) \mod 2^a.$$

Proof. Let $\gamma = 3^{b-1}$, $r = r_1 + r_2$ and $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ given by $\mathcal{H}(x) = \dfrac{3\gamma x + r}{2^a}$ by Proposition 18 we have

$$\frac{3\gamma x + r}{2^a} = \sigma_{3\gamma}^{a}(r) = \sigma_{3\gamma}^{a}(r_1) + \sigma_{3\gamma}^{a}(r_2) - 3^{3\gamma} u \text{ with } u \in \mathbb{Z}$$

Then

$$x = \frac{2^a \sigma_{3\gamma}^{a}(r_1) - r_1}{3\gamma} + \frac{2^a \sigma_{3\gamma}^{a}(r_2) - r_2}{3\gamma} - 2^a u$$

Let $k \in \mathbb{N}$ such that $r < 2^{2\gamma k}$ and $2\gamma k > a$ and let $\delta_j, \varepsilon_j, \eta_j \in \{0,1\}$ such that

$$r_1\left\{\frac{2^{2\gamma k} - 1}{3\gamma}\right\} = \sum_{j=1}^{2\gamma k-1} \delta_j 2^j$$
$$r_2\left\{\frac{2^{2\gamma k} - 1}{3\gamma}\right\} = \sum_{j=1}^{2\gamma k-1} \varepsilon_j 2^j \text{ and}$$
$$r\left\{\frac{2^{2\gamma k} - 1}{3\gamma}\right\} = \sum_{j=1}^{2\gamma k-1} \eta_j 2^j$$

by Proposition 25 we have

$$\left(\frac{2^a \sigma_{3\gamma}^{a}(r_1) - r_1}{3\gamma}\right)(\text{Base } 2) = \{\delta_j\}_{j=1}^{a-1}$$
$$\left(\frac{2^a \sigma_{3\gamma}^{a}(r_2) - r_2}{3\gamma}\right)(\text{Base } 2) = \{\varepsilon_j\}_{j=1}^{a-1} \text{ and}$$
$$\left(\frac{2^a \sigma_{3\gamma}^{a}(r) - r}{3\gamma}\right)(\text{Base } 2) = \{\eta_j\}_{j=1}^{a-1}$$

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Then we have

$$\frac{2^a\sigma_{3\gamma}^a(r) - r}{3\gamma} = \frac{2^a\sigma_{3\gamma}^a(r_1) - r_1}{3\gamma} + \frac{2^a\sigma_{3\gamma}^a(r_2) - r_2}{3\gamma} - 2^a u$$
$$\text{if and only if } \sum_{j=1}^{a-1} \eta_j 2^j = \sum_{j=1}^{a-1} \delta_j 2^j + \sum_{j=1}^{a-1} \varepsilon_j 2^j - 2^a u$$

Then we have $\sum_{j=1}^{a-1} \eta_j 2^j = \sum_{j=1}^{a-1} \delta_j 2^j + \sum_{j=1}^{a-1} \varepsilon_j 2^j \mod 2^a$, therefore

$$Cod\sigma_{3\gamma}^a(r) = Cod\sigma_{3\gamma}^a(r_1) + Cod\sigma_{3\gamma}^a(r_2) \mod 2^a$$

□

Proposition 26. *Let $r_1, r_2 \in \mathbb{N}$. Then $Cod\sigma_{3^b} : \mathbb{N} \to \mathbb{Z}_2$ is linear:*

$$Cod\sigma_{3^b}(r_1 + r_2) = Cod\sigma_{3^b}(r_1) + Cod\sigma_{3^b}(r_2) \in \mathbb{Z}_2$$

Proof. By Corollary 5 we have $Cod\sigma_{3^b}^n(r_1 + r_2) = Cod\sigma_{3^b}^n(r_1) + Cod\sigma_{3^b}^n(r_2) \mod 2^n$ for all $n \in \mathbb{N}$, this is equivalent to

$$||Cod\sigma_{3^b}^n(r_1 + r_2) - Cod\sigma_{3^b}^n(r_1) + Cod\sigma_{3^b}^n(r_2)||_2 < 2^{-n} \text{ for all } n \in \mathbb{N}$$

Therefore

$$Cod\sigma_{3^b}(r_1 + r_2) = Cod\sigma_{3^b}(r_1) + Cod\sigma_{3^b}(r_2) \in \mathbb{Z}_2$$

□

Example 11. *1. Let $S(x) = \dfrac{9x+5}{2^4}$, we have*

$$\sigma_9^4(5) = \sigma_9^4(3) + \sigma_9^4(2) \mod 9$$

(a) $\mathcal{O}\sigma_9^4(3) = \{3, 6, 3, 6, 3\}$ *then* $Cod\sigma_9^4(3) = 0101$.
(b) $\mathcal{O}(\sigma_9^4(2)) = \{2, 1, 5, 7, 8\}$ *then* $Cod\sigma_9^4(2) = 1110$.

$$\begin{aligned} Cod\sigma_9^4(5) &= Cod\sigma_9^4(3) + Cod\sigma_9^4(2) \mod 2^4 \\ &= 1110 + 0101 \mod 2^4 \\ &= 0011 \end{aligned}$$

$\mathcal{O}(\sigma_9^4(5)) = \{5, 7, 8, 4, 2\}$ *then* $Cod\sigma_9^4(5) = 0011$, *Then we have* $\rho_0(S) = 1 + 2 = 3$

$$S(3) = \frac{9 \cdot 3 + 5}{2^4} = 2$$

2. *Let $S(x) = \dfrac{9x + 3 + 2^{\theta_1}}{2^{\theta_1+\theta_2}}$ with θ_1 even. We have*

$$\begin{aligned} \sigma_9^{\theta_1+\theta_2}(3 + 2^{\theta_1}) &= \sigma_9^{\theta_1+\theta_2}(3) + \sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) \mod 9 \\ &= 3\sigma_3^{\theta_1+\theta_2} + \sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) \mod 9 \end{aligned}$$

Then

$$\begin{aligned} Cod\sigma_9^{\theta_1+\theta_2}(3+2^{\theta_1}) &= Cod\sigma_9^{\theta_1+\theta_2}(3) + Cod\sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) \mod 2^{\theta_1+\theta_2} \\ &= Cod\sigma_3^{\theta_1+\theta_2} + Cod\sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) \mod 2^{\theta_1+\theta_2} \end{aligned}$$

(a) $\mathcal{O}(\sigma_3^{\theta_1+\theta_2}) = \{1,2,1,2,1,2,\dots\}$ *then* $Cod\sigma_3^{\theta_1+\theta_2} = \underbrace{\dots 010101}_{\theta_1+\theta_2}$.

(b) $\mathcal{O}(\sigma_9^{\theta_1+\theta_2}(2^{\theta_1})) = \{2^{\theta_1}, 2^{\theta_1-1}, \dots, 1, 5, 7, 8, 4, 2, 1, 5, 7, 8, 4, 2, 1, \dots\}$ *then*

$$Cod\sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) = \underbrace{\dots 000111000111}_{\theta_2}\underbrace{\dots 00000}_{\theta_1}$$

.

Then we have

$$\begin{aligned} Cod\sigma_9^{\theta_1+\theta_2}(3+2^{\theta_1}) &= \underbrace{\dots 000111000111}_{\theta_2}\underbrace{\dots 00000}_{\theta_1} + \underbrace{\dots 010101}_{\theta_1+\theta_2} \mod 2^{\theta_1+\theta_2} \\ &= \underbrace{\dots 000111000111000111100}_{\theta_2}\underbrace{\dots 1010101010}_{\theta_1} \mod 2^{\theta_1+\theta_2} \\ &= Cod\sigma_9^{\theta_2-2}00Cod\sigma_3^{\theta_1} \mod 2^{\theta_1+\theta_2} \end{aligned}$$

Then we have $\rho_0(S)(base2) = Cod\sigma_9^{\theta_2-2}00Cod\sigma_3^{\theta_1} \mod 2^{\theta_1+\theta_2}$.

3. *Let* $S_n(x) = \dfrac{3^n x + 4^n - 3^n}{4^n}$

$$\sigma_{3^n}^{2n}(4^n - 3^n) = \sigma_{3^n}^{2n}(4^n) - \sigma_{3^n}^{2n}(3^n) = 1 - 3^n\sigma_1^{2n} = 1 \mod 3^n$$

On the other hand

(a) $\mathcal{O}(\sigma_{3^n}^{2n}(2^{2n})) = \{2^{2n}, 2^{2n-1}, 2^{2n-2}, \dots, 1\}$ *then* $Cod\sigma_{3^n}^{2n}(2^{2n}) = \underbrace{\dots 00000}_{2n}$.

(b) $\mathcal{O}(\sigma_{3^n}^{2n}(3^n)) = \{3^n, 3^n, 3^n, 3^n \dots\}$ *then* $Cod\sigma_{3^n}^{2n}(3^n) = \underbrace{\dots 11111111}_{2n}$.

Then we have

$$\begin{aligned} &\underbrace{0\dots0}_{2n} - \underbrace{1\dots1}_{2n} \mod 2^{2n} \\ =&1\underbrace{0\dots0}_{2n} - 0\underbrace{1\dots1}_{2n} \mod 2^{2n} \\ =&1 \mod 2^{2n} \\ =&1 \end{aligned}$$

Then

$$Cod\sigma_{3^n}^{2n}(4^n - 3^n) = 00\dots001$$

That is, $Cod\sigma_{3^n}^{2n}(4^n - 3^n)$ *has a constant coding equal to* 1*. This is natural, since* S_n *is stable.*

Lemma 5 (Rational equivalence of the Cod-Sigma function.)**.** *Let* $b \in \mathbb{N}$*, then we have*

$$Cod\sigma_{3^b} = -\frac{1}{3^b} \in \mathbb{Z}_2$$

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Furthermore. Let $u \in \mathbb{N}$, then

$$Cod\sigma_{3^b}(u) = -\frac{u}{3^b} \in \mathbb{Z}_2$$

Proof. Let $u = \overline{\gamma_b} \in \mathbb{Z}_2$ with $\gamma_b = Cod\sigma_{3^b}^{2\cdot 3^{b-1}}$. Let $u = x$ then multiplying by $2^{2\cdot 3^{b-1}}$ we have $u\underbrace{0\dots0}_{2\cdot 3^{b-1}} = 2^{2\cdot 3^{b-1}}x$ and subtracting, we have

$$u - u\underbrace{0\dots0}_{2\cdot 3^{b-1}} = -(2^{2\cdot 3^{b-1}} - 1)x$$
$$\gamma_b = -(2^{2\cdot 3^{b-1}} - 1)x$$

On the other hand we have that $\gamma_b = \dfrac{2^{2\cdot 3^{b-1}} - 1}{3^b}$, then

$$\frac{2^{2\cdot 3^{b-1}} - 1}{3^b} = -(2^{2\cdot 3^{b-1}} - 1) \text{ then we have } x = -\frac{1}{3^b}$$

Other way for proof it, is

$$\begin{aligned}
u = \sum_{n=0}^{\infty} u_n 2^n &= \left\{\sum_{n=0}^{2\cdot 3^{b-1}-1} u_n 2^n\right\} + \left\{\sum_{n=2\cdot 3^{b-1}}^{4\cdot 3^{b-1}-1} u_n 2^n\right\} + \left\{\sum_{n=4\cdot 3^{b-1}}^{6\cdot 3^{b-1}-1} u_n 2^n\right\}\dots \\
&= Cod\sigma_{3^b}^{2\cdot 3^{b-1}} + 2^{2\cdot 3^{b-1}} Cod\sigma_{3^b}^{2\cdot 3^{b-1}} + \left(2^{2\cdot 3^{b-1}}\right)^2 Cod\sigma_{3^b}^{2\cdot 3^{b-1}} + \dots \\
&= Cod\sigma_{3^b}^{2\cdot 3^{b-1}} \left\{\sum_{n=0}^{\infty} \left(2^{2\cdot 3^{b-1}}\right)^n\right\} \\
&= Cod\sigma_{3^b}^{2\cdot 3^{b-1}} \left\{\frac{1}{1 - 2^{2\cdot 3^{b-1}}}\right\} \\
&= -\left\{\frac{1 - 2^{2\cdot 3^{b-1}}}{3^b}\right\}\left\{\frac{1}{1 - 2^{2\cdot 3^{b-1}}}\right\} \\
&= -\frac{1}{3^b}
\end{aligned}$$

Now let's demonstrate the second part. Let $v \in \mathbb{N}$

$$Cod\sigma_{3^b}^k(v) = -v\left\{\frac{1 - 2^{2\cdot 3^{b-1}k}}{3^b}\right\} = -\frac{v}{3^b} + \frac{v 2^{2\cdot 3^{b-1}k}}{3^b}$$

On the other hand

$$\left|\left|Cod\sigma_{3^b}^k(v) - \left(-\frac{v}{3^b}\right)\right|\right|_2 \leq 2^{-2\cdot 3^{b-1}k} \to 0 \text{ as } k \to \infty$$

Therefore $Cod\sigma_{3^b}(v) = -\dfrac{v}{3^b}$ □

Definition 17 (Null Tail). *we will say that $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$ has a null tail of index J if $J > 0$ the smallest index such that $j > J$ we have $\xi_j = 0$.*

Proposition 27. *Let $\xi = 0^{\theta_1}\prod_{j=2}^{\infty} 10^{\theta_j} \in \Sigma_2^*$ with $\theta_1 \geq 0$ and $\theta_k > 0$ for $k > 1$ and $a_n = \sum_{j=1}^{n}\theta_j$. We define $\pi^1_{k(\xi)} = \sum_{j=1}^{k} \frac{2^{a_j}}{3^j}$. Let $\xi \in \Sigma_2^*$ with null tail of index K given by $\xi = 0^{\theta_1}\prod_{j=2}^{K} 10^{\theta_j} \prod_{j=K+1}^{\infty} 0^{\theta_j}$ with $\theta_1 \geq 0$, $\theta_k > 0$*

for $K \geq k > 1$ *and* $\theta_j = 1$ *for* $j > K$. *We define* $\pi_k^1(\xi) = \sum_{j=1}^{k} \frac{2^{a_j}}{3^j}$ *if* $k < K$ *and* $\pi_k^1(\xi) = \sum_{j=1}^{K-1} \frac{2^{a_j}}{3^j}$ *if* $k \geq K$. *We define the* π^1 *function given by* $\pi^1 : \Sigma_2^* \to \mathbb{Z}_2$ *and definite by* $\pi^1(\xi) = \sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$ *if* ξ *is not null tail and* $\pi^1(\xi) = \sum_{j=1}^{K-1} \frac{2^{a_j}}{3^j}$ *if* ξ *has a null tail of index K Then we have* $\pi^1(\xi), \pi_k^1(\xi) \in \mathbb{Z}_2$ *and*

$$Cod\sigma(\pi^1(\xi)) = -\pi^1(\xi) \in \mathbb{Z}_2.$$

Proof. Without losing generality, let's assume that ξ does not have a null tail.

Claim 1: Let $\xi \in \Sigma_2^*$ and $k \in \mathbb{N}$, the we have $Cod\sigma\left(\pi_k^1(\xi)\right) = -\pi_k^1(\xi) \in \mathbb{Z}_2$.
Indeed. Let $k \in \mathbb{N}$ and $\xi \in \Sigma_2^*$. By Lemma 4, Proposition 26 and Lemma 5, we have

$$\begin{aligned}
Cod\sigma\left(\pi_k^1(\xi)\right) &= Cod\sigma\left(\sum_{j=1}^{k} \frac{2^{a_j}}{3^j}\right) \\
&= Cod\sigma_{3^k}\left(\sum_{j=1}^{k} 2^{a_j} 3^{k-j}\right) \\
&= \sum_{j=1}^{k} Cod\sigma_{3^k}\left(2^{a_j} 3^{k-j}\right) \\
&= \sum_{j=1}^{k} Cod\sigma_{3^j}(2^{a_j}) \\
&= -\sum_{j=1}^{k} \frac{2^{a_j}}{3^j} \\
&= -\pi_k^1(\xi)
\end{aligned}$$

Since $-\frac{2^{a_j}}{3^j} \in \mathbb{Z}_2$ then $-\pi_k^1(\xi) \in \mathbb{Z}_2$.

Claim 2: $\lim_{k \to \infty} \pi_k^1(\xi) = \pi^1(\xi) \in \mathbb{Z}_2$.

Indeed. We have the following equivalence on $\mathbb{Q}_2$ (Proposition 3.3 page 76 of [7])

$$\pi^1(\xi) = \sum_{n=1}^{\infty} \frac{2^{a_n}}{3^n} \in \mathbb{Q}_2 \text{ if and only if } \lim_{k \to \infty} \frac{2^{a_k}}{3^k} = 0 \text{ on } \mathbb{Q}_2$$

On the other hand

$$\left\| \frac{2^{a_k}}{3^k} \right\|_2 = \|Cod\sigma_{3^b}(2^{a_k})\|_2 = \|Cod\sigma_{3^b} 0^{a_k}\|_2 = 2^{-a_k} \to 0 \text{ since } a_k \text{ it is increasing}$$

Therefore $\pi^1(\xi) = \sum_{n=1}^{\infty} \frac{2^{a_n}}{3^n} \in \mathbb{Q}_2$. On the other hand, we have that $\pi_k^1(\xi)$ is a Cauchy sequence on $\mathbb{Z}_2$. Indeed

$$\left\| \pi_{k+1}^1(\xi) - \pi_k^1(\xi) \right\|_2 = \left\| \frac{2^{a_{k+1}}}{3^{k+1}} \right\|_2 = 2^{-a_{k+1}} \to 0 \text{ as } k \to \infty$$

and by Proposition 2.10 page 59 of [7], we also have that $\mathbb{Z}_2$ is a complete metric space, then $\lim_{k \to \infty} \pi_k^1(\xi) = \pi^1(\xi) \in \mathbb{Z}_2$. $\square$

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We will now show a connection between the minimum positive integer value and the encoding of the sigma function.

Lemma 6. *Let $S_k \in \langle \theta, \psi \rangle$ with $Cod(S_k) = 0^{\theta_1}10^{\theta_2}\dots 10^{\theta_k}10^{\theta_{k+1}}$. Then $\rho_0(S_k) = Cod\sigma^{a_{k+1}}\left(\frac{N_k}{3^k}\right)$*

Proof. Let $S_k \in \langle \theta, \psi \rangle$. Then

$$\rho_0(S_k) = \frac{2^{a_{k+1}}}{3^k}\sigma_{3^k}^{a_{k+1}}(N_k) - \frac{N_k}{3^k} = \frac{2^{a_{k+1}}}{3^k}\left\{\frac{N_k}{2^{a_{k+1}}} + \frac{3^k Cod\sigma^{a_{k+1}}\left(\frac{N_k}{3^k}\right)}{2^{a_{k+1}}}\right\} - \frac{N_k}{3^k}$$
$$= Cod\sigma_{3^k}^{a_{k+1}}(N_k) = Cod\sigma^{a_{k+1}}\left(\frac{N_k}{3^k}\right).$$

□

As a consequence of the next proposition, we have that if $-\pi^1$ is a negative or non-integer number, the minimum value diverges, since we have that the dyadic representation of these numbers always has an infinite amount of numbers.

Proposition 28. *Let $\{S_k\}_{k\in\mathbb{N}}$ and $\xi = Cod\{S_k\}_{k\in\mathbb{N}}$. Then $\rho_0(S_k) = Cod\sigma\left(\pi^1(\xi)\right) \mod 2^{A_k}$, where A_k is the quantity of 0 of $Cod(S_k)$*

Proof. Let us assume without loss of generality that ξ is not null tail.Let $\xi = Cod(\{S_k\}_{k\in\mathbb{N}} = 0^{\theta_1}\prod_{j=2}^{\infty}10^{\theta_j}$. So

$$\left\|Cod\sigma\left(\pi^1(\xi)\right) - Cod\sigma^{a_{k+1}}\left(\frac{N_k}{3^k}\right)\right\|_2 = \left\|\pi^1(\xi) - \pi_k^1(\xi)\right\|_2 \leq 2^{-a_{k+1}}, \text{ since } \pi_k^1(\xi)$$
$$= \pi^1(\xi) \mod 2^{a_{k+1}}$$

Then $Cod\sigma^{a_{k+1}}\left(\frac{N_k}{3^k}\right) = Cod\sigma\left(\pi^1(\xi)\right) \mod 2^{a_{k+1}}$. □

7. The G_0, G_∞ and G_1 sets

Let $\xi \in \Sigma_2^*$. Let $b_k$the quantity of 1 of $\{\xi_j\}_{j=1}^k$. Define the function $\frac{3^{b_k(\xi)}}{2^{a_{k+1}(\xi)}}$. This function corresponds to the slope of the function S_k such that $Cod^k(S_k) = \{\xi_j\}_{j=1}^k$. Let us consider three subsets that will be relevant to study the non-existence of divergent orbits. G_1, G_∞ and G_1 which correspond to the subset of the sequences such that $\frac{3^{b_k(\xi)}}{2^{a_{k+1}(\xi)}}$ converges to $0, \infty$ and some real respectively.

7.1. Summary of Propositions in the Section

1. **Definition 18:** Sets G_0, G_∞ and G_1.
2. **Lemma 7:** Characterization of G_0 and G_∞ through accumulation points of $\frac{a_{k+1}}{k}$.
3. **Proposition 29:** Let $\xi \in G_1$ then exists $M, m > 0$ such that $m < \frac{3^k}{2^{a_{k+1}}} < M$.
4. **Lemma 8:** Let $0 < \varepsilon < \frac{L}{7}$, then ε satisfies the following inequalities $L + \varepsilon < \frac{3}{2}(L - \varepsilon)$ and $\frac{3}{2^{\theta_{k+2}}}(L + \varepsilon) < L - \varepsilon$ if $\theta_{k+2} \geq 2$.
5. **Proposition 30:** Let $\xi \in G_1$, then $\frac{3^k}{2^{a_{k+1}}}$ is not convergent.

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7.2. The G_0, G_∞ and G_1 sets

We can consider the following set of Σ_2^*:

Definition 18 (The G_0, G_∞ and G_1 sets)**.** *Let $\xi = 0^{\theta_1}\prod_{j=2}^{\infty}10^{\theta_j} \in \Sigma_2^*$ with $\theta_1 \geq 0$ and $\theta_k > 0$ for $k > 1$ or $\xi = 0^{\theta_1}\prod_{j=2}^{K}10^{\theta_j}\prod_{j=K+1}^{\infty}0^{\theta_j} \in \Sigma_2^*$ with $\theta_1 \geq 0$, $\theta_k > 0$ for $K \geq k > 1$ and $\theta_j = 1$ for $j > K$. Let $a_n = \sum_{j=1}^{n}\theta_j$. We will consider the following subsets of Σ_2^**

$$G_0 = \left\{\xi \in \Sigma_2^* : \liminf_{k\to\infty}\frac{a_{k+1}}{k} > \frac{\ln(3)}{\ln(2)} \text{ on } \mathbb{R} \text{ or } \xi \text{ is Null Tail}\right\},$$

$$G_\infty = \left\{\xi \in \Sigma_2^* : \limsup_{k\to\infty}\frac{a_{k+1}}{k} < \frac{\ln(3)}{\ln(2)} \text{ on } \mathbb{R} \text{ or } \xi \text{ is Null Tail}\right\}$$

and

$$G_1 = \left\{\xi \in \Sigma_2^* : \lim_{k\to\infty}\frac{a_{k+1}}{k} = \frac{\ln(3)}{\ln(2)} \text{ on } \mathbb{R}\right\}$$

Example 12. *1. Let $\xi_1 \in \Sigma_2^*$ such that $a_k = 3k + (-1)^k$, then $\xi_1 \in G_0$.*

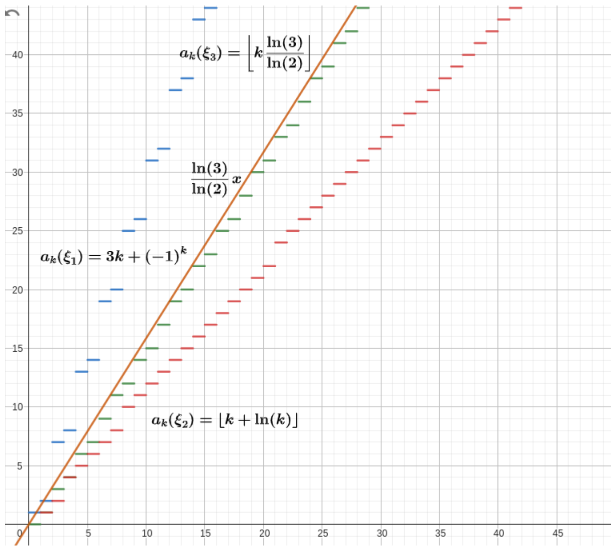

Figure 2. *the sequences in G_0 have their counting function 0 above the graph of the function $\frac{\ln(3)}{\ln(2)}x$, the sequences in G_∞ have their counting function 0 below the graph of $\frac{\ln(3)}{\ln(2)}x$ and the sequences in G_1 have their counting function 0 tending to the graph of the function $\frac{\ln(3)}{\ln(2)}x$.*

2. *$\xi_2 \in \Sigma_2^*$ such that $a_k = \lfloor k + \ln(k)\rfloor$, then $\xi_2 \in G_\infty$*
3. *Let $\xi \in \Sigma_2^*$ such that $a_k = \left\lfloor k\frac{\ln(3)}{\ln(2)}\right\rfloor$, then $\xi \in G_1$*

The following result establishes a characterization of the elements of G_0 and G_∞. We begin by providing a technical definition that we will use for the subsequent results.

Lemma 7. *Let $\xi \in \Sigma_2^*$ such that ξ is not null tail, then*

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1. *$\xi \in G_0$ if and only if $\frac{3^k}{2^{a_{k+1}}} \to 0$ as $k \to \infty$.*
2. *$\xi \in G_\infty$ if and only if $\frac{3^k}{2^{a_{k+1}}} \to \infty$ as $k \to \infty$.*

Proof.

1. Let's suppose that $\frac{3^k}{2^{a_{k+1}}} \to 0$ as $k \to \infty$ and suppose that $\liminf_{k\to\infty} \frac{a_{k+1}}{k} < \frac{\ln(3)}{\ln(2)}$. Then let $\frac{a_{k_l+1}}{k_l} = \inf_{k>k_l} \frac{a_{k+1}}{k}$, then exist $K, \varepsilon > 0$ such that $\frac{a_{k_l+1}}{k_l} + \varepsilon < \frac{\ln(3)}{\ln(2)}$ for $k > K$, then we have,

$$\frac{3^{k_l}}{2^{a_{k_l+1}}} = \left(\frac{3}{2^{\frac{a_{k_l+1}}{k_l}}}\right)^{k_l} > \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)} - \varepsilon}}\right)^{k_l} = 2^{k_l\varepsilon} \to \infty \text{ as } l \to \infty$$

which is a contradiction.

Now, let $\xi \in G_0$ then $\liminf_{k\to\infty} \frac{a_{k+1}}{k} > \frac{\ln(3)}{\ln(2)}$. Then exist $K, \varepsilon > 0$ such that $\frac{a_{k+1}}{k} - \varepsilon > \frac{\ln(3)}{\ln(2)}$ for all $k > K$. Then

$$\frac{3^k}{2^{a_{k+1}}} = \left(\frac{3}{2^{\frac{a_{k+1}}{k}}}\right)^k < \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)} + \varepsilon}}\right)^k = 2^{-k\varepsilon} \to 0 \text{ as } k \to \infty$$

2. Let's suppose that $\frac{3^k}{2^{a_{k+1}}} \to \infty$ as $k \to \infty$ and $\limsup_{k\to\infty} \frac{a_{k+1}}{k} > \frac{\ln(3)}{\ln(2)}$. Let $\frac{a_{k_l+1}}{k_l} = \sup_{k>k_l} \frac{a_{k+1}}{k}$, then exist $K, \varepsilon > 0$ such that $\frac{a_{k_l+1}}{k_l} - \varepsilon > \frac{\ln(3)}{\ln(2)}$ for $k > K$, then we have,

$$\frac{3^{k_l}}{2^{a_{k_l+1}}} = \left(\frac{3}{2^{\frac{a_{k_l+1}}{k_l}}}\right)^{k_l} < \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)} + \varepsilon}}\right)^{k_l} = 2^{-k_l\varepsilon} \to 0 \text{ as } l \to \infty$$

which is a contradiction.
Now, Let $\xi \in G_\infty$ then $\limsup_{k\to\infty} \frac{a_{k+1}}{k} < \frac{\ln(3)}{\ln(2)}$, Then exist $K, \varepsilon > 0$ such that $\frac{a_{k+1}}{k} + \varepsilon < \frac{\ln(3)}{\ln(2)}$ for all $k > K$. Then

$$\frac{3^k}{2^{a_{k+1}}} = \left(\frac{3}{2^{\frac{a_{k+1}}{k}}}\right)^k > \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)} - \varepsilon}}\right)^k = 2^{k\varepsilon} \to \infty \text{ as } k \to \infty$$

□

Now we will show what happens when $\xi \in G_1$.

Proposition 29. *Let $\xi \in G_1$ then exists $M, m > 0$ such that $m < \frac{3^k}{2^{a_{k+1}}} < M$ on $\mathbb{R}$.*

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Proof. Let $\xi \in G_1$, then $\lim_{k\to\infty} \frac{a_{k+1}}{k} = \frac{\ln(3)}{\ln(2)}$. Let's prove that exists $M, m > 0$ such that $m < \frac{3^k}{2^{a_{k+1}}} < M$. If there exists a subsequence a_{k_l+1} such that $\frac{3^{k_l}}{2^{a_{k_l+1}}} \to 0$ as $l \to \infty$ then $\lim_{l\to\infty} \frac{a_{k_l+1}}{k_l} > \frac{\ln(3)}{\ln(2)}$ which contradicts the fact that $\lim_{k\to\infty} \frac{a_{k+1}}{k} = \frac{\ln(3)}{\ln(2)}$. On the other hand if there exists a subsequence a_{k_l+1} such that $\frac{3^{k_l}}{2^{a_{k_l+1}}} \to \infty$ as $k \to \infty$ then $\lim_{l\to\infty} \frac{a_{k_l+1}}{k_l} < \frac{\ln(3)}{\ln(2)}$ which contradicts the fact that $\lim_{k\to\infty} \frac{a_{k+1}}{k} = \frac{\ln(3)}{\ln(2)}$, then exists $M, m > 0$ such that $m < \frac{3^k}{2^{a_{k+1}}} < M$. □

Lemma 8. *Let $0 < \varepsilon < \frac{L}{7}$, then ε satisfies the following inequalities $L + \varepsilon < \frac{3}{2}(L - \varepsilon)$ and $\frac{3}{2^{\theta_{k+2}}}(L + \varepsilon) < L - \varepsilon$ if $\theta_{k+2} \geq 2$.*

Proof. Let us verify that ε satisfying $0 < \varepsilon < \frac{L}{7}$ satisfies both inequalities.

1. For the first inequality: $L + \varepsilon < \frac{3}{2}(L - \varepsilon)$, expanding and simplifying, we have $\varepsilon < \frac{L}{5}$. Since ε is less than $\frac{L}{7} < \frac{L}{5}$, this inequality is satisfied.
2. For the second inequality: First we will prove the following inequality

$$\frac{1}{7} \leq \frac{2^{\theta_{k+2}} - 3}{3 + 2^{\theta_{k+2}}}$$

for $\theta_{k+2} \geq 2$, indeed

$$\frac{1}{7} \leq \frac{2^{\theta_{k+2}} - 3}{3 + 2^{\theta_{k+2}}} \text{ this is equivalent to } 24 \leq 6 \cdot 2^{\theta_{k+2}}$$

this inequality is also satisfied.

Now $\frac{3}{2^{\theta_{k+2}}}(L + \varepsilon) < L - \varepsilon$, then

$$\varepsilon < \frac{L(2^{\theta_{k+2}} - 3)}{3 + 2^{\theta_{k+2}}}.$$

Since ε is less than $\frac{L(2^2 - 3)}{3 + 2^2}$ and $\frac{L(2^2 - 3)}{3 + 2^2} \leq \frac{L(2^{\theta_{k+2}} - 3)}{3 + 2^{\theta_{k+2}}}$.

□

Proposition 30. *Let $\xi \in G_1$, then $\frac{3^k}{2^{a_{k+1}}}$ is not convergent on $\mathbb{R}$.*

Proof. Suppose that $\xi = 0^{\theta_1} \prod_{i=2}^{\infty} 10^{\theta_i}$ and exist $L > 0$ such that $\lim_{k\to\infty} \frac{3^k}{2^{a_{k+1}}} = L$. Let $0 < \varepsilon < \frac{L}{7}$ then exist $K > 0$ such that $L - \varepsilon < \frac{3^k}{2^{a_{k+1}}} < L + \varepsilon$. On the other hand we have $\frac{3^{k+1}}{2^{a_{k+2}}} = \frac{3}{2^{\theta_{k+2}}} \frac{3^k}{2^{a_{k+1}}}$, so

$$\frac{3}{2^{\theta_{k+2}}}(L - \varepsilon) < \frac{3^{k+1}}{2^{a_{k+2}}} < \frac{3}{2^{\theta_{k+1}}}(L + \varepsilon)$$

By Lemma 8 we have

1. if $\theta_{k+2} = 1$, then $L + \varepsilon < \frac{3}{2}(L - \varepsilon)$. Therefore $\frac{3^{k+1}}{2^{a_{k+2}}} \notin (L - \varepsilon, L + \varepsilon)$
2. if $\theta_{k+2} \geq 2$. then $\frac{3}{2^{\theta_{k+2}}}(L + \varepsilon) < L - \varepsilon$. Therefore $\frac{3^{k+1}}{2^{a_{k+2}}} \notin (L - \varepsilon, L + \varepsilon)$

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Therefore $\frac{3^k}{2^{a_{k+1}}}$ is not convergent. □

8. The Extension of Collatz Function on $\mathbb{Z}_2$

In this section we will study the extension of the Collatz function on $\mathbb{Z}_2$ proposed by Lagaria in [9], and in an analogous way we will define the dyadic integer sets and the encoding set. We will prove that given an coding there exists a unique dyadic integer with this coding. We will show that this extension is topologically conjugate to the shift function in Σ_2^* and we will use this result to prove that codings in G_1 are unstable.

8.1. Summary of Propositions in the Section

1. **Lemma 9**: Equivalence of the parity of fractions and their dyadic representation.
2. **Definition 19:** Extension of the Collatz function on the set of dyadic numbers and the definitions of dyadic integer set and coding set.
3. **Proposition 31:** Characterization of the dyadic integer set.
4. **Proposition 32:** Establishes that the Coding set and the Dyadic Integer Set are the same.
5. **Proposition 33:** It establishes that given a coding there is a unique dyadic number with said coding.
6. **Theorem 5:** The Collatz function on the set of dyadic numbers is topologically conjugate to the Shift function.
7. **Corollary 6:** The periodic points of the Collatz function in $\mathbb{Z}_2$ are dense in $\mathbb{Z}_2$.
8. **Proposition 34:** The periodic sequences of G_0 correspond to positive periodic points of the Collatz functions and the periodic sequences of G_∞ correspond to negative periodic points of the Collatz functions.

8.2. Extension of the Collatz Function on $\mathbb{Z}_2$.

Now we are going to extend the Collatz function to the set of $\mathbb{Z}_2$. In order for the extension to be compatible with the results obtained in the previous sections, we will first show that the parity of the elements of $\mathbb{Q}_{odd}$ is preserved in $\mathbb{Z}_2$.

Lemma 9. *Let $\beta = \sum_{i=1}^{\infty} \delta_i 2^i \in \mathbb{Z}_2$ the dyadic representation of $\frac{p}{q} \in \mathbb{Q}_{odd}$, then p is even if and only if $\delta_0 = 0$ and p is odd if and only if $\delta_0 = 1$.*

Proof. Let p a even number, so we have

$$\frac{p}{q} = \frac{2k}{q} = \delta_0 + 2M \text{ for any } M \in \mathbb{Z}_2, \text{ so } 2k = q\delta_0 + 2Mq$$
$$\text{then } \delta_0 = 0. \text{ Since } q \text{ is odd number}$$

Let p a odd number, so we have

$$\frac{p}{q} = \frac{2k+1}{q} = \delta_0 + 2M \text{ for any } M \in \mathbb{Z}_2, \text{ so } 2k+1 = q\delta_0 + 2Mq$$
$$\text{then } \delta_0 = 1. \text{ Since } q \text{ is odd number}$$

□

Now let us consider the following extension of the Collatz function on $\mathbb{Z}_2$.

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Definition 19. *Let $Col : \mathbb{Z}_2 \to \mathbb{Z}_2$ given by*

$$Col(\beta) = \begin{cases} \dfrac{3\beta+1}{2} & \text{if } \beta \mod 2 = 1 \\ \\ \dfrac{\beta}{2} & \text{if } \beta \mod 2 = 0 \end{cases}$$

$Cod^k(\beta) = \{\eta_i\}_{i=1}^k$ with $\eta_i = 0$ if $Cod^{i-1}(\beta) = 0 \mod 2$ and $\eta_i = 10$ if $Cod^{i-1}(\beta) = 1 \mod 2$. Let $\xi \in \Sigma_2^*$ then we defined the $k-$Coding set of ξ

$$Cod^k(\xi) = \left\{\beta \in \mathbb{Z}_2; Cod^k(\beta) = \{\xi\}_{i=1}^k\right\}$$

Let $S \in \langle\theta, \psi\rangle$, we define the dyadic integer set of S as

$$\mathbb{D}(S) = \{\beta \in \mathbb{Z}_2; S(\beta) \in \mathbb{Z}_2\}$$

Next we will show the version in $\mathbb{Z}_2$ to the results seen in previous Sections. The following Proposition characterizes the set of dyadic integers of $S_k \in \langle\theta, \psi\rangle$ analogously to the entire set.

Proposition 31. *Let $\xi = 0^{\theta_1}\prod_{j=2}^{\infty} 10^{\theta_j} \in \Sigma_2^*$ and $S_k \in \langle\theta, \psi\rangle$ such that $Cod(S_k) = 0^{\theta_1}\prod_{j=2}^{k+1} 10^{\theta_j}$. Let $\beta \in \mathbb{D}(S_k)$, then $\mathbb{D}(S_k) = \beta_0 + 2^{a_{k+1}}\mathbb{Z}_2$.*

Proof. Let $\beta_0 \in \mathbb{D}(S_k)$ then we have $\beta_0 + 2^{a_{k+1}}\mathbb{Z}_2 \subset \mathbb{D}(S_k)$. Indeed. Let $\xi \in \Sigma_2^*$ and $S_k \in \langle\theta, \psi\rangle$ such that $Cod(S_k) = 0^{\theta_1}\prod_{j=2}^{k+1} 10^{\theta_j}$, then

$$S_k(x) = \frac{3^k x + N_k}{2^{a_{k+1}}}$$

so

$$S_k(\beta_0 + 2^{a_{k+1}}\mathbb{Z}_2) = \frac{3^k(\beta_0 + 2^{a_{k+1}}\mathbb{Z}_2) + N_k}{2^{a_{k+1}}} = \frac{3^k\beta_0 + N_k}{2^{a_{k+1}}} + \mathbb{Z}_2 \subset \mathbb{Z}_2$$

Now $\mathbb{D}(S_k) \subset \beta_0 + 2^{a_{k+1}}\mathbb{Z}_2$. Let $\beta \in \mathbb{D}(S_k)$, so $3^r\beta + N_k = 0 \mod 2^{a_{k+1}}$, so

$$\begin{aligned} 3^k\beta + N_r - (3^k\beta_0 + N_k) &= 0 \mod 2^{a_{k+1}} \\ 3^k(\beta - \beta_0) &= 0 \mod 2^{a_{k+1}} \\ \beta - \beta_0 &= 0 \mod 2^{a_{k+1}} \end{aligned}$$

Therefore, we have

$$\beta - \beta_0 = \underbrace{0\ldots0}_{2^{a_{k+1}}}\alpha = 2^{a_{k+1}}\alpha, \text{ with } \alpha \in \mathbb{Z}_2 \Rightarrow \beta = \beta_0 + 2^{a_{k+1}}\alpha$$

□

The following proposition states that the entire dyadic set is equal to the coding set of S_k.

Proposition 32. *Let $\xi = 0^{\theta_1}\prod_{j=2}^{\infty} 10^{\theta_j} \in \Sigma_2^*$ and $S_k \in \langle\theta, \psi\rangle$ such that $Cod(S_k) = 0^{\theta_1}\prod_{j=2}^{k+1} 10^{\theta_j}$. Then $\mathbb{D}(S_k) = Cod^{a_{k+1}}(\xi)$.*

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Proof. Let $\beta \in Cod^{a_{k+1}}(\xi)$, so by definition we have $Cod^{a_{k+1}}(\beta) = 0^{\theta_1}\prod_{j=2}^{k+1} 10^{\theta_j}$. We can rewrite as $\beta = \beta_k + 2^{a_{k+1}}\alpha_k$ for any $\alpha_k \in \mathbb{Z}_2$ and $\beta_{a_{k+1}} = \beta \mod 2^{a_{k+1}}$. We Claim that $\beta_{a_{k+1}} \in Cod^{a_{k+1}}(\xi)$. Indeed

$$Col^l(\beta) = Col^l(\beta_k) + 2^{a_{k+1}-l}\alpha_k \text{ with } l \leq a_{k+1},$$

We have that the parity on the right side only depends on $Col^l(\beta_{a_{k+1}})$, then the latter must have the same $j-$coding as $Col^j(\beta)$ by Proposition 11 we have $\beta_{a_{k+1}} \in \mathbb{E}(S_k)$. So $S_k(\beta) = S_k(\beta_k) + \alpha_k \in \mathbb{Z}_2$, so $\beta \in \mathbb{D}(S_k)$.

Let $\beta \in \mathbb{D}(S_k)$ and let $\beta_k = \beta \mod 2^{a_{k+1}}$, then $\beta_k \in \mathbb{D}(S_k)$. On the other hand, since $\beta_{a_{k+1}}$ trivially has a representation as a natural number, we have that $S_k(\beta_{a_{k+1}})$ it is also a natural number, so $\beta_{a_{k+1}} \in \mathbb{E}(S_k)$ by Proposition 11 we have that $\beta_{a_{k+1}} \in Cod^{a_{k+1}}(\xi)$. By hypothesis we have that $\beta_{a_{k+1}} = \beta \mod 2^{a_{k+1}}$, then exist $\alpha_k \in \mathbb{Z}_2$ such that $\beta = \beta_{a_{k+1}} + 2^{a_{k+1}}\alpha_k$, applying S_k $l-$times with $l \leq a_{k+1}$, we have

$$Col^l(\beta) = Col^l(\beta_k) + 2^{a_{k+1}-l}\alpha_k$$

Since $2^{a_{k+1}-l}\alpha_k$ is always even for all $l < a_{k+1}$, we have that the parity of each iteration only depends on $Col^l(\beta_k)$, then $\beta \in Cod^{a_{k+1}}(\xi)$. □

In theorem 3 we saw that given $\xi \in \Sigma_2^*$ if there exists a rational whose encoding is exactly ξ, then it is the only rational solution. However we could not guarantee the existence of such a number. The following Proposition guarantees us that there exists a solution in the set of dyadic numbers.

Proposition 33. *Let $\xi \in \Sigma_2^*$, then exist an unique $\beta \in \mathbb{Z}_2$ such that $Cod(\beta) = \xi$.*

Proof. Let $\xi = 0^{\theta_1}\prod_{j=2}^{\infty} 10^{\theta_j} \in \Sigma_2^*$ by Lemma 27 we have $-\pi^1(\xi) \in \mathbb{Z}_2$. Now we are going to prove that:

Claim $-\pi^1(\xi) \in Cod^{a_{k+1}}(\xi)$ for all $k \in \mathbb{N}$. Indeed, let $S_k \in \langle \theta, \psi \rangle$ such that $Cod(S_k) = 0^{\theta_1}\prod_{j=2}^{k+1} 10^{\theta_j}$ then $S_k(x) = \dfrac{3^k x + N_k}{2^{a_{k+1}}}$, we have $S_k(-\pi_k^1(\xi)) = 0$ for all $k \in \mathbb{N}$, Then

$$\begin{aligned}
-\pi^1(\xi) &= -\pi_k^1(\xi) - \sum_{j=k+1}^{\infty} \frac{2^{a_j}}{3^j} \\
\text{so } S_k(-\pi^1(\xi)) &= S_k(-\pi_k^1(\xi)) - \frac{3^k}{2^{a_{k+1}}} \sum_{j=k+1}^{\infty} \frac{2^{a_j}}{3^j} \\
&= -\sum_{j=k+1}^{\infty} \frac{2^{a_j - a_{k+1}}}{3^{j-k}} \in \mathbb{Z}_2 \text{ since } \left\| \sum_{j=k+1}^{\infty} \frac{2^{a_j - a_{k+1}}}{3^{j-k}} \right\|_2 = 2^0 = 1
\end{aligned}$$

By Proposition 32 we have $-\pi^1(\xi) \in Cod^{a_{k+1}}(\xi)$ for all $k \in \mathbb{N}$. Since $Cod^k(\xi) \subset Cod^{k-1}(\xi)$, we have

$$-\lim_{k\to\infty} \pi_k^1(\xi) \in \bigcap_{k\in\mathbb{N}} Cod^{a_{k+1}}(\xi) = Cod(\xi)$$

Suppose there exists another dyadic integer α such that it is also in $Cod(\xi)$, then

$$\alpha, -\pi^1(\xi) \in Cod(S_k) \text{ for all } k \in \mathbb{N} \text{ so } \|\pi^1(\xi) - \alpha\|_2 < 2^{-a_{k+1}} \to 0 \text{ as } k \to \infty$$

Therefore $\alpha = -\pi^1(\xi)$. □

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The existence of solutions to the equation $Cod(\beta) = \xi$ in the dyadic numbers does not guarantee the existence of rational solutions. This will depend primarily on whether the dyadic solution can be represented as a rational number or, more generally, as a real number. Based on the nature of this solution, we can determine whether or not a rational solution exists.

8.3. Topological Conjugation

The Shift function ω on Σ_2^* is defined as the mapping that deletes the first term of the sequence. The following theorem states that the Collatz function on $\mathbb{Z}_2$ is dynamically equivalent to the Shift function on Σ_2^* and that the function $-\pi^1$ is a homeomorphism between these two spaces. A similar result can be found in [9] where the Shift function is defined on $\mathbb{Z}_2$ instead of Σ_2^*.

Theorem 5 (Col is topologically conjugate to ω)**.** *Let $\omega : \Sigma_2^* \to \Sigma_2^*$ given by*

$$\omega\left(\{\xi_j\}_{j=1}^{\infty}\right) = \begin{cases} \{\xi_{j+2}\}_{j=1}^{\infty} & \text{if } \xi_1 = 1 \\ \{\xi_{j+1}\}_{j=1}^{\infty} & \text{if } \xi_1 = 0 \end{cases}$$

Col is topologically conjugate to ω i.e the following diagram is commutative

$$\begin{tikzcd} \Sigma_2^* \arrow[r, "\omega"] \arrow[d, "-\pi^1"'] \arrow[dr, phantom, "\circlearrowright"] & \Sigma_2^* \arrow[d, "-\pi^1"] \\ \mathbb{Z}_2 \arrow[r, "Col"'] & \mathbb{Z}_2 \end{tikzcd}$$

and $-\pi^1$ is a homemorphism.

Proof. Let's first prove that the diagram is commutative.

Let $\{\xi_j\}_{j=1}^{\infty} = 10^{\theta_1}10^{\theta_2}\ldots = \prod_{j=1}^{\infty} 10^{\theta_j} \in \Sigma_2^*$ with $\theta_j \in \mathbb{N}$. Writing this way, we get an explicit form for the function $a_k = \sum_{j=1}^{k} \theta_j$. If $a_1 = 0$ we have

$$\begin{aligned} Col \circ (-\pi^1)\left(\prod_{j=1}^{\infty} 10^{\theta_j}\right) &= Col\left(-\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=1}^{\infty})}}{3^k}\right) \\ &= \frac{1}{2}\left(-3\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=1}^{\infty})}}{3^k} + 1\right) \\ &= \frac{1}{2}\left(-\sum_{k=2}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=1}^{\infty})}}{3^{k-1}} + 1\right) \\ &= \frac{1}{2}\left(-1-\sum_{k=2}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=1}^{\infty})}}{3^{k-1}} + 1\right) \\ &= -\sum_{k=2}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=1}^{\infty})} - 1}{3^{k-1}} \\ &= -\sum_{k=1}^{\infty} \frac{2^{a_{k+1}(\{\xi_j\}_{j=1}^{\infty})} - 1}{3^k} \end{aligned}$$

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and

$$-\pi^1 \circ \omega\left(\prod_{j=1}^{\infty} 10^{\theta_j}\right) = -\pi^1\left(0^{\theta_1 - 1}\prod_{j=2}^{\infty} 10^{\theta_j}\right) = -\sum_{k=1}^{\infty} \frac{2^{a_{k+1}(\{\xi_i\}_{i=1}^{\infty})} - 1}{3^k}$$

where both parts are equal. On the other hand. suppose that $\{\xi_j\}_{j=1}^{\infty} = 0^{\theta_1}10^{\theta_2}1\ldots = \prod_{j=1}^{\infty} 0^{\theta_j}1 \in \Sigma_2^*$ with $\theta_j \in \mathbb{N}$ and $\theta_1 > 0$. We have

$$Col \circ (-\pi^1)\left(\prod_{j=1}^{\infty} 0^{\theta_j}1\right) = Col\left(-\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_i\}_{i=1}^{\infty})}}{3^k}\right) = -\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_i\}_{i=1}^{\infty})} - 1}{3^k}$$

and

$$(-\pi^1) \circ \omega\left(\prod_{j=1}^{\infty} 0^{\theta_j}1\right) = (-\pi^1) \circ \left(\theta^{\theta_1 - 1}\prod_{j=1}^{\infty} 0^{\theta_j}1\right) = -\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=1}^{\infty})} - 1}{3^k}$$

where again both parts are equal. Then we conclude that the diagram is commutative.

Now we are going to prove that $-\pi^1$ is a bijection.

Let $Cod : \mathbb{Z}_2 \to \Sigma_2^*$ given by $\xi_j = 0$ if $Col^{j-1}(\beta) = 0 \mod 2$ and $\xi_j = 10$ if $Col^{j-1}(\beta) = 1 \mod 2$. Let's prove that $Cod \circ -\pi^1(\xi) = \xi$ and $-\pi^1 \circ Cod(\beta) = \beta$

1. $Cod \circ -\pi^1 = Id_{\Sigma_2^*}$: By Corollary 33 we have $Cod(-\pi^1(\xi)) = \xi$.
2. $-\pi^1 \circ Cod = Id_{\mathbb{Z}_2}$: Let $\beta \in \mathbb{Z}_2$ and $\xi \in \Sigma_2^*$ such that $\xi = Cod(\beta)$, so $\beta \in Cod^k(\xi)$ for all $k \in \mathbb{N}$. On the other hand we have $-\pi^1(\xi) \in Cod^k(\xi)$ for all $k \in \mathbb{N}$. Let A_k the number of 0 of $Cod^k(\xi)$ Exists $\gamma_k \in \mathbb{Z}_2$ for all $k \in \mathbb{N}$ such that $\beta = -\pi^1(\xi) + 2^{A_k}\gamma_k$ so $\beta = -\pi^1(\xi) \mod 2^{A_k}$ then $\|\beta - \{-\pi^1(\xi)\}\|_2 \leq 2^{-A_k} \to 0$ as $k \to \infty$ Thus $\beta = -\pi^1(Cod(\beta))$.

Let us show that the applications $-\pi^1$ and Cod are uniformly continuous. Let $D : \Sigma_2^* \times \Sigma_2^* \to [0, \infty)$ the symbolic metric of two symbols given by

$$D(\xi, \eta) = \sum_{j=1}^{\infty} \frac{1}{2^j}\Delta(\xi_j, \eta_j), \text{ with } \xi_j, \eta_j \in \{0, 10\}$$

and

$$\Delta(\xi_j, \eta_j) = \begin{cases} 1 & \text{if} \quad \xi_j \neq \eta_j \\ 0 & \text{if} \quad \xi_j = \eta_j \end{cases}$$

The space (Σ_2^*, D) is a complete metric space with the property that if two sequences are arbitrarily close if and only if their first terms are equal.

$$D(\xi, \eta) < \frac{1}{2^r} \text{ if and only if } \xi_j = \eta_j \text{ for all } j < r.$$

π^1 is uniformly continuous: Let $\varepsilon > 0$ and $A \in \mathbb{N}$ such that $2^{-A} < \varepsilon$ and $\xi, \eta \in \Sigma_2^*$. So let $\delta = \dfrac{1}{2^{A+2}}$

$$D(\xi, \eta) < \delta = \frac{1}{2^{A+2}} \text{ then we have } \xi_j = \eta_j \text{ for } j \leq A + 1$$

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Let b_{A+1} the number from 1 to the $A+1$-th term of ξ, then $a_k(\xi) = a_k(\eta)$ for $k \leq b_{A+1}$, so $-\sum_{k=1}^{b_{A+1}} \frac{2^{a_k(\xi)}}{3^k} = -\sum_{k=1}^{b_{A+1}} \frac{2^{a_k(\eta)}}{3^k} \in Cod^{A+1}(\xi)$, then in particular

$$-\sum_{k=1}^{b_{A+1}} \frac{2^{a_k(\xi)}}{3^k} = -\sum_{k=1}^{b_{A+1}} \frac{2^{a_k(\eta)}}{3^k} \mod 2^{A+1}$$

so

$$-\pi^1(\xi) = -\pi^1(\eta) \mod 2^{A+1} \text{ then we have } \|\pi^1(\xi) - \pi^1(\eta)\|_2 \leq 2^{-A-1} < 2^{-A} < \varepsilon.$$

Cod **is uniformly continuous:** Let $\varepsilon > 0$ and $A \in \mathbb{N}$ such that $\delta = 2^{-A} < \varepsilon$ and $\alpha, \beta \in \mathbb{Z}_2$ such that $\|\alpha - \beta\|_2 < \delta = 2^{-A}$ then $\alpha = \beta \mod 2^A$, so $Cod^A(\alpha) = Cod^A(\beta)$, so $D(Cod(\alpha), Cod(\beta)) < \frac{1}{2^A} < \varepsilon$.

Therefore π^1 is a homemorphism and *Col* is topologically conjugate to ω. □

8.4. Periodic Point

As a first consequence of topological conjugation, we have that the set of periodic points of the Collatz function is dense in $\mathbb{Z}_2$.

Corollary 6. *Let $Per(\omega)$ the set of periodic point of ω, then we have $\overline{-\pi^1(Per(\omega))} = \mathbb{Z}_2$*

Proof. consequence of the continuity of the function $-\pi^1$ and the fact that the periodic sequences of the Shift function are dense in Σ_2^*. □

As a consequence of the topological conjugation we have that $\pi^1(\xi)$ is a periodic point if ξ is periodic, we are going to show that the periodic points in G_0 are positive and the periodic points in G_∞ are negative.

Proposition 34. *Let $\xi \in \Sigma_2^*$ periodic and $\pi^1 : \Sigma_2^* \to \mathbb{Q}_2$, then we have:*

1. *If $\xi \in G_0$ so $-\pi^1(\xi)$ has positive rational representation.*
2. *If $\xi \in G_\infty$ so $-\pi^1(\xi)$ has negative rational representation.*

Proof. Let $\xi = 10^{\theta_2}10^{\theta_3}\ldots10^{\theta_K}10^{\theta_{K+1}}10^{\theta_2}10^{\theta_3}\ldots = \prod_{j=2}^{\infty} 10^{\theta_j} \in \Sigma_2^*$ with $\theta_{K+2} = \theta_2$. Considering that $\theta_1 = 0$ so we have $a_k = \sum_{j=1}^{k} \theta_j$, Let us first show that $a_{r(K+1)+l} = ra_{k+1} + a_l$ with $l < K$ for all $r \in \mathbb{N}$:

$$\begin{aligned}
a_{r(K+1)+l} = &\sum_{j=1}^{r(K+1)+l} \theta_j \\
&= (\theta_2 + \theta_3 + \ldots + \theta_{K+1}) + (\theta_{K+2} + \theta_{K+3} + \ldots + \theta_{K+(K+1)}) \\
&+ \ldots + (\theta_{rK+2} + \theta_{rK+3} + \ldots + \theta_{rK+l}) \\
&= (\theta_2 + \theta_3 + \ldots + \theta_{K+1}) + (\theta_2 + \theta_3 + \ldots + \theta_{K+1}) \\
&+ \ldots + (\theta_2 + \theta_3 + \ldots + \theta_l) \\
&= ra_{K+1} + a_l
\end{aligned}$$

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We are going to show that $-\pi^1(\xi)$ is rational

$$
\begin{aligned}
&-\sum_{j=1}^{\infty}\frac{2^{a_j}}{3^j}\\
&=-\left\{\frac{1}{3}+\frac{2^{a_2}}{3^2}+\ldots+\frac{2^{a_K}}{3^K}\right\}-\left\{\frac{2^{a_{K+1}}}{3^{K+1}}+\frac{2^{a_{K+2}}}{3^{K+2}}+\ldots+\frac{2^{a_{2K}}}{3^{2K}}\right\}\\
&-\ldots-\left\{\frac{2^{a_{rK+1}}}{3^{rK+2}}+\frac{2^{a_{rK+2}}}{3^{rK+2}}+\ldots+\frac{2^{a_{(r+1)K}}}{3^{(r+1)K}}\right\}-\ldots\\
&=-\left\{\frac{1}{3}+\frac{2^{a_2}}{3^2}+\ldots+\frac{2^{a_K}}{3^K}\right\}-\left\{\frac{2^{a_{K+1}}}{3^{K+1}}+\frac{2^{a_{K+1}+a_2}}{3^{K+2}}+\ldots+\frac{2^{a_{K+1}+a_K}}{3^{2K}}\right\}\\
&-\ldots-\left\{\frac{2^{ra_{K+1}}}{3^{rK+1}}+\frac{2^{ra_{K+1}+a_2}}{3^{rK+2}}+\ldots+\frac{2^{ra_{K+1}+a_K}}{3^{(r+1)K}}\right\}-\ldots\\
&=-\left\{\frac{1}{3}+\frac{2^{a_1}}{3^2}+\ldots+\frac{2^{a_K}}{3^K}\right\}-\frac{2^{a_{K+1}}}{3^K}\left\{\frac{1}{3}+\frac{2^{a_1}}{3^2}+\ldots+\frac{2^{a_K}}{3^K}\right\}\\
&-\ldots-\left(\frac{2^{a_{K+1}}}{3^K}\right)^r\left\{\frac{1}{3}+\frac{2^{a_1}}{3^2}+\ldots+\frac{2^{a_K}}{3^K}\right\}-\ldots\\
&=-\left\{\frac{1}{3}+\frac{2^{a_1}}{3^2}+\ldots+\frac{2^{a_K}}{3^K}\right\}\sum_{r=0}^{\infty}\left(\frac{2^{a_{K+1}}}{3^K}\right)^r\\
&=-\left\{\frac{1}{3}+\frac{2^{a_1}}{3^2}+\ldots+\frac{2^{a_K}}{3^K}\right\}\left(\frac{1}{1-\frac{2^{a_{K+1}}}{3^K}}\right)\\
&=\pi_k^1(\xi)\left(\frac{1}{1-\frac{2^{a_{K+1}}}{3^K}}\right)\\
&=\frac{3^{K-1}+2^{a_1}\cdot 3^{K-2}+\ldots+2^{a_K}}{2^{a_{K+1}}-3^K}\in\mathbb{Q}
\end{aligned}
$$

which corresponds to a rational number. For the case $\xi=0^{\theta_1}\prod_{j=2}^{\infty}10^{\theta_j}$ with $\theta_0>0$, we take $\eta=\prod_{j=1}^{\infty}10^{\theta_j}$, we have that even η is of period, then

$$
-\pi^1(\eta)=-\sum_{j=1}^{\infty}\frac{2^{a_j}}{3^j}=\frac{3^{K-1}+2^{a_1}\cdot 3^{K-2}+\ldots+2^{a_K}}{2^{a_{K+1}}-3^K}
$$

then

$$
\begin{aligned}
-\pi^1(\xi)=-\pi^1(0^{\theta_1}\eta)=-2^{\theta_1}\sum_{j=1}^{\infty}\frac{2^{a_j(\eta)}}{3^j}&\\
&=-2^{\theta_1}\left\{\frac{1}{3}+\frac{2^{a_1(\eta)}}{3^2}+\ldots+\frac{2^{a_K(\eta)}}{3^K}\right\}\left(\frac{1}{1-\frac{2^{a_{K+1}(\eta)}}{3^K}}\right)\\
&=-\left\{\frac{2^{a_1(\xi)}}{3}+\frac{2^{a_2(\xi)}}{3^2}+\ldots+\frac{2^{a_K(\xi)}}{3^K}\right\}\left(\frac{1}{1-\frac{2^{(a_{K+1}-a_1)(\xi)}}{3^K}}\right)\in\mathbb{Q}
\end{aligned}
$$

where its sign depends on the denominator $1-\frac{2^{a_{K+1}}}{3^K}$, if ξ are in G_0 if and only if $\left(\frac{3^K}{2^{a_{K+1}}}\right)^r\to 0$ as $r\to\infty$ if and only if $\frac{3^K}{2^{a_{K+1}}}<1$ so the denominator is negative, so $-\pi^1(\xi)$ is positive, analogously

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when ξ is in G_∞ if and only if $\left(\frac{3^K}{2^{a_{K+1}}}\right)^r \to \infty$ as $r \to \infty$ if and only if $\frac{3^K}{2^{a_{K+1}}} > 1$ so the denominator is positive, so $-\pi^1(\xi)$ is negative. □

9. Real Function π^1 and π^2 Function

Let's define a new function π^1 defined on $\pi^1 : \Sigma_2^* \to [0,\infty]$ given by $\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$, unlike $\pi^1 : \Sigma_2^* \to \mathbb{Q}_2$ case ξ does not have a null tail and $\sum_{j=1}^{K} \frac{2^{a_j}}{3^j}$ when it has a null tail with index J. It is not always convergent. Does this mean that when it is divergent, then there is no solution to the encoding problem? The answer is no, for example if we take the encoding 1001000100... which corresponds to the encoding of 1, however the function $\pi^1(\xi)$ is divergent. As we will show in the next section, when it is convergent, it is in fact the only solution to the encoding problem. In addition to the real function π^1, we will define the function π^2 which unlike π^1 which is a series, this is a function on the natural numbers to the rational numbers. We will show that the function $\pi^1 : \Sigma_2^* \to [0,\infty]$ is convergent if and only if $\xi \in G_\infty$ and that the function π^2 is bounded if and only if $\xi \in G_0$.

9.1. Summary of Propositions in the Section

1. **Definition 20:** We will give the definition of the functions π^1 and π^2.
2. **Proposition 35** Characterization of G_0 and G_∞ through functions π^1 and π^2.
   (a) $\xi \in G_\infty$ if and only if $\pi^1(\xi) < \infty$.
   (b) $\xi \in G_0$ if and only if $\pi^2(\xi)(k)$ is bounded.
3. **Lemma 10:** Let $\liminf_{k\to\infty} \frac{a_{k+1}}{k} = \lambda \in (0,\infty]$. Then exist $T > 0$ such that if $T < k - j$ we have $\frac{a_{k+1} - a_j}{k-j} \geq \lambda$.
4. **Lemma 11:** Let $k, j \in \mathbb{N}$, if $k > j$ then, we have $\frac{a_{k+1} - a_j}{k-j} \geq 1$.
5. **Corollary 7:** Let $\xi \in \Sigma_2^*$. If exist a sub-sequence such that $\lim_{j\to\infty} \frac{a_{k_l+1}(\xi)}{k_l} > \frac{\ln(3)}{\ln(2)}$. Then exist $\Omega > 0$ such that $\pi^2(\xi)(k_l) < \Omega$.
6. **Definition 21:** Definition fix function.
7. **Proposition 36:** Let $\xi \in \Sigma_2^*$ and $\pi^1 : \Sigma_2^* \to \mathbb{R}$ and $\pi^2, \text{fix} : \Sigma_2^* \to \{f : \mathbb{N} \to \mathbb{Q}\}$. Then we have:
   (a) if $\xi \in G_\infty$ then exist $\lim_{k\to\infty} \text{fix}(\xi)$ and $\lim_{k\to\infty} \text{fix}(\xi) = -\pi^1(\xi)$.
   (b) if $\xi \in G_0$ then $\text{fix}(\xi)$ is bounded and if $\pi^2(\xi)(k_j)$ is a subsequence convergent then $\lim_{j\to\infty} \text{fix}(\xi)(k_j) = \lim_{j\to\infty} \pi^2(\xi)(k_j)$.
   (c) if $\xi \in G_1$ then $\lim_{k\to\infty} |\text{fix}(k)| = \infty$.
8. **Theorem 6:** Let $\xi \in G_1$, then not exist $\frac{p}{q} \in \mathbb{Q}$ such that $Cod\left(\frac{p}{q}\right) = \xi$. In particular ξ is positive and negative unstable.

9.2. The π^1 and π^2 Functions

We will give the definition of the functions π.

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Definition 20 (The π^1 and π^2 functions)**.** *Let $\xi \in \Sigma_2^*$ without null tail given by $\xi = 0^{\theta_1}\prod_{j=2}^{\infty} 10^{\theta_j}$ with $\theta_1 \geq 0$ and $\theta_k > 0$ for $k > 1$. Let $a_k = \sum_{j=1}^{k} \theta_j$. Define the function $N_k : \Sigma_2^* \to \mathbb{N}_0$ given by*

$$N_k\{\xi_j\}_{j=1}^{\infty} = \begin{cases} 3^{k-1}2^{a_1} + 3^{k-2}2^{a_2} + \ldots + 2^{a_k} & \text{if} \quad k > 1 \\ 0 & \text{if} \quad k = 1 \end{cases}$$

and defined $\pi_k^1 : \Sigma_2^* \to \mathbb{Q}$ by

$$\pi_k^1(\xi) = \frac{N_k(\xi)}{3^k} = \sum_{j=1}^{k} \frac{2^{a_j}}{3^j}$$

and defined the π^1-function as $\pi^1 : \Sigma_2^* \to \mathbb{R} \cup \{\infty\}$

$$\pi^1(\xi) = \lim_{k\to\infty} \frac{N_k(\xi)}{3^k} = \sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$$

and $\pi^2 : \Sigma_2^* \to \{f : \mathbb{N} \to \mathbb{Q} : f \text{ function}\}$ by

$$\pi^2(\xi)(k) = \frac{N_k(\xi)}{2^{a_{k+1}}(\xi)} = \frac{3^k}{2^{a_{k+1}}}\pi_k^1(\xi) = \frac{3^k}{2^{a_{k+1}}}\sum_{j=1}^{k} \frac{2^{a_j}}{3^j}$$

Let $\xi \in \Sigma_2^*$ with null tail and index $K \geq 0$ given by $\xi = 0^{\theta_1}\prod_{j=2}^{K} 10^{\theta_j} \prod_{j=K+1}^{\infty} 0$ with $\theta_1 \geq 0$ and $\theta_j > 0$ for $j > 1$. Define $\pi^1 : \Sigma_2^* \to \mathbb{R} \cup \{\infty\}$

$$\pi_k^1(\xi) = \sum_{j=1}^{k} \frac{2^{a_j}}{3^j} \text{ if } k < K \text{ and}$$

$$\pi_k^1(\xi) = \sum_{j=1}^{K-1} \frac{2^{a_j}}{3^j} \text{ if } k \geq K$$

and $\pi^2 : \Sigma_2^* \to \{f : \mathbb{N} \to \mathbb{Q} : f \text{ function}\}$ by

$$\pi^2(\xi)(k) = \frac{N_k(\xi)}{2^{a_{k+1}}(\xi)} = \frac{3^k}{2^{a_{k+1}}}\pi_k^1(\xi) = \frac{3^k}{2^{a_{k+1}}}\sum_{j=1}^{k} \frac{2^{a_j}}{3^j} \text{ if } k < K \text{ and}$$

$$\pi^2(\xi)(K+k) = \frac{N_k(\xi)}{2^{a_K}+k} = \frac{3^K}{2^{a_K}+k}\pi_K^1(\xi) = \frac{3^{K-1}}{2^{a_K}+k}\sum_{j=1}^{K-1} \frac{2^{a_j}}{3^j} \text{ for } k \geq 0$$

Example 13. *We will give some examples of the functions π^1 and π^2:*

1. *Let $\xi_0 = 00101001000000\ldots0000\ldots \in \Sigma_2^*$*
   (a) $\pi^1(\xi_0) = \frac{2^2}{3} + \frac{2^3}{9} + \frac{2^5}{27}$.
   (b) $\pi^2(\xi_0)(k) = \frac{27}{2^{k+2}}\left\{\frac{2^2}{3} + \frac{2^3}{9} + \frac{2^5}{27}\right\}$ *for $k \geq 4$ and* $\lim_{k\to\infty} \pi^2(\xi_0)(k) = 0$
2. *Let $\xi_1 = 100100100\ldots \in \Sigma_2^*$ then we have*
   (a) $\pi^1(\xi_1) = \sum_{j=1}^{\infty} \frac{2^{2(j-1)}}{3^j} = \frac{1}{3}\sum_{j=1}^{\infty} \frac{2^{2j}}{3^j} = \frac{1}{3}\sum_{j=1}^{\infty}\left(\frac{4}{3}\right)^j = \infty.$

(b) $$\pi^2(\xi_1)(k) = \frac{3^k}{2^{2k}} \sum_{j=1}^{k} \frac{2^{2(j-1)}}{3^j} = \frac{3^{k-1}}{2^{2k}} \sum_{j=1}^{k-1} \frac{2^{2j}}{3^j}$$
$$= \frac{3^{k-1}}{2^{2k}} \left\{ \frac{\frac{2^{2k}}{3^k} - 1}{\frac{4}{3} - 1} \right\} = \left\{ 1 - \left\{ \frac{3}{4} \right\}^k \right\}.$$

3. *Let* $\xi_2 = 10101010\ldots \in \Sigma_2^*$ *then we have*

(a) $$\pi^1(\xi_2) = \sum_{j=1}^{\infty} \frac{2^{j-1}}{3^j} = \frac{1}{3} \sum_{j=1}^{\infty} \frac{2^j}{3^j} = \frac{1}{3} \sum_{j=1}^{\infty} \left(\frac{2}{3} \right)^j = 1.$$

(b) $$\pi^2(\xi_2)(k) = \frac{3^k}{2^k} \sum_{j=1}^{k} \frac{2^{j-1}}{3^j} = \frac{3^{k-1}}{2^k} \sum_{j=1}^{k-1} \frac{2^j}{3^j}$$
$$= \frac{3^{k-1}}{2^k} \left\{ \frac{\frac{2^k}{3^k} - 1}{\frac{3}{3} - 1} \right\} = \left\{ \left\{ \frac{3}{2} \right\}^k - 1 \right\} \to \infty.$$

4. *Let* $\xi_3 = 10100100010000\ldots \in \Sigma_2^*$

(a) $$\pi^1(\xi_3) = \sum_{j=1}^{\infty} \frac{2^{\frac{j(j+1)}{2} - 1}}{3^j} \to \infty. \text{ Since, } \frac{2^{\frac{j(j+1)}{2}}}{3^j} \to \infty.$$

(b) $$\pi^2(\xi_3)(k) = \frac{3^k}{2^{\frac{k(k+1)}{2}}} \sum_{j=1}^{k} \frac{2^{\frac{j(j+1)}{2} - 1}}{3^j} < 1$$

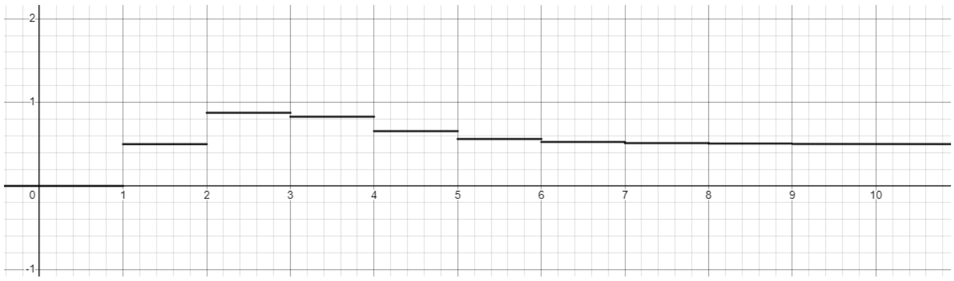

Figure 3. $\pi^2(\xi_3)$

5. *Let* $\xi_4 = 101000101000\ldots \in \Sigma_2^*$ *then we have*

(a) $$\pi^1(\xi_4) = \frac{1}{3} + \frac{2}{3^2} + \frac{2^4}{3^3} + \frac{2^5}{3^4} + \frac{2^8}{3^5} + \frac{2^9}{3^6} + \frac{2^{12}}{3^7} + \ldots$$
$$= \frac{1}{3} + \left\{ \frac{2}{3^2} + \frac{2^4}{3^3} \right\} + \frac{2^4}{3^2} \left\{ \frac{2}{3^2} + \frac{2^4}{3^3} \right\} + \left(\frac{2^4}{3^2} \right)^2 \left\{ \frac{2}{3^2} + \frac{2^4}{3^3} \right\} + \ldots$$
$$= \frac{1}{3} + \left\{ \frac{2}{3^2} + \frac{2^4}{3^3} \right\} \sum_{j=1}^{\infty} \left(\frac{2^4}{3^2} \right)^j \to \infty.$$

(b) $$\pi^2(\xi_4)(1) = \frac{1}{2}, \pi^2(\xi_4)(2) = \frac{5}{2^4},$$
$$\pi^2(\xi_4)(2k+1) = \frac{3^{2k+1}}{2^{4k+1}} \left(\frac{1}{3} + \left\{ \frac{2}{3^2} + \frac{2^4}{3^3} \right\} \sum_{j=1}^{k-1} \left(\frac{2^4}{3^2} \right)^j \right)$$
and $$\pi^2(\xi_4)(2k) = \frac{3^{2k}}{2^{4(k+1)}} \left(\frac{1}{3} + \left\{ \frac{2}{3^2} + \frac{2^4}{3^3} \right\} \sum_{j=1}^{k-2} \left(\frac{2^4}{3^2} \right)^j + \frac{2^{4k+1}}{3^{2k}} \right)$$

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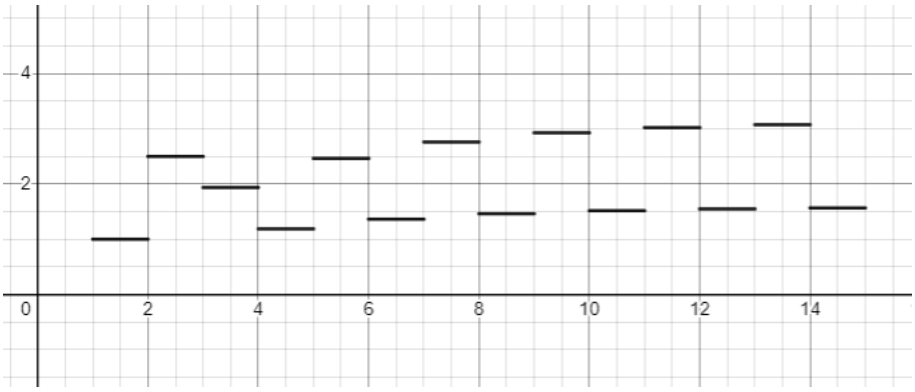

Figure 4. $\pi^2(\xi_4)$

The following result establishes a characterization of the elements of G_0 and G_∞ for the behaviour of the functions π.

Proposition 35 (Characterization of the G_0 and G_∞ sets)**.** *Let $\xi \in \Sigma_2^*$ $\pi^1 : \Sigma_2^* \to \mathbb{R}$ and $\pi^2 : \Sigma_2^* \to \{f : \mathbb{N} \to \mathbb{Q}\}$. Then*

1. *$\xi \in G_\infty$ if and only if $\pi^1(\xi) < \infty$.*
2. *$\xi \in G_0$ if and only if $\pi^2(\xi)(k)$ is bounded.*

Proof. *Proof of the first statement.* Is obvious for the case of null tails with index J, since we have π^1 it is automatically finite, and as we see in the examples π^2 would be of the form $\frac{3^{k_0}}{2^k}A \to 0$ when $k \to \infty$ which implies that π^2 is finite.

So we are going to assume that ξ has no tail null.

Suppose that $\xi \in G_\infty$, then by Lemma 7 we have $\limsup_{j\to\infty} \frac{a_j}{j} \le \limsup_{j\to\infty} \frac{a_{j+1}}{j} < \frac{\ln(3)}{\ln(2)}$, so, there exists $J, \varepsilon > 0$ such that for all $j > J$ we have that $\frac{a_j}{j} < \frac{\ln(3)}{\ln(2)} - \varepsilon$

$$\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j} = \sum_{j=1}^{J} \frac{2^{a_j}}{3^j} + \sum_{j=J+1}^{\infty} \frac{2^{a_j}}{3^j} = \sum_{j=1}^{J} \frac{2^{a_j}}{3^j} + \sum_{j=J+1}^{\infty} \left(\frac{2^{\frac{a_j}{j}}}{3} \right)^j$$

$$< \sum_{j=1}^{J} \frac{2^{a_j}}{3^j} + \sum_{j=J+1}^{\infty} \left(\frac{2^{\frac{\ln(3)}{\ln(2)} - \varepsilon}}{3} \right)^j = \sum_{j=1}^{J} \frac{2^{a_j}}{3^j} + \sum_{j=J+1}^{\infty} 2^{-\varepsilon j} < \infty$$

Let's suppose $\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j} < \infty$, then

$$\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j} = \frac{1}{3} \sum_{j=1}^{\infty} \frac{2^{a_j}}{3^{j-1}} = \frac{1}{3} \sum_{j=0}^{\infty} \frac{2^{a_{j+1}}}{3^j}$$

so $\lim_{j\to\infty} \frac{2^{a_{j+1}}}{3^j} = 0$ then $\xi \in G_\infty$.

□ **of the first statement.**

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Proof of the second statement. Suppose $\pi^2(\xi)(j)$ is bounded, we will prove that $\dfrac{3^k}{2^{a_{k+1}(\xi)}}$ converges to 0. Suppose $\dfrac{3^k}{2^{a_{k+1}(\xi)}} > \varepsilon > 0$ for $k > K$ for any $K \in \mathbb{N}$. Then we have

$$\pi^2(\xi)(k) = \frac{N_k}{2^{a_{k+1}}} = \frac{3^k}{2^{a_{k+1}}} \sum_{j=1}^{k} \frac{2^{a_j}}{3^j}$$

We have that the sum on the right is divergent.

$$\frac{3^k}{2^{a_{k+1}}} \sum_{j=1}^{k} \frac{2^{a_j}}{3^j} > \varepsilon \sum_{j=1}^{k} \frac{2^{a_j}}{3^j} = \infty$$

which generates a contradiction to the fact that $\pi^2\{\xi\}(k)$ is bounded.

To demonstrate the other implication, let us consider the following lemmas:

Lemma 10. *Let* $\liminf_{k\to\infty} \dfrac{a_{k+1}}{k} = \lambda \in (0, \infty]$. *Then exist* $T > 0$ *such that if* $T < k - j$ *we have*

$$\frac{a_{k+1} - a_j}{k - j} \geq \lambda.$$

Proof. Let $\liminf_{k\to\infty} \dfrac{a_{k+1}}{k} = \lambda \in (0, \infty]$ and $d = k - j$, then we have

$$\liminf_{(k-j)\to\infty} \frac{a_{k+1} - a_j}{k - j} = \liminf_{d\to\infty} \frac{a_{j+d+1} - a_j}{d} = \liminf_{d\to\infty} \frac{a_{d+1}}{d} = \lambda$$

On the other hand, by definition of lower limit, we have

$$\liminf_{(k-j)\to\infty} \frac{a_{k+1} - a_j}{k - j} = \lim_{t\to\infty} \left\{ \inf_{(k-j)>t} \frac{a_{k+1} - a_j}{k - j} \right\}$$

Then exist $T > 0$ such that if $T < k - j$ we have

$$\frac{a_{k+1} - a_j}{k - j} \geq \inf_{(k-j)>T} \frac{a_{k+1} - a_j}{k - j} = \lambda$$

□

Lemma 11. *Let* $k, j \in \mathbb{N}$, *if* $k > j$ *then, we have* $\dfrac{a_{k+1} - a_j}{k - j} \geq 1$.

Proof. Let $\xi \in \Sigma_2^*$ writing explicitly, we have $\xi = 0^{\theta_1} \prod_{i=2}^{\infty} 10^{\theta_i}$ with $\theta_1 \geq 0$ and $\theta_i > 0$ for $i > 1$, then we can write:

$$a_k = \sum_{i=1}^{k} \theta_i$$

Suppose $k > j$. Since the minimum value that θ can take is 1, we have

$$\frac{a_{k+1} - a_j}{k - j} = \frac{\sum_{i=j}^{k+1} \theta_i}{k - j} \geq \frac{k + 1 - j}{k - j} > \frac{k - j}{k - j} = 1$$

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□

By Claim 3 and 4 we have, exist $T, \varepsilon \in \mathbb{N}$ such that if $k - T > j$ we have

$$\frac{a_{k+1} - a_j}{k - j} > \frac{\ln(3)}{\ln(2)} + \varepsilon$$

Then by claim 5, Let $k > T$ so

$$\begin{aligned}
\frac{3^k}{2^{a_{k+1}}} \sum_{j=1}^{k} \frac{2^{a_j}}{3^j} &= \sum_{j=1}^{k} \frac{2^{a_j} - a_{k+1}}{3^j - k} \\
&= \sum_{j=1}^{k} \left(\frac{3}{2^{\frac{a_{k+1} - a_j}{k-j}}} \right)^{k-j} \\
&= \sum_{j=1}^{k-T} \left(\frac{3}{2^{\frac{a_{k+1} - a_j}{k-j}}} \right)^{k-j} + \sum_{j=k-T+1}^{k} \left(\frac{3}{2^{\frac{a_{k+1} - a_j}{k-j}}} \right)^{k-j} \\
&< \sum_{j=1}^{k-T} \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)} + \varepsilon}} \right)^{k-j} + \sum_{j=k-T+1}^{k} \left(\frac{3}{2} \right)^{k-j} \\
&< \sum_{j=1}^{k} \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)} + \varepsilon}} \right)^{k-j} + \sum_{j=k-T+1}^{k} \left(\frac{3}{2} \right)^{k-j} \\
&< \frac{1}{2^{\varepsilon k}} \sum_{j=1}^{k} 2^{\varepsilon j} + \left(\frac{3}{2} \right)^{k} \sum_{j=k-T+1}^{k} \left(\frac{2}{3} \right)^{j} \\
&= \frac{1}{2^{\varepsilon k}} \left(\frac{2^{\varepsilon} - 2^{\varepsilon(k+1)}}{1 - 2^{\varepsilon}} \right) + \left(\frac{3}{2} \right)^{k} \left\{ \frac{\left(\frac{2}{3} \right)^{k+1} - \left(\frac{2}{3} \right)^{k-T+1}}{\frac{2}{3} - 1} \right\} \\
&< \frac{\frac{2^{\varepsilon}}{2^{\varepsilon k}} - 2^{\varepsilon}}{1 - 2^{\varepsilon}} + \frac{\frac{2}{3} - \left(\frac{3}{2} \right)^{T-1}}{\frac{2}{3} - 1} \\
&\leq \frac{2^{\varepsilon}}{2^{\varepsilon} - 1} + 3 \left\{ \left(\frac{3}{2} \right)^{T-1} - \frac{2}{3} \right\}
\end{aligned}$$

Let $M = \frac{2^{\varepsilon}}{2^{\varepsilon} - 1} + 3 \left\{ \left(\frac{3}{2} \right)^{T-1} - \frac{2}{3} \right\} > 0$. Then we have $\frac{3^k}{2^{a_{k+1}}} \sum_{j=1}^{k} \frac{2^{a_j}}{3^j} < M$. Then we conclude that π^2 is bounded.

□ **of the second statement.**

□

The following result is the version of the previous theorem for sub-sequence.

Corollary 7. *Let $\xi \in \Sigma_2^*$. If exist a sub-sequence such that $\lim_{j\to\infty} \frac{a_{k_l+1}(\xi)}{k_l} > \frac{\ln(3)}{\ln(2)}$. Then exist $\Omega > 0$ such that $\pi^2(\xi)(k_l) < \Omega$.*

Proof. Let $j < k_l$ then

$$\lim_{k_l - j\to\infty} \frac{a_{k_l+1} - a_j}{k_l - j} = \lim_{k_l - 1\to\infty} \frac{a_{k_l+1} - a_0}{k_l - 1} = \lim_{l\to\infty} \frac{a_{k_l+1}}{k_l} > \frac{\ln(3)}{\ln(2)}$$

then exist $T, \varepsilon > 0$ such that if $k_l - j > T$ we have

$$\frac{a_{k_l+1} - a_j}{k_l - j} > \frac{\ln(3)}{\ln(2)} + \varepsilon$$

Using the lemma 11 we have

$$\begin{aligned}
\pi^2(\xi)(k_l) &= \frac{3^{k_l}}{2^{a_{k_l+1}}} \sum_{j=1}^{k_l} \frac{2^{a_j}}{3^j} \\
&= \sum_{j=1}^{k_l} \frac{2^{a_j - a_{k_l+1}}}{3^{j - k_l}} \\
&= \sum_{j=1}^{k_l} \left(\frac{3}{2^{\frac{a_{k_l+1} - a_j}{k_l - j}}} \right)^{k_l - j} \\
&= \sum_{j=1}^{k_l - T} \left(\frac{3}{2^{\frac{a_{k_l+1} - a_j}{k_l - j}}} \right)^{k_l - j} + \sum_{j=k_l - T+1}^{k_l} \left(\frac{3}{2^{\frac{a_{k_l+1} - a_j}{k_l - j}}} \right)^{k_l - j} \\
&< \sum_{j=1}^{k_l - T} \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)} + \varepsilon}} \right)^{k_l - j} + \sum_{j=k_l - J+1}^{k_l} \left(\frac{3}{2} \right)^{k_l - j} \\
&< \sum_{j=1}^{k_l} \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)} + \varepsilon}} \right)^{k_l - j} + \sum_{j=k_l - J+1}^{k_l} \left(\frac{3}{2} \right)^{k_l - j} \\
&< \frac{1}{2^{\varepsilon k_l}} \sum_{j=1}^{k_l} 2^{\varepsilon j} + \left(\frac{3}{2} \right)^{k_l} \sum_{j=k_l - J+1}^{k_l} \left(\frac{2}{3} \right)^{j} \\
&= \frac{1}{2^{\varepsilon k_l}} \left(\frac{2^{\varepsilon} - 2^{\varepsilon(k_l+1)}}{1 - 2^{\varepsilon}} \right) + \left(\frac{3}{2} \right)^{k_l} \left\{ \frac{\left(\frac{2}{3} \right)^{k_l+1} - \left(\frac{2}{3} \right)^{k_l - J+1}}{\frac{2}{3} - 1} \right\} \\
&\leq \frac{2^{\varepsilon}}{2^{\varepsilon} - 1} + 3 \left\{ \left(\frac{3}{2} \right)^{J-1} - \frac{2}{3} \right\} = \Omega
\end{aligned}$$

Therefore $\pi^2(\xi)(k_l) < \Omega$ □

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9.3. The Set G_1 Is Unstable

Definition 21. *Let $\xi \in \Sigma_2^*$ without null tail, define the* fix *function as* $\text{fix} : \Sigma_2^* \to \{f : \mathbb{N} \to \mathbb{Q}\}$ *given by* $\text{fix}(\xi) : \mathbb{N} \to \mathbb{Q}$

$$\text{fix}(\xi)(k) = -\left(\frac{1}{1 - \frac{2^{a_{k+1}}}{3^k}}\right) \sum_{j=1}^{k} \frac{2^{a_j}}{3^j} = -\left(\frac{1}{1 - \frac{2^{a+k}}{3^k}}\right) \pi_k^1(\xi)$$

This functions associates to each sequence ξ the fixed point associated with the function $S \in \langle \theta, \psi \rangle$ such that $Cod^k(S) = (\xi_j)_{j=1}^k$. Next we will see the behavior of the function $\text{fix}(\xi)$ when ξ is in G_0, G_∞ and G_1.

Proposition 36. *Let $\xi \in \Sigma_2^*$ and $\pi^1 : \Sigma_2^* \to \mathbb{R}$ and $\pi^2, \text{fix} : \Sigma_2^* \to \{f : \mathbb{N} \to \mathbb{Q}\}$. Then we have:*

1. *if $\xi \in G_\infty$ then exist $\lim_{k\to\infty} \text{fix}(\xi)(k)$ and $\lim_{k\to\infty} \text{fix}(\xi)(k) = -\pi^1(\xi)$.*
2. *if $\xi \in G_0$ then $\text{fix}(\xi)$ is bounded and if $\pi^2(\xi)(k_j)$ is a subsequence convergent then $\lim_{j\to\infty} \text{fix}(\xi)(k_j) = \lim_{j\to\infty} \pi^2(\xi)(k_j)$.*
3. *if $\xi \in G_1$ then there exists a positive monotonic and divergent sequence γ_k such that $\gamma_k \leq |\,\text{fix}(k)|$*

Proof. Let $\xi \in \Sigma_2^*$, then:

1. If $\xi \in G_\infty$ and let $\pi^1 : \Sigma_2^* \to \mathbb{R}$ by Proposition we have

$$\left|\text{fix}(\xi)(k) - (-\pi_k^1(\xi))\right| = \left|-\left(\frac{1}{1 - \frac{2^{a_{k+1}}}{3^k}}\right) \pi_k^1(\xi) + \pi_k^1(\xi)\right| = |\pi_k^1(\xi)| \left|\left(\frac{1}{1 - \frac{2^{a_{k+1}}}{3^k}}\right) - 1\right|$$
$$< \Omega \left|\left(\frac{1}{1 - \frac{2^{a_{k+1}}}{3^k}}\right) - 1\right| \to 0 \text{ as } k \to \infty, \text{ since } \lim_{k\to\infty} \frac{2^{a_{k+1}}}{3^k} = 0$$

then $\lim_{k\to\infty} \text{fix}(\xi) = -\lim_{k\to\infty} \pi_k^1(\xi) = -\pi^1(\xi) \in \mathbb{R}$

2. If $\xi \in G_0$ then $|\pi^2(\xi)| < \Omega$. On the other hand, we have the following.

$$|\,\text{fix}(\xi)(k) - \pi^2(\xi)(k)| = \left|-\left(\frac{1}{1 - \frac{2^{a_{k+1}}}{3^k}}\right) \pi_k^1(\xi) - \frac{3^k}{2^{a_{k+1}}} \pi_k^1(\xi)\right|$$
$$= \left|\frac{3^k}{2^{a_{k+1}}} \pi_k^1(\xi)\right| \left|\frac{\frac{2^{a_{k+1}}}{3^k}}{1 - \frac{2^{a_{k+1}}}{3^k}} + 1\right|$$
$$< \Omega \left|\frac{\frac{2^{a_{k+1}}}{3^k}}{1 - \frac{2^{a_{k+1}}}{3^k}} + 1\right| \to 0 \text{ as } k \to \infty \text{ since } \frac{\frac{2^{a_{k+1}}}{3^k}}{1 - \frac{2^{a_{k+1}}}{3^k}} \to -1$$

Then if $\pi^2(\xi)(k_j)$ is a convergent subsequence, then $\lim_{j\to\infty} \pi^2(\xi)(k_j) = \lim_{j\to\infty} \text{fix}(\xi)(k_j)$. Now we will demonstrate that $\text{fix}(\xi)$ is bounded. Let $K > 0$ such that $|\,\text{fix}(\xi)(k) - \pi^2(\xi)(k)| < 1$ then

$$|\,\text{fix}(\xi)(k)| < \max_{k \leq K}\{\text{fix}(\xi)(k)\} + \Omega + 1$$

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3. If $\xi \in G_1$, then exist $\varepsilon > 0$ such that

$$\left| \frac{1}{1 - \frac{2^{a_{k_j+1}}}{3^{k_j}}} \right| > \varepsilon$$

Suppose that exist a sequence $\{a_{k_j}\}_{j\in\mathbb{N}}$ such that $\left| \frac{1}{1 - \frac{2^{a_{k_j+1}}}{3^{k_j}}} \right| \to 0$ as $j \to \infty$, this is equivalent to $\left| 1 - \frac{2^{a_{k_j+1}}}{3^{k_j}} \right| \to \infty$ as $j \to \infty$ so

$$\lim_{j\to\infty} \frac{2^{a_{k_j+1}}}{3^{k_j}} = \infty$$

This contradicts Proposition 29 where it states that there exist $M, m > 0$ such that $m < \frac{2^{a_{k+1}}}{3^k} < M$ for all $k \in \mathbb{N}$.

Therefore, we have

$$|\operatorname{fix}(k)| = \left| \left(\frac{1}{1 - \frac{2^{a_{k+1}}}{3^k}} \right) \pi_k^1(\xi) \right| > \varepsilon \pi_k^1(\xi)$$

Let us denote by $\gamma_k = \varepsilon \pi_k^1(\xi)$, as $\pi_k^1(\xi) = \sum_{j=1}^{k} \frac{2^{a_j}}{3^j}$, then we have that γ_k is monotonically increasing, on the other hand as $\xi \notin G_\infty$, then by Proposition 35 we have that $\pi_k^1(\xi) \to \infty$ as $k \to \infty$

□

Theorem 6. *Let $\xi \in G_1$, then not exist $\frac{p}{q} \in \mathbb{Q}$ such that $Cod\left(\frac{p}{q}\right) = \xi$. In particular ξ is positive and negative unstable.*

Proof. Let $\xi \in G_1$, for simplicity we will assume that $a_0 = 0$. Let's assume that it is stable, that is, it exists $\frac{p}{q} \in \mathbb{Q}$ such that $Cod\left(\frac{p}{q}\right) = \xi$ and let $-\pi^1 : \Sigma_2^* \to \mathbb{Z}_2$ then we have

$$-\pi^1(\xi) = -\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j} = \frac{p}{q}$$

Let $\xi_k \in \Sigma_2^*$ be a periodic sequence of period $(a_k(\xi) + k - 1)$ such that the first $(a_k(\xi) + k - 1)$ terms of ξ_k coincide with the first $(a_k(\xi) + k - 1)$ terms of ξ. For example:

$$\begin{aligned}
\xi &= 1000101000100101000000000100\cdots \\
\xi_1 &= 000000000000000000000000\cdots \\
\xi_2 &= 100010001000100010001000\cdots \\
\xi_3 &= 100010100010100010100010\cdots \\
\xi_4 &= 1000101000100010100010001000101000\cdots
\end{aligned}$$

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Due to periodicity, we have $\xi_k \in G_0 \cup G_\infty$. By Proposition 34 we admit a rational representation

$$-\pi^1(\xi_k) = -\sum_{j=1}^{\infty} \frac{2^{a_j(\xi_k)}}{3^k} = -\left\{\frac{2^{a_0}}{3} + \frac{2^{a_1}}{3^2} + \ldots + \frac{2^{a_k}}{3^k}\right\}\left(\frac{1}{1 - \frac{2^{a_{k+1}}}{3^k}}\right) = \text{fix}(\xi)(k) \in \mathbb{Q}$$

By the continuity of the function $-\pi^1 : \Sigma_2^* \to \mathbb{Z}_2$ we have that $-\pi^1(\xi_k) \to -\pi^1(\xi)$ as $k \to \infty$ on $\mathbb{Z}_2$. On the other hand. By Proposition 36 we have

$$|\pi^1(\xi_k)| = |\,\text{fix}(\xi_k)| > \gamma_k$$

Where γ_k is a monotonic and divergent sequence real. This means that the sequence fix(ξ_k) is unbounded in $\mathbb{R}$. Due to the uniqueness of the limit we have to $\frac{p}{q} = \infty$, this is a contradiction to assumption that ξ is stable, we have that G_1 is unstable. □

10. The Coding of π^1

Now we are going to prove that there is a complete metric on G_∞. We will use this result to prove that if $\pi^1\left(\{\xi_j\}_{j=1}^{\infty}\right) \in \mathbb{Q}_{odd}$ then $Cod\left(\pi^1\left(\{\xi_j\}_{j=1}^{\infty}\right)\right) = \{\xi_j\}_{j=1}^{\infty}$ and in the case that $\pi^1\left(\{\xi_j\}_{j=1}^{\infty}\right) \in \mathbb{R} \setminus \mathbb{Q}$, then there is no rational r such that $Cod(r) = \{\xi_j\}_{j=1}^{\infty}$. We also show that the parity of the Collatz function on $\pi^1(G_\infty)$ depends solely on the first term. Building upon this insight, we extend the Collatz function to $\pi^1(G_\infty)$ and conclude the section by showing that the Collatz function is topologically conjugate to the Shift function in Σ_2^*. We will use this result to establish that the set of periodic orbits is dense.

10.1. Summary of Propositions in the Section

1. **Lemma 12:** Established that when the function $d(\pi^1(\xi), \pi^1(\eta)) = \sum_{j\in\mathbb{N}} \frac{1}{3^j}|2^{a_j(\xi)} - 2^{a_j(\eta)}| \leq \frac{1}{3^r}$ then ξ and η share at least the first $k-1$ terms.
2. **Proposition 37:** Establishes that $(\pi^1(G_\infty), d)$ is a complete metric space.
3. **Corollary 8:** Established that the $k-$coding set is an open set.
4. **Theorem 7:** Established that the full coding set is a singleton set or an empty set depending on whether $\pi^1(\xi)$ is rational or not.
5. **Proposition 38:** $-\pi^1 : G_\infty \to -\pi^1(G_\infty)$ is continuous.
6. **Corollaty 9:** The $\pi^1 : G_\infty \to \pi^1(G_\infty) \subset \mathbb{R}$ is a continuous function with the usual metric of $\mathbb{R}$.
7. **Proposition 39:** It establishes that the parity of $\pi^1(\xi)$ depends only on the first term of the series.
8. **Definition 22:** Defines an extension of the Collatz function on all $\pi^1(G_\infty)$.
9. **Proposition 40:** The Collatz functions are continuous.
10. **Proposition 41:** The Collatz function on $\pi^1(G_\infty)$ is topological conjugacy to Shift map on Σ_2^*
11. **Corollary 10:** It is stable that the periodic points of the Collatz function in $(\pi^1(G_\infty), d)$ are dense.

10.2. $\pi^1(G_\infty)$ as Complete Metric Space

To ensure the coherent definition of a metric in $\pi^1(G_\infty)$, we need to "complete" the missing terms of the series to enable the calculation of the difference $|2^{a_k(\xi)} - 2^{a_k(\eta)}|$ for all $k \in \mathbb{N}$, irrespective of whether ξ or η has a null tail. To accomplish this, we define that when the sequence of 1s in ξ ends, the function a_k will take on the value $-\infty$. Hence, we have $|2^{a_k(\xi)} - 2^{a_k(\eta)}| = |2^{-\infty} - 2^{a_k(\eta)}| = 2^{a_k(\eta)}$ from the index of ξ. Let $\xi \in \Sigma_2^*$ with null tail with index J. We will write a short description.

$$\pi^1\left(\{\xi\}_{j=1}^{\infty}\right) := \sum_{j=1}^{J-1} \frac{2^{a_k}}{3^j} + \sum_{j=J+1}^{\infty} \frac{2^{-\infty}}{3^j}$$

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In the following lemma, we are going to introduce a new function, which, as we will see later, corresponds to a metric in the space $\pi^1(G_\infty)$. Additionally, we will present another result that we will examine more closely in this section and essentially indicates to us that, since the parity of $\pi^1(\xi)$ depends only on the first term, we can interpret this in the following way: If two sequences are arbitrarily close, then they share the first terms of their encoding. This is of great importance for understanding the behavior of the orbits of the Collatz function, since, if we consider the Euclidean metric in $\mathbb{Q}$ or that of the absolute value, we observe the phenomenon that even though two numbers are arbitrarily close, their dynamics are completely different. One may converge to a cycle in a few iterations, while the other may take a very long time.

Lemma 12 (Convergence and Coincidence Lemma)**.** *Let $\xi, \eta \in G_\infty$, then*

1. $\displaystyle\sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_j(\xi)} - 2^{a_j(\eta)}\right|$ *is well defined.*
2. $\displaystyle\sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_j(\xi)} - 2^{a_j(\eta)}\right| \leq \frac{1}{3^r} \Rightarrow a_j(\xi) = a_j(\eta)$ *for all $j < r$. In addition, we have to $\xi_{j-1} = \eta_j$ for all $j < r$.*

Proof. Let $\xi, \eta \in G_\infty$, then we have

Claim Let, $a, b \in \mathbb{N}$ then $|2^a - 2^b| < 2^{\max\{a,b\}}$. If $a = b$, then $0 < 2^a$. Suppose that $a > b$ then

$$|2^a - 2^b| = 2^a\left|1 - \frac{1}{2^{a-b}}\right| < 2^a.$$

□ **of the Claim.**

Now we prove that it is well-defined, by *Claim* we have:

$$d\left(\sum_{j=1}^{\infty} \frac{2^{a_j(\xi)}}{3^j}, \sum_{j=1}^{\infty} \frac{2^{a_j(\eta)}}{3^j}\right) = \sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_j(\xi)} - 2^{a_j(\eta)}\right| < \sum_{j=1}^{\infty} \frac{2^{\max\{a_j(\xi), a_j(\eta)\}}}{3^j}$$

as $\dfrac{2^{a_j(\xi)}}{3^j}$ and $\dfrac{2^{a_j(\eta)}}{3^j}$ converge to 0, then for $n > \mathbb{N}$ exist $J \in \mathbb{N}$ such that if $j > J$ we have

$$\left|\frac{2^{a_j(\xi)}}{3^j}\right|, \left|\frac{2^{a_j(\eta)}}{3^j}\right| < \frac{1}{n}$$

Then for $j > J$ we also have

$$\frac{2^{\max\{a_j(\xi), a_j(\eta)\}}}{3^j} < \frac{1}{n}$$

Then we have that $\dfrac{2^{\max\{a_j(\xi), a_j(\eta)\}}}{3^j}$ also converges to 0. Then by Proposition 35 we have $\displaystyle\sum_{j=1}^{\infty} \frac{2^{\max\{a_j(\xi), a_j(\eta)\}}}{3^j} < \infty$. To prove the statement, we will consider whether the sequences ξ and η in Σ_2^* have a null tail or not.

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Let us first assume that the sequence does not have a null tail, then if we have

$$\sum_{j=1}^{\infty} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\eta)} \right| \leq \frac{1}{3^r}$$

All terms less than r must be null. Suppose there exists some non-zero term between 1 and $r-1$, then we have that

$$\frac{1}{3^{r-1}} \leq \sum_{j=1}^{r-1} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\eta)} \right| \leq \sum_{j=1}^{\infty} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\eta)} \right| \leq \frac{1}{3^r} \text{ Then } r-1 \geq r$$

which is absurd. Then we have that $\sum_{j=1}^{r-1} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\eta)} \right| = 0$. Which implies that

$$a_j(\xi) = a_j(\eta) \text{ for all } j < r.$$

Now we will prove that the sequences coincide up to r.

$$\xi = 0^{\theta_1^1} \prod_{i=2}^{\infty} 10^{\theta_i^1} \text{ and } \eta = 0^{\theta_1^2} \prod_{i=2}^{\infty} 10^{\theta_i^2}$$

writing this way, we have to

$$a_j(\xi) = \sum_{i=1}^{j} \theta_i^1 \text{ and } a_j(\eta) = \sum_{i=1}^{j} \theta_i^2$$

then

$$\theta_j^1 = a_j(\xi) - a_{j-1}(\xi) = a_j(\eta) - a_{j-1}(\eta) = \theta_j^2 \text{ for all } 1 \leq j < r$$

which means that ξ and η share the first $(r-1) - 10^{\theta_j}$ blocks Now suppose that ξ has a null tail of index $I+1$ and η has no tail null. Then we have

$$\sum_{j=1}^{I} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\eta)} \right| + \sum_{j=I+1}^{\infty} \frac{1}{3^j} \left| 2^{a_j(\eta)} \right| \leq \frac{1}{3^r}$$

if $r \leq I$ then we have the previous case, then $a_j(\xi) = a_j(\eta)$ for all $j < r$. Now if $r > I$ we have

$$\sum_{j=1}^{I} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\eta)} \right| + \sum_{j=I+1}^{\infty} \frac{1}{3^j} \left| 2^{a_j(\eta)} \right| \leq \frac{1}{3^r}$$

$$\Rightarrow \sum_{j=1}^{I} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\eta)} \right| + \sum_{j=I+1}^{r-1} \frac{1}{3^j} \left| 2^{a_j(\eta)} \right| = 0$$

The latter makes sense if η also has a null tail of index $I+1$, then

$$\xi_j = \eta_j \text{ for all } j \in \mathbb{N}$$

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In particular $\xi_j = \eta_j$ for all $j < r$. Finally, suppose that ξ and η have a null tail of index $I+1$ and $L+1$ respectively, without loss of generality we can assume that $I \leq L$. Then

$$\sum_{j=1}^{I} \frac{1}{3^j}\left|2^{a_j(\xi)} - 2^{a_j(\eta)}\right| + \sum_{j=I+1}^{L} \frac{1}{3^j}\left|2^{a_j(\eta)}\right| + \sum_{j=L+1}^{\infty} \frac{1}{3^j}\left|2^{-\infty}\right| \leq \frac{1}{3^r}$$

1. If $1 \leq r \leq I$ then all terms with an index less than r are null and in particular we have $a_j(\xi) = a_j(\eta)$ for $j < r$. and as we already saw in the proofs above, this implies that $\xi_j = \eta_j$ for all $j < r$.
2. If $r \geq I$. Then $\sum_{j=I+1}^{L} \frac{1}{3^j}\left|2^{a_j(\eta)}\right| = 0$ we have that $I = J$ therefore $\xi = \eta$. particular we have $a_j(\xi) = a_j(\eta)$ for $j < r$.

□

Now we are going to show that the function we defined above is a complete metric on $\pi^1(G_\infty)$.

Proposition 37 (Metric space complete)**.** *Let $d : \pi^1(G_\infty) \times \pi^1(G_\infty) \to [0, \infty)$ given by*

$$d\left(\sum_{j=1}^{\infty} \frac{2^{a_j(\xi)}}{3^j}, \sum_{j=1}^{\infty} \frac{2^{a_j(\eta)}}{3^j}\right) = \sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_j(\xi)} - 2^{a_j(\eta)}\right|$$

Then $(\pi^1(G_\infty), d)$ is a metric space complete.

Proof. Let's prove that d is a metric through the axioms of metric:

1. $d(\pi^1(\xi), \pi^1(\eta)) = 0$ if and only if $\pi^1(\xi) = \pi^1(\eta)$ for all $\pi^1(\xi), \pi^1(\eta) \in \pi^1(G_\infty)$: Trivially we have that if $\pi^1(\xi) = \pi^1(\eta)$, then

$$d\left(\sum_{j=1}^{\infty} \frac{2^{a_j(\xi)}}{3^j}, \sum_{j=1}^{\infty} \frac{2^{a_j(\xi)}}{3^j}\right) = \sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_j(\xi)} - 2^{a_j(\xi)}\right| = 0.$$

   Let $\pi^1(\xi), \pi^1(\eta) \in \pi^1(G_\infty)$ such that

$$d\left(\sum_{j=1}^{\infty} \frac{2^{a_j(\xi)}}{3^j}, \sum_{j=1}^{\infty} \frac{2^{a_j(\eta)}}{3^j}\right) = \sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_j(\xi)} - 2^{a_j(\eta)}\right| = 0$$

   by lemma 12 we have

$$\sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_j(\xi)} - 2^{a_j(\eta)}\right| \leq \frac{1}{3^r} \Rightarrow a_j(\xi) = a_j(\eta) \text{ for } j < r$$

   In particular, for $r \to \infty$ we have $\xi = \eta$.
2. $d(\pi^1(\xi), \pi^1(\eta)) = d(\pi^1(\eta), \pi^1(\xi))$ for all $\pi^1(\xi), \pi^1(\eta) \in \pi^1(G_\infty)$:

$$\sum_{j=1}^{k} \frac{1}{3^j}\left|2^{a_j(\xi)} - 2^{a_j(\eta)}\right| = \sum_{j=1}^{k} \frac{1}{3^j}\left|2^{a_j(\eta)} - 2^{a_j(\xi)}\right|$$

   then

$$\sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_j(\xi)} - 2^{a_j(\eta)}\right| = \sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_j(\eta)} - 2^{a_j(\xi)}\right|$$

3. $d(\pi^1(\xi),\pi^1(\eta)) \leq d(\pi^1(\xi),\pi^1(\kappa)) + d(\pi^1(\kappa),\pi^1(\eta))$ for all, $\pi^1(\xi),\pi^1(\eta),\pi^1(\kappa) \in \pi^1(G_\infty)$

$$\sum_{j=1}^{k}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\eta)}\right| \leq \sum_{j=1}^{k}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\kappa)}\right| + \sum_{j=1}^{k}\frac{1}{3^j}\left|2^{a_j(\kappa)}-2^{a_j(\eta)}\right|$$

then

$$\sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\eta)}\right| \leq \sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\kappa)}\right| + \sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_j(\kappa)}-2^{a_j(\eta)}\right|$$

then $(\pi^1(G_\infty), d)$ is a metric space. Now we are going to prove that it is a complete metric space. Let $\{\{\pi^1(\xi_j^k)\}_{j=1}^{\infty}\}_{k=1}^{\infty}$ be a Cauchy sequence on $\pi^1(G_\infty)$ then for any $\varepsilon > 0$ exist $K > 0$ such that

$$d(\{\pi^1(\xi_j^n)\}_{j=1}^{\infty},\{\pi^1(\xi_j^m)\}_{j=1}^{\infty}) = \sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_j(\xi^n)}-2^{a_j(\xi^m)}\right| < \varepsilon \text{ for all } n,m > K$$

Let $r > 0$ such that $\sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_j(\xi^n)}-2^{a_j(\xi^m)}\right| < \frac{1}{3^r} < \varepsilon$ by lemma 12 we have

$$\xi_j^n = \xi_j^m \text{ for all } j < r$$

On the other hand let $D : \Sigma_2^* \times \Sigma_2^* \to [0,\infty)$ the symbolic metric of two symbols given by

$$D(\xi,\eta) = \sum_{j=1}^{\infty}\frac{1}{2^j}\Delta(\xi_j,\eta_j)$$

with

$$\Delta(\xi_j,\eta_j) = \begin{cases} 1 & \text{if} \quad \xi_j \neq \eta_j \\ 0 & \text{if} \quad \xi_j = \eta_j \end{cases}$$

The space (Σ_2^*, D) is a complete metric space with the property that if two sequences are arbitrarily close if and only if their first terms are equal.

$$D(\xi,\eta) < \frac{1}{2^r} \text{ if and only if } \xi_j = \eta_j \text{ for all } j < r.$$

then given a Cauchy sequence in $\pi_1(G_\infty)$ by the observation above we obtain a Cauchy sequence in Σ_2^* and the latter being complete there is a $\xi \in \Sigma_2^*$ such that

$$\xi^n \to \xi \text{ as } n \to \infty$$

We will now prove that ξ is in G_∞ and that the sequence $\pi_1(\xi^n)$ converges to $\pi_1(\xi)$.

1. Let's prove that G_∞ is complete. Let $\{\xi_j\}_{j\in\mathbb{N}}$ a Cauchy sequence on G_∞ and let $\lim_{j\to\infty}\xi_j = \xi \in \Sigma_2^*$. Let's show that $\xi \in G_\infty$. Let $R = a_j + b_j$, then by definition we have exist $N > 0$ such that $D(\xi_n,\xi_m) < \frac{1}{R}$ for all $m,n > N$, so we have ξ_m and ξ_n have the first R terms equal, Suppose ξ has no null tail, then $R = a_j + j$ then

$$\lim_{j\to\infty}\frac{3^j}{2^{a_{j+1}}} = \infty \text{ so } \xi \in G_\infty$$

If ξ has a null tail, then by definition it is in G_∞.

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2. Let's prove that $\pi^1(G_\infty)$ is complete. Let $\pi^1(\xi_k)$ a Cauchy sequence in $\pi^1(G_\infty)$. Let $r \in \mathbb{N}$, then exist N such that
$$d(\pi^1(\xi_n), \pi^1(\xi_m)) < \frac{1}{3^r} \text{ for all } m, n > N$$
by Lemma 12 we have $D(\xi_n, \xi_m) < \dfrac{1}{2^r}$ for all $m, n > N$. In other words we have that $\{\xi_j\}$ is a Cauchy sequence in G_∞. As we proved in point 1, we have that G_∞ is a complete metric subspace of Σ_2^*, so exist $\xi \in G_\infty$ such that $\lim\limits_{j\to\infty} \xi_j = \xi$. Let us now demonstrate that $\lim\limits_{j\to\infty} \pi^1(\xi_j) = \pi^1(\xi)$
$$\begin{aligned} d(\pi^1(\xi_j), \pi^1(\xi)) &= \sum_{k\in\mathbb{N}} \frac{1}{3^k}\left|2^{a_k(\xi_j)} - 2^{a_k(\xi)}\right| \\ &= \underbrace{\sum_{k=1}^{r} \frac{1}{3^k}\left|2^{a_k(\xi_j)} - 2^{a_k(\xi)}\right|}_{=0} + \sum_{k=j+1}^{\infty} \frac{1}{3^k}\left|2^{a_k(\xi_j)} - 2^{a_k(\xi)}\right| \\ &\leq \sum_{k=r+1}^{\infty} \frac{1}{3^k} 2^{\max\{a_k(\xi_j), a_k(\xi)\}} \to 0 \text{ as } r \to \infty \end{aligned}$$
Therefore $\pi^1(G_\infty)$ it is a complete space.

then we can conclude that the metric space $(\pi^1(G_\infty), d)$ is complete. □

With this metric, we have that the coding sets are open sets.

Corollary 8 (Cod^k is an open set)**.** *Let $\xi \in G_\infty$ and $k \in \mathbb{N}$. Then $Cod^k(\xi)$ is an open set on $(\pi^1(G_\infty), d)$*

Proof. Let $\xi \in G_\infty$ and $u \in \pi^1(G_\infty)$ such that $u \in Cod^k(\xi)$. Let us consider v in $\pi^1(G_\infty)$ such that $d(u, v) < \dfrac{1}{3^{k+1}}$, by definition, exist $\mu, \tau \in G_\infty$ such that $\pi^1(\mu) = u$ and $\pi^1(\tau) = v$. By Lemma 12 have $\mu_j = \tau_j$ for $j \leq k$, then we have that $v \in Cod^k(\xi)$. Therefore, then $B\left(u, \dfrac{1}{3^{k+1}}\right)$ i.e. the ball of radius $\dfrac{1}{3^{k+1}}$ and center u is a subset of $Cod^k(\xi)$, therefore $Cod^k(\xi)$ is an open set. □

Proposition 38. $-\pi^1 : G_\infty \to -\pi^1(G_\infty)$ *is continuous.*

Proof. Let $\xi \in G_\infty$ and $\{\eta_j\}_{l\in\mathbb{N}}$ a sequence on G_∞ such that $D(\eta_j, \xi) < \dfrac{1}{2^r}$, so $a_k(\eta_j) = a_k(\xi)$ for all $j < r$, then we have
$$d(\pi^1(\eta_j), \pi^1(\xi)) = \sum_{k=r}^{\infty} \frac{|2^{a_k(\eta_j))} - 2^{a_k(\eta))}|}{3^k} \to 0 \text{ as } r \to \infty$$
Therefore $d(\pi^1(\eta_j), \pi^1(\xi)) \to 0$ as $D(\eta_j, \xi) \to 0$, that is to say that $-\pi^1$ es □

Corollary 9. *The $\pi^1 : G_\infty \to \pi^1(G_\infty) \subset \mathbb{R}$ is a continuous function with the usual metric of $\mathbb{R}$.*

Proof. Let $\xi \in G_\infty$ a sequence $\{\xi_j\}$ on G_∞ such that $\xi_j \to \xi$. By continuity of $\pi^1 : G_\infty \to \pi^1(G_\infty)$ we have
$$\begin{aligned} \left|\pi^1(\xi_j) - \pi^1(\xi)\right| &= \left|\sum_{j\in\mathbb{N}} \frac{2^{a_j(\xi_j)}}{3^j} - \sum_{j\in\mathbb{N}} \frac{2^{a_j(\xi)}}{3^j}\right| \\ &\leq \sum_{j\in\mathbb{N}} \frac{1}{3^j}\left|2^{a_j(\xi_j)} - 2^{a_j(\xi)}\right| \\ &= d(\pi^1(\xi_j), \pi^1(\xi)) \to 0 \text{ as } D(\xi_j, \xi) \to 0 \end{aligned}$$

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□

10.3. Characterization of the Full Coding Sets Through the Function π^1

Theorem 7 (Asymptotic Solutions Theorem). *Let $\{\xi_j\}_{j=1}^{\infty} \in G_\infty$, then if $\pi^1(\{\xi_j\}_{j=1}^{\infty})$ it is rational, then the $-\pi^1(\{\xi_j\}_{j=1}^{\infty}) = -\dfrac{p}{q}$ only rational that satisfies $Cod\left(-\dfrac{p}{q}\right) = \{\xi_j\}_{j=1}^{\infty}$, in particular $\pi^1(\{\xi_j\}_{j=1}^{\infty}) \in \mathbb{Q}_{odd}$. If $\pi^1(\{\xi_j\}_{j=1}^{\infty})$ it is irrational, then there is no rational $\dfrac{p}{q}$ such that $Cod\left(\dfrac{p}{q}\right) = \{\xi_j\}_{j=1}^{\infty}$.*

Proof. Let $\xi \in G_\infty$, We have by definition that $\pi^1(\xi) = \lim_{k\to\infty} \pi^1_k(\xi) \in \mathbb{Q}$. First, we will prove that this limit also makes sense in $(\pi^1(G_\infty), d)$. We have:
Claim 1: $d(\pi^1_k(\xi), \pi^1(\xi)) \to 0$ as $k \to \infty$.

Indeed, we have

$$d(\pi^1_k(\xi), \pi^1(\xi)) = \underbrace{\sum_{j=1}^{n} \frac{1}{3^j}|2^{a_j(\xi)} - 2^{a_j(\xi)}|}_{0} + \sum_{j=k+1}^{\infty} \frac{2^{a_j(\xi)}}{3^j} = \sum_{j=k+1}^{\infty} \frac{2^{a_j(\xi)}}{3^j}$$

By Proposition 35, we have $\displaystyle\sum_{j=k+1}^{\infty} \frac{2^{a_j(\xi)}}{3^j} \to 0$ as $k \to \infty$, then $d(\pi^1_k(\xi), \pi^1(\xi)) \to 0$ as $k \to \infty$ □

We will now prove, using the completeness of $(\pi^1(G_\infty), d)$, that $\pi^1(\xi) \in \pi^1(G_\infty)$:
Claim 2: $\pi^1(\xi) \in \pi^1(G_\infty)$.
We can be rewritten,

$$\pi^1_k(\xi) = \pi^1\left(\{\xi_j\}_{j=1}^{k} 000\ldots\right)$$

let's prove that $\pi^1_k(\xi)$ is a Cauchy sequence in $\pi^1(G_\infty)$. Let $\varepsilon > 0$ and $n, m \in \mathbb{N}$ with $n < m$, then

$$d(\pi^1_n(\xi), \pi^1_m(\xi)) = \underbrace{\sum_{j=1}^{n} \frac{1}{3^j}|2^{a_j(\xi)} - 2^{a_j(\xi)}|}_{0} + \sum_{j=n+1}^{m} \frac{2^{a_j(\xi)}}{3^j} = \sum_{j=n+1}^{m} \frac{2^{a_j(\xi)}}{3^j}$$

Since $\xi \in G_\infty$ we have by Proposition 35 $\pi^1(\xi)$ is convergent, then $\displaystyle\sum_{j=n+1}^{m} \frac{2^{a_j(\xi)}}{3^j} \to 0$ as $n, m \to \infty$. Therefore, exists $N \in \mathbb{N}$ such that $n, m > N$ we have $\displaystyle\sum_{j=n+1}^{m} \frac{2^{a_j(\xi)}}{3^j} < \varepsilon$. Then the sequence is Cauchy and since $\pi^1(G_\infty)$ is a complete metric space, we have that $\pi^1(\xi) = \lim_{k\to\infty} \pi^1_k(\xi) \in \pi^1(G_\infty)$.

□ **of the Claim.**

Claim 3: Let $\xi = 0^{\theta_1}10^{\theta_2}\ldots10^{\theta_{k+1}}\ldots \in Sigma_2^*$, then $-\pi^1_k(\xi) \in Cod^{a_{k+1}}(\xi)$.

Indeed, let $S_k \in \langle\theta, \psi\rangle$ such that $Cod(S_k) = 0^{\theta_1}10^{\theta_2}\ldots10^{\theta_{k+1}}$ by Proposition 12 we have

$$S_k(-\pi^1_k\left(\{\xi_j\}_{j=1}^{\infty}\right) = \frac{3^k(-\pi^1_k\left(\{\xi_j\}_{j=1}^{\infty}\right) + N_k}{2^{a_{k+1}}} = \frac{3^k\left(-\dfrac{N_k}{3^k}\right) + N_k}{2^{a_{k+1}}} = 0.$$

In other hand, we have

$$\frac{3^k\left(-\dfrac{N_k}{3^k}\right)+N_k}{2^{a_{k+1}}}=\frac{3^k(-N_k)+3^k(N_k)}{3^k2^{a_{k+1}}}=\frac{1}{3^k}\left(\frac{3^k(-N_k)+3^k(N_k)}{2^{a_{k+1}}}\right)=\frac{1}{3^k}S^{3^k}$$

then $-N_k\in\mathbb{E}(S^{3^k})$, by Proposition 6 we have $Cod_{3^k}(S_k^{3^k})=Cod(S_k)$. Therefore $Cod^{a_{k+1}}\left(\dfrac{-N_k}{3^k}\right)=Cod^{a_{k+1}}(-\pi_k^1(\xi))=Cod_{3^k}(S^{3^k})=Cod(S_k)$.

$\square$ **of the Claim.**

Claim 4: $-\pi^1(\xi)\in Cod^k(\xi)$ for all $k\in\mathbb{N}$.

Let $L+1\in\mathbb{N}$, we have by *Claim 1*, exist $K\in\mathbb{N}$ such that if $k>K>L+1$ so $d(\pi_k^1(\xi),\pi^1(\xi))<\dfrac{1}{3^{L+1}}$, By Lemma 12 we have to share the first L terms of the coding. On the other hand, by *Claim 3*, we have that $-\pi_k^1(\xi)\in Cod^{a_{k+1}}(\xi)$ and since we have $a_{k+1}>K>L+1$, due to the monotony of Cod, we have $-\pi_k^1(\xi)\in Cod^L(\xi)$. Therefore $-\pi^1(\xi)\in Cod^L(\xi)$.

$\square$ **of the Claim.**

Claim 5: $\pi^1(\xi)\in Cod(\xi)$.

Since $-\pi^1(\xi)\in Cod^k(\xi)$ for all $k\in\mathbb{N}$, we have to

$$-\pi^1(\xi)\in\bigcap_{k\in\mathbb{N}}Cod^k(\xi)=Cod(\xi).$$

$\square$ **of the Claim.**

Claim 6: $Cod\left(-\pi^1\left(\{\xi_j\}_{j=1}^\infty\right)\right)=\{\xi_j\}_{j=1}^\infty$.

We first show that $\pi^1(\xi)\in\mathbb{Q}_{odd}$. Let's assume that $\pi^1\left(\{\xi_j\}_{j=1}^\infty\right)$ converges to a fraction with an even denominator; then its coding is $1111\ldots$. However, $1111\ldots$ is not an element of Σ_2^*, which leads to a contradiction with *claim 4*. Then we have $\pi^1(\xi)\in\mathbb{Q}_{odd}$ and, by Theorem 3, we have $Cod\left(-\pi^1\left(\{\xi_j\}_{j=1}^\infty\right)\right)=\{\xi_j\}_{j=1}^\infty$.

$\square$ **of the Claim.**

For the next part of the proposition, we will leverage the results presented in Section 6. In this section, we introduce the Sigma function along with its main properties and applications in solving linear Diophantine equations. It serves as an alternative to classical methods for solving this type of equation.

Claim 7: If $\pi^1(\{\xi_j\}_{j=1}^\infty)$ it is irrational, then there is no rational then there is no rational $\dfrac{p}{q}$ such that $cod\left(\dfrac{p}{q}\right)=\{\xi_j\}_{j=1}^\infty$.

We will prove that there is no rational solution, By the theorem 2 we have that if there is another rational solution it must be a minimum positive integer value or a maximum negative integer value for $S_k(x)\in\langle\theta,\psi^q\rangle$ such that $Cod(S_k)=\{\xi_j\}_{j=1}^{k+1}=0^{\theta_1}10^{\theta_2}\ldots0^{\theta_k}10^{\theta_{k+1}}$ for unique q not null, by proposition 5 we have

$$S_k(x)=\frac{3^kx+qN_k(\{\xi_j\}_{j=1}^\infty)}{2^{a_{k+1}}}$$

by Propositions 4 and 21 the minimum positive integer value is

$$\rho_0(S_k) = \frac{1}{3^k}\left(2^{a_{k+1}}\sigma_{3^k}^{a_{k+1}}(qN_k) - qN_k\right) = 2^{a_{k+1}}\sigma^{a_{k+1}}(q\pi_k^1(\{\xi_j\}_{j=1}^{\infty}) - q\pi_k^1(\{\xi_j\}_{j=1}^{\infty})$$
$$= Cod\sigma^{a_{k+1}}(q\pi_k^1(\{\xi_j\}_{j=1}^{\infty})$$

and the maximum negative integer value is

$$\rho_1(S_k) = \frac{1}{3^k}\left(2^{a_{k+1}}\sigma_{-3^k}^{a_{k+1}}(qN_k) - qN_k\right) = \frac{1}{3^k}\left(2^{a_{k+1}}\sigma_{3^k}^{a_{k+1}}(qN_k) - qN_k\right) - 2^{a_{k+1}}$$
$$= Cod\sigma^{a_{k+1}}(q\pi_k^1(\{\xi_j\}_{j=1}^{\infty}) - 2^{a_{k+1}}$$

when

$$Cod\sigma^{a_{k+1}}(q\pi_k^1(\{\xi_j\}_{j=1}^{\infty}) = \sum_{j=1}^{a_{k+1}-1} \delta_j 2^j \text{ with } \delta_j \in \{0,1\}$$

On the other hand, by Proposition 28 we have that the coding of the $\sigma^{a_{k+1}}(q\pi_k^1(\{\xi_j\}_{j=1}^{\infty})$ corresponds to the dyadic expansion of $-q\pi^1(\{\xi_j\}_{j=1}^{\infty})$. Since $\pi^1(\{\xi_j\}_{j=1}^{\infty})$ is irrational, then the dyadic expansion of $q\pi^1(\{\xi_j\}_{j=1}^{\infty})$ will never have a tail of 0 or 1 for all $q \in \mathbb{Z}$, so $Cod\sigma^{a_k}(q\pi_k^1(\{\xi_j\}_{j=1}^{\infty})$ has an infinite number of non-zero digits for all $q \in \mathbb{Z}$. In particular we have to $\rho_0(S_k) \to \infty$. Now for $\rho_1(S_k)$, we have

$$\rho_1(S_k) = \sum_{j=1}^{a_{k+1}-1} \delta_j 2^j - 2^{a_{k+1}} = \sum_{j=1}^{a_{k+1}-1} \delta_j 2^j - \sum_{j=1}^{a_{k+1}-1} 2^j = \sum_{j=1}^{a_{k+1}-1} (\delta_j - 1)2^j$$

for this sum to be finite it is necessary exist $J > 0$ such that δ_j are all 1 for $j > J$, however as $q\pi_k^1(\{\xi_j\}_{j=1}^{\infty})$ is irrational for all $q \in \mathbb{Z}$, then there are infinitely many terms of δ_j that are null, then this sum is divergent.

Then if $\pi_k^1(\{\xi_j\}_{j=1}^{\infty})$ is irrational then there is no rational $\frac{p}{q}$ such that $cod\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{\infty}$.

□ **of the Claim.**

Example 14. 1. *Let* $\xi_1 = 101010\ldots \in \Sigma_2^*$, *then* $\pi^1(\xi) = 1$ *(see Example 13). Therefore* $Cod(-1) = 101010\ldots$.

2. *Let* $\xi_2 = 1010010100\ldots \in \Sigma_2^*$, *then*

$$\pi^1(\xi_2) = \frac{1}{3} + \frac{2}{3^2} + \frac{2^3}{3^3} + \frac{2^4}{3^4} + \frac{2^6}{3^5} + \frac{2^7}{3^6} + \frac{2^9}{3^7} + \ldots$$
$$= \frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\} + \frac{2^3}{3^2}\left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\} + \left(\frac{2^3}{3^2}\right)^2\left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\} + \ldots$$
$$= \frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\}\sum_{j=1}^{\infty}\left(\frac{2^3}{3^2}\right)^j$$
$$= \frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\}\frac{1}{1 - \frac{2^3}{3^2}} = 5$$

Therefore $Cod(-5) = 1010010100\ldots$

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10.4. Extension of the Collatz Function on $\pi^1(G_\infty)$

Proposition 39 (Parity Preservation Proposition). *Let $\{\xi_j\}_{j=1}^\infty \in G_\infty$ such that $\pi^1(\{\xi_j\}_{j=1}^\infty) \in \mathbb{Q}_{odd}$, then*

$$Col\left(-\pi^1(\{\xi_j\}_{j=1}^\infty)\right) = \begin{cases} \dfrac{-3\pi^1(\{\xi_j\}_{j=1}^\infty)+1}{2} & \text{if } a_1 = 0 \\ -\dfrac{1}{2}\pi^1(\{\xi_j\}_{j=1}^\infty) & \text{if } a_1 > 0 \end{cases}$$

or equivalent if $\pi^1(\{\xi_j\}_{j=1}^\infty) = \sum_{j=1}^\infty \frac{2^{a_j}}{3^j}$ then

$$Col\left(-\sum_{j=1}^\infty \frac{2^{a_j}}{3^j}\right) = \begin{cases} -\sum_{j=1}^\infty \dfrac{2^{a_j-1}}{3^j} & \text{if } a_1 = 0 \\ -\sum_{j=1}^\infty \dfrac{2^{a_j-1}}{3^j} & \text{if } a_1 > 0 \text{ or } -\infty \end{cases}$$

Proof. Let's prove that the parity of $\pi^1(\{\xi_j\}_{j=1}^\infty)$ only depends on the first term

$$\pi^1(\{\xi_j\}_{j=1}^\infty) = \frac{2^{a_1}}{3} + \sum_{j=2}^\infty \frac{2^{a_j}}{3^j} = \frac{2^{a_1}}{3} + 2\left(\sum_{j=2}^\infty \frac{2^{a_j-1}}{3^j}\right)$$

Claim: The series $\sum_{j=2}^\infty \frac{2^{a_j-1}}{3^j}$ cannot converge to a fraction with an even denominator.

Let us assume by contradiction that have $\sum_{j=2}^\infty \frac{2^{a_j-1}}{3^j} = \frac{p}{2q}$ with $(q,p) = (p,2) = 1$. Let $\{\xi_j\}_{j=1}^l 0 \ldots \in G_\infty$ such that $\pi^1\{\xi_j\}_{j=1}^l 0 \ldots = \frac{2^{a_1}}{3}$. Let $\eta \in G_\infty$ given by $\eta = \{\xi_j\}_{j=l+1}^\infty$ so $\pi^1(\eta) = \pi^1\{\xi_j\}_{j=l+1}^\infty = \sum_{j=2}^\infty \frac{2^{a_j-1}}{3^j} = \frac{p}{2q}$ and since η are in G_∞, this generates a contradiction to the Theorem 7. □

Let $\sum_{j=2}^\infty \frac{2^{a_j-1}}{3^j} = \frac{p}{q}$ with $(p,q) = (q,2) = 1$. We have:

$$\pi^1(\{\xi_j\}_{j=1}^\infty) = \frac{2^{a_1}}{3} + \sum_{j=2}^\infty \frac{2^{a_j}}{3^j} = \frac{2^{a_1}}{3} + 2\left(\sum_{j=2}^\infty \frac{2^{a_j-1}}{3^j}\right) = \frac{2^{a_1}}{3} + \frac{2p}{q} = \frac{2^{a_1}q+6p}{3q}$$

if $a_1 = 0$ then $2^0 q + 6p = q + 6p$ is odd, since q is odd. Then $\pi^1(\{\xi_j\}_{j=1}^\infty)$ is odd, and if $a_1 > 0$ then $2^{a_1}q + 6p$ is even. Then $\pi^1(\{\xi_j\}_{j=1}^\infty)$ is even. In the case that $a_1 = -\infty$ we have that $\xi = 0$ or equivalent $\xi_j = 0$ for all $j \in \mathbb{N}_0$, so $\pi^1(0) = 0$, then $Col(0) = \frac{0}{2} = 0$.

From the result above, we can extend the Collatz function over the entire set G_∞, thus obtaining a way of defining Collatz for the cases where the function π^1 is irrational.

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Definition 22 (Extension Collatz functions on $-\pi^1(G_\infty)$)**.** *We defined $Col : -\pi^1(G_\infty) \to -\pi^1(G_\infty)$ by*

$$Col\left(-\pi^1(\{\xi_j\}_{j=1}^{\infty})\right) = \begin{cases} \dfrac{-3\pi^1(\{\xi_j\}_{j=1}^{\infty})+1}{2} & \text{if } a_1 = 0 \\ -\dfrac{1}{2}\pi^1(\{\xi_j\}_{j=1}^{\infty}) & \text{if } a_1 > 0 \end{cases}$$

or equivalent if $\pi^1(\{\xi_j\}_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$ then

$$Col\left(-\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}\right) = \begin{cases} -\displaystyle\sum_{j=1}^{\infty} \frac{2^{a_j-1}}{3^j} & \text{if } a_1 = 0 \\ -\displaystyle\sum_{j=1}^{\infty} \frac{2^{a_j-1}}{3^j} & \text{if } a_1 > 0 \text{ or } -\infty \end{cases}$$

The extension of the Collatz function on -$\pi^1(G_\infty)$ is continuous.

Proposition 40 (Collatz function is continuous)**.** *$Col : -\pi^1(G_\infty) \to -\pi^1(G_\infty)$ is continuous.*

Proof. Let us consider the metric induced in $-\pi^1(G_\infty)$ by $-id : \pi^1(G_\infty) \to -\pi^1(G_\infty)$, that is, $d_{-\pi^1(G_\infty)}(-u,-v) = d(u,v)$ on $\pi^1(G_\infty)$, we will use the same notation for both metrics.

Let $u \in -\pi^1(G_\infty)$ and $\{u_j\}_{j\in\mathbb{N}}$ sequence of $-\pi^1(G_\infty)$ such that $u_j \to u$. Let $\xi, \xi_j \in G_\infty$ such that $Cod(u) = \xi$ and $Cod(\xi_j) = u_j$.

$$d(u_j, u) = \sum_{k\in\mathbb{N}} \frac{|2^{a_k(\xi_j)} - 2^{a_k(\xi)}|}{3^k} \to 0 \text{ as } j \to \infty.$$

Since $r > 3$ then at least the first term must coincide.
Suppose that, $a_0 > 0$ then

$$d(Col(u_j), Col(u)) = \sum_{k\in\mathbb{N}} \frac{|2^{a_k(\xi_j)-1} - 2^{a_k(\xi)-1}|}{3^k} \to 0 \text{ as } j \to \infty$$

Suppose now that $a_0 = 0$

$$d(Col(u_j), Col(u)) = \sum_{k\in\mathbb{N}} \frac{|2^{a_k(\xi_j)-1} - 2^{a_k(\xi)-1}|}{3^k} \to 0 \text{ as } j \to \infty$$

Therefore Col is continuous. □

Now we will prove that the extension of the function Collatz on G_∞ is topologically conjugate to the shift function in Σ_2^*.

10.5. Topological Conjugation

In a dynamic system, there is a well-studied dynamics in the space of sequences of two symbols, known as the Shift map. This map acts on the sequences by eliminating the first term. It is known that with the metric D, this map is continuous, and its periodic orbits form a dense set. In the following proposition, we will show that the extension of the Collatz function on $-\pi^1(G_\infty)$ is, in fact, topologically conjugate to the dynamics of the Shift map.

Proposition 41. *Let us consider the following function $\omega : \Sigma_2^* \to \Sigma_2^*$ given by,*

$$\omega\left(\{\xi_j\}_{j=1}^{\infty}\right) = \begin{cases} \{\xi_{j+2}\}_{j=1}^{\infty} & \text{if } \xi_1 = 10 \\ \{\xi_{j+1}\}_{j=1}^{\infty} & \text{if } \xi_1 = 0 \end{cases}$$

Then Col and ω are Topologically Conjugacy.

Proof. We are going to prove that this diagram is commutative

$$\begin{tikzcd} G_\infty \arrow[r, "\omega"] \arrow[d, "-\pi^1"'] & G_\infty \arrow[d, "-\pi^1"] \\ -\pi^1(G_\infty) \arrow[r, "Col"'] & -\pi^1(G_\infty) \end{tikzcd}$$

and that $-\pi^1$ is a homeomorphism.

Claim 1: The diagram is commutative. Suppose that $\{\xi_j\}_{j=1}^{\infty} = 10^{\theta_1}10^{\theta_2}\ldots = \prod_{j=1}^{\infty} 10^{\theta_j} \in G_\infty$ with $\theta_j \in \mathbb{N}$. Writing this way, we get an explicit form for the function $a_k = \sum_{j=1}^{k} \theta_j$. If $a_0 = 0$ we have

$$\begin{aligned} Col \circ (-\pi^1)\left(\prod_{j=1}^{\infty} 10^{\theta_j}\right) &= Col\left(-\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=1}^{\infty})}}{3^k}\right) \\ &= \frac{1}{2}\left(-3\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=1}^{\infty})}}{3^k} + 1\right) \\ &= \frac{1}{2}\left(-\sum_{k=2}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=1}^{\infty})}}{3^{k-1}} + 1\right) \\ &= \frac{1}{2}\left(-1 - \sum_{k=2}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=1}^{\infty})}}{3^{k-1}} + 1\right) \\ &= -\sum_{k=2}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=1}^{\infty})} - 1}{3^{k-1}} \\ &= -\sum_{k=1}^{\infty} \frac{2^{a_{k+1}(\{\xi_j\}_{j=1}^{\infty})} - 1}{3^k} \end{aligned}$$

and

$$-\pi^1 \circ \omega\left(\prod_{j=1}^{\infty} 10^{\theta_j}\right) = -\pi^1\left(0^{\theta_1 - 1}\prod_{j=2}^{\infty} 10^{\theta_j}\right) = -\sum_{k=1}^{\infty} \frac{2^{a_{k+1}(\{\xi_i\}_{i=1}^{\infty})} - 1}{3^k}$$

where both parts are equal. On the other hand. suppose that $\{\xi_j\}_{j=1}^{\infty} = 0^{\theta_1}10^{\theta_2}1\ldots = \prod_{j=1}^{\infty} 0^{\theta_j}1 \in G_\infty$ with $\theta_j \in \mathbb{N}$ and $\theta_1 > 0$. We have

$$Col \circ (-\pi^1)\left(\prod_{j=1}^{\infty} 0^{\theta_j}1\right) = Col\left(-\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_i\}_{i=1}^{\infty})}}{3^k}\right) = -\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_i\}_{i=1}^{\infty})} - 1}{3^k}$$

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and

$$(-\pi^1) \circ \omega\left(\prod_{j=1}^{\infty} 0^{\theta_j} 1\right) = (-\pi^1) \circ \left(\theta^{\theta_0 - 1} \prod_{j=1}^{\infty} 0^{\theta_j} 1\right) = -\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=1}^{\infty})} - 1}{3^k}$$

where again both parts are equal. Then we conclude that the diagram is commutative.
Claim 2: $-\pi^1 : G_\infty \to -\pi^1(G_\infty)$ **with It is a bijective function**. Let $Cod : -\pi^1(G_\infty) \to G_\infty$ let's prove that $Cod \circ -\pi^1 = Id_{G_\infty}$ and $-\pi^1 \circ Cod = Id_{-\pi^1(G_\infty)}$.

1. $Cod \circ -\pi^1 = Id_{G_\infty}$: Let $\xi = \xi_1\xi_2\xi_3 \ldots \in G_\infty$ with $\xi_j \in \{0, 10\}$. Since the parity of $-\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$ depends only on the first term, if ξ starts with 0 then $a_1 > 0$, then $-\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$ is even then the first term of its coding is 0, and if ξ starts with 1 then $a_1 = 0$ then $-\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$ is odd then the first term of coding is 10. By applying the Collatz function we obtain the same result as applying a translation of the terms of ξ. Indeed
   (a) if $a_1 = 0$

$$Col\left(-\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}\right) = \sum_{j=1}^{\infty} \frac{2^{a_{j+1}-1}}{3^j} = -\pi^1(\xi_2\xi_3\ldots).$$

   .
   (b) if $a_1 > 0$

$$Col\left(-\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}\right) = \sum_{j=1}^{\infty} \frac{2^{a_j-1}}{3^j} = -\pi^1(\xi_2\xi_3\ldots).$$

   Then applying the function $-\pi^1$. Then we can repeat the same procedure and we recover ξ. Therefore

$$Cod(-\pi^1(\xi)) = \xi$$

2. $-\pi^1 \circ Cod = Id_{-\pi^1(G_\infty)}$: Let $u \in -\pi^1(G_\infty)$ and $\xi \in G_\infty$ such that $-\pi^1(\xi) = u$. On the other hand we have $Cod(-\pi^1(\xi)) = Cod(u)$ and $Cod(-\pi^1(\xi)) = \xi$ then $Cod(u) = \xi$, applying $-\pi^1$ on both sides we have $-\pi^1(Cod(u)) = -\pi^1(\xi) = u$.

□ **of the Claim.**

Claim 3: $-\pi^1 : G_\infty \to -\pi^1(G_\infty)$ **is continuous.** Consequence of the Proposition 38.

□ **of the Claim.**

Claim 4: $Cod : -\pi^1(G_\infty) \to G_\infty$ **is continuous.** Let $u \in -\pi^1(G_\infty)$ and let $\varepsilon > 0$. Take $r \in \mathbb{N}$ such that $\frac{1}{2^r} < \varepsilon$. then by Lemma 12 we have

$$d(u, v) < \frac{1}{3^r} \Rightarrow Cod(u)_j = Cod(v)_j \text{ for all } j < r \Rightarrow D(Cod(u), Cod(v)) < \frac{1}{2^r} < \varepsilon$$

□ **of the Claim.**

Therefore, $-\pi^1$ is a homeomorphism and therefore a topological conjugation. □

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10.6. Periodic Point

Corollary 10 (The Periodic orbit of Collatz function)**.** *The set of periodic orbits of Collatz function is dense in* $\pi^1(G_\infty)$

Proof. Direct consequence of the proposition 41 □

Corollary 11. *Let* $\xi \in G_\infty$*. Suppose that* $-\lim_{k\to\infty} \pi^1(\xi) \in \mathbb{Z}_2$ *admits real representation and let us denote the representation by* Γ*. Then* $\Gamma = -\lim_{k\to\infty} \pi^1(\xi) \in \mathbb{R}$*.*

Proof. Consequence of uniqueness, since $-\lim_{k\to\infty} \pi^1(\xi) \in \mathbb{R}$. is the only solution in $\mathbb{R}$ whose coding is ξ and we know that Γ is the only dyadic solution whose coding is ξ . □

11. The Problem of Divergence

In this section, we address the fundamental aspects of divergence of the Collatz function. The primary focus is on the behavior of sequences and orbits, especially those with divergent slopes and their stability properties. The main results are summarized in the following key theorems:

1. **Theorem 8:** This theorem states that all sequences S_k with a divergent slope are positively unstable, defining the sufficient condition under which a sequence becomes unstable.
2. **Theorem 9:** This theorem shows that all orbits with codings in G_0 are bounded.
3. **Theorem 10:** This theorem concludes that all natural numbers have bounded orbits, implying the non-existence of divergent orbits for natural numbers.

First, we examine the conditions under which the slope of a function S_k diverges, leading to instability. Next, we explore the boundedness of orbits coded within G_0 and G_∞, providing proofs and corollaries to support these findings. Finally, we demonstrate the non-existence of divergent orbits for natural numbers.

11.1. Summary of Propositions in the Section

1. **Theorem 8:** It is stated that all sequences S_k with a divergent slope are positively unstable.
2. **Theorem 9:** It is stated that all orbits with codings in G_0 are bounded.
3. **Corollary 7:** If exist a sub-sequence such that $\lim_{j\to\infty} \frac{a_{k_l+1}(\xi)}{k_l} > \frac{\ln(3)}{\ln(2)}$. Then exist $\Omega > 0$ such that $\pi^2(\xi)(k_l) < \Omega$.
4. **Theorem 10:** There are no divergent orbits for the Collatz function on natural numbers.
5. **Lemma 13:** Let $\frac{p}{q} \in \mathbb{Q}$ then $Cod\left(\frac{p}{q}\right) \in G_0 \cup G_\infty$. In particular, if there is $J > 0$ such that $Col_q^J(p) > 0$ then $Cod_q(p) \in G_0$ and if $Col_q^j(p) < 0$ for all $j \in \mathbb{N}$, then $Cod_q(p) \in G_\infty$.
6. **Theorem 11:** Consider the extension of Collatz's function on $\mathbb{Q}$, then all orbit fall into some cycle.

11.2. The Problem of Divergence

The following theorem shows that if the slope of the function S_k diverges, then so does the minimum value, this is because the only value that satisfies the encoding of S_k is negative.

Theorem 8 (Positively unstable Theorem)**.** *Let* $\{S_j\}_{j=1}^\infty \subset \langle \theta, \psi^q \rangle$ *such that* $\frac{3^{b_j}}{2^{a_{j+1}}} \to \infty$*, then* $S_j(x)$ *is positively unstable for all* $q \in \mathbb{Z}$*.*

Proof. Let $\xi = Cod(\{S_j\}_{j=1}^\infty)$. Since $\frac{3^{b_j}}{2^{a_{j+1}}} \to \infty$ then $\xi \in G_\infty$ by Proposition 35, 33 and 28 and Theorem 7. We have $-\pi^1(\xi) = -\sum_{j=1}^\infty \frac{2^{a_j}}{3^j} \in \mathbb{R}$ is the only value whose coding is ξ. On the other hand, regardless

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of rationality, this number is always negative. Therefore, the minimum value must necessarily be divergent. □

We are going to show a series of results referring to the bounds of the orbits of numbers whose coding is in G_0 and G_∞.

Theorem 9 (Bounded Orbit Theorem)**.** *Let $n \in \mathbb{Z}$ such that $Cod_q(n) \in G_0$ then the orbit of n is bounded for all $q \in \mathbb{N}$ odd.*

Proof. Let $\{S_j\}_{j\in\mathbb{N}}$ on $\langle \theta, \psi^q \rangle$ such that $Cod_q\{S_j\}_{j\in\mathbb{N}} = Cod(n)$. Without loss of generality we can assume that n is positive, because in the case that n is negative we have that

$$\begin{aligned}\lim_{j\to\infty} Col_q^{a_{j+1}}(n) &= \lim_{j\to\infty} S_{a_{j+1}}(n) = \lim_{j\to\infty} \frac{3^j n + qN_j}{2^{a_{j+1}}} \\ &= \lim_{j\to\infty} \frac{3^j}{2^{a_{j+1}}} n + \lim_{j\to\infty} q\pi^2(Cod(n))(j) = \lim_{j\to\infty} q\pi^2(Cod(n))(j) \geq 0\end{aligned}$$

then eventually its orbit will fall into a non-negative number.
If the coding of $\{S_j\}_{j\in\mathbb{N}}$ has a null tail, the result is trivial. Suppose $\{S_j\}_{j\in\mathbb{N}}$ has no null tail, then:

$$Col_q^{a_{j+1}}(n) = S_{a_{j+1}}(n) = \frac{3^j n + qN_j}{2^{a_{j+1}}} = \frac{3^j}{2^{a_{j+1}}} n + q\pi^2(Cod(n))(j)$$

Since $\dfrac{3^j}{2^{a_{j+1}}} \to 0$ we have by Proposition 35 we have that $q\pi^2(Cod(n))(j)$ is bounded and let $M > 0$ such that $q\pi^2(Cod(n))(j) < M$ for all $j \in \mathbb{N}$. On the other hand, since $\dfrac{3^j}{2^{a_{j+1}}} \to 0$ we have $\dfrac{3^j}{2^{a_{j+1}}}$ is bounded and let $H > 0$ such that $\dfrac{3^j}{2^{a_{j+1}}} < H$ for all $j \in \mathbb{N}$, then

$$\frac{3^j}{2^{a_{j+1}}} n + q\pi^2(Cod(n))(j) < Hn + M$$

Therefore $Col_q^j(n) < \infty$. □

Monks and Yazinski [13] also extend the results of Eliahou [5] (1993) and Lagarias [9] (1985) concerning the density of "odd" points in an orbit. Let $b_n(x)$ denote the number of ones in the first n digits of $x \in \Sigma_2^*$. If $x \in Q_{odd}$ eventually enters an n-periodic orbit, then

$$\frac{\ln(2)}{\ln(3 + 1/m)} \leq \lim_{n\to\infty} \frac{b_n(x)}{n} \leq \frac{\ln(2)}{\ln(3 + 1/M)}$$

where m, M are the least and greatest cyclic elements in the eventual cycle. If $x \in Q_{odd}$ diverges, then

$$\frac{\ln(2)}{\ln(3)} \leq \liminf_{n\to\infty} \frac{b_n(x)}{n}.$$

We will now show the main theorem of this work, where we finally show the non-existence of divergent orbits for every positive integer. We will show that the necessary and sufficient condition for an orbit to be divergent is

$$\limsup_{j\to\infty} \frac{a_{j+1}}{j} < \frac{\ln(3)}{\ln(2)}$$

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which implies that the only solution if it exists must be

$$-\infty < -\pi^1(\xi) \leq -\frac{1}{3}.$$

Theorem 10 (Divergent Orbits Theorem)**.** *There are no divergent orbits for the Collatz function on natural numbers.*

Proof. Let $n \in \mathbb{N}$ such that $Col^j(n) \to \infty$ as $j \to \infty$ and $\{S_j\}_{j=1}^{\infty} \subset \langle\theta, \psi\rangle$ such that $Cod\{S_j\}_{j=1}^{\infty} = Cod(n)$. We are going to prove that the necessary and sufficient condition for an orbit to be divergent is that the coding does not have a null tail and $\dfrac{3^j}{2^{a_{j+1}}} \to \infty$ as $j \to \infty$.

If $Cod\{S_j\}_{j=1}^{\infty}$ has a null tail, it means that from a certain iteration, the orbit of n must always be even, which implies that this orbit must be decreasing. This contradicts the fact that we have assumed that $Col^j(n) \to \infty$.

Now without loss of generality, we can assume that $Cod\{S_j\}_{j=1}^{\infty}$ does not have a null tail

$$Col^{a_j}(n) = \frac{3^j n + N_j}{2^{a_{j+1}}} \to \infty \text{ then } \frac{3^j}{2^{a_{j+1}}} \to \infty \text{ or } \pi_k^2(Cod\{S_j\}_{j=1}^{\infty}) = \frac{N_j}{2^{a_{j+1}}} \to \infty$$

If $\dfrac{3^j}{2^{a_{j+1}}} \to \infty$, then by Proposition 35, we have that $Cod(n) \in G_\infty$. Thus, we have that $-\pi^1(Cod(n))$ is the only real number that satisfies the coding, and $-\pi^1(Cod(n)) \neq n$ since $-\pi^1(Cod(n)) < 0$, which contradicts the hypothesis that S_j is positively stable, since $\rho_0(S_j) \to \infty$ as $j \to \infty$.

If $\pi_k^2(Cod\{S_j\}_{j=1}^{\infty}) \to \infty$, by Proposition 35 we have that $Cod\{S_j\}_{j=1}^{\infty} \notin G_0$ then $\liminf\limits_{j\to\infty} \dfrac{a_{j+1}}{j} \leq \dfrac{\ln(3)}{\ln(2)}$. Let's show now in fact $\limsup\limits_{k\to\infty} \dfrac{a_{k+1}}{k} \leq \dfrac{\ln(3)}{\ln(2)}$. Suppose that exist $\{j_k\}_{k\in\mathbb{N}}$ such that $\lim\limits_{j\to\infty} \dfrac{a_{j_k+1}}{j_k} > \dfrac{\ln(3)}{\ln(2)} + \varepsilon$ with $\varepsilon > 0$, so using the estimated bound in the demonstration of the Lemma 7, we have

$$\pi^2(\xi)(j_k) < \frac{2^\varepsilon}{2^\varepsilon - 1} + 3\left\{\left(\frac{3}{2}\right)^{T-1} - \frac{2}{3}\right\} = \Omega$$

Since $\dfrac{3^{j_k}}{2^{a_{j_k+1}}} \to 0$ as $j \to \infty$. So $\max\limits_k \left\{\dfrac{3^{j_k}}{2^{a_{j_k+1}}}\right\} < \infty$ then we have

$$Col^{j_k}(n) = \frac{3^{j_k}}{2^{a_{j_k+1}}} n + \frac{N_{j_k}}{2^{j_k+1}} < \max_k \left\{\frac{3^{j_k}}{2^{a_{j_k+1}}}\right\} n + \Omega$$

Since $Col^j(n) \in \mathbb{N}$ for all $j \in \mathbb{N}$, then exist $K \in \mathbb{N}$ such that $Col^{j_k}(n) = Col^{j_{k+K}}(n)$. So we have that the orbit of n must fall into a cycle, which implies that $\lim\limits_{j\to\infty} \dfrac{3^j}{2^{a_{j+1}}} = 0$ which implies that $\pi^2(\xi)(j)$ is bounded, which contradicts the hypothesis that $Col^j(n) \to \infty$. Therefore $\limsup\limits_{j\to\infty} \dfrac{a_{j+1}}{j} \leq \dfrac{\ln(3)}{\ln(2)}$. This is equivalent to $Cod\{S_j\}_{j=1}^{\infty} \in G_1 \cup G_\infty$.

On the other hand if $Cod\{S_j\}_{j=1}^{\infty} \in G_1$ by Theorem 6 we have $\rho_0\{S_j\}_{j=1}^{\infty} \to \infty$ which contradicts the hypothesis that $\{S_j\}_{j=1}^{\infty}$ is positively stable, so $Cod\{S_j\}_{j=1}^{\infty} \in G_\infty$

Since the Theorem 9 above, to have $Col^j(u)$ divergent it is necessary that $\dfrac{3^k}{2^{a_{k+1}}} \to \infty$, by Theorem 8 we have $\{S_j\}_{j=1}^{\infty}$ is positively unstable, by Theorem, 2 we have $\mathbb{E}\left(\{S_j\}_{j=1}^{\infty}\right) \cap \mathbb{N} = \emptyset$. Therefore, cannot exist $n \in \mathbb{N}$ such that its orbit is divergent. $\square$

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Next we will present a more general result. Indeed, all rational ones have orbits that fall into some cycle.

Lemma 13. *Let $\frac{p}{q} \in \mathbb{Q}$ then $Cod\left(\frac{p}{q}\right) \in G_0 \cup G_\infty$. In particular, if there is $J > 0$ such that $Col_q^J(p) > 0$ then $Cod_q(p) \in G_0$ and if $Col_q^j(p) < 0$ for all $j \in \mathbb{N}$, then $Cod_q(p) \in G_\infty$.*

Proof. Let $\frac{p}{q} \in \mathbb{Q}$ with $q > 0$. If $p > 0$ then we can repeat the same argument from the proof of the Theorem 10 to show that $Cod_q(p) \in G_0$. Now if $p < 0$. Suppose that $Cod_q(p) \notin G_0 \cup G_\infty$. Additionally, let us assume that the orbit of p is always negative, since if there exists $J > 0$ such that $Col^J(p) > 0$, then $Col^j(p) > 0$ for all $j > J$ and we can repeat the argument of the Theorem 10 again using $n = Col^J(p)$. Then assuming that the orbit of p is always negative we have

$$Col^{a_{j+1}}(p) = \frac{3^j}{2^{a_{j+1}}}p + \frac{N_j}{2^{a_{j+1}}} < 0 \text{ for all } j \in \mathbb{N}$$

so

$$\frac{N_j}{2^{a_{j+1}}} < -\frac{3^j}{2^{a_{j+1}}}p \text{ so we have } \frac{N_j}{3^j} = \pi_j^1(Cod_q(p)) < -p$$

Since π_j^1 is monotonous, then $\lim_{j\to\infty} \pi_j^1(Cod_q(p)) \in \mathbb{R}$ and by Proposition 35 we have that $Cod_q(p) \in G_\infty$ □

Theorem 11. *Consider the extension of Collatz's function on $\mathbb{Q}$, then all orbit fall into some cycle.*

Proof. By Lemma 13 we only have to analyse the cases where the coding is in G_0 or in G_∞. Let $\frac{p}{q} \in \mathbb{Q}$ with $q > 0$. Suppose that $Cod\left(\frac{p}{q}\right) = \xi \in G_\infty$ and that ξ does not have a null tail, otherwise its orbit falls at point 0, then $\frac{p}{q} = -\sum_{j\in\mathbb{N}} \frac{2^{a_j}}{3^j}$. On the other hand, since $\xi \in G_\infty$ exist $\varepsilon > 0$ such that $\limsup_{k\to\infty} \frac{a_{k+1}}{k} < \frac{\ln(3)}{\ln(2)} - \varepsilon$. Exist $J > 0$ such that $\frac{a_{j+1} - a_k}{j-k} < \frac{\ln(3)}{\ln(2)} - \varepsilon$ for all $k > J$. so

$$\begin{aligned}
Col^{a_{k+1}}\left(-\pi^1(\xi)\right) &= Col^{a_{k+1}}\left(-\sum_{j\in\mathbb{N}} \frac{2^{a_j}}{3^j}\right) = -\frac{3^k}{2^{a_{k+1}}}\sum_{j\in\mathbb{N}} \frac{2^{a_j}}{3^j} + \frac{N_k}{2^{a_{k+1}}} \\
&= -\frac{3^k}{2^{a_{k+1}}}\sum_{j=1}^{k} \frac{2^{a_j}}{3^j} - \frac{3^k}{2^{a_{k+1}}}\sum_{j=k+1}^{\infty} \frac{2^{a_j}}{3^j} + \frac{N_k}{2^{a_{k+1}}} = -\frac{N_k}{2^{a_{k+1}}} - \frac{3^k}{2^{a_{k+1}}}\sum_{j=k+1}^{\infty} \frac{2^{a_j}}{3^j} + \frac{N_k}{2^{a_{k+1}}} \\
&= -\frac{3^k}{2^{a_{k+1}}}\sum_{j=k+1}^{\infty} \frac{2^{a_j}}{3^j} = -\sum_{j=k+1}^{\infty} \frac{2^{a_j - a_{k+1}}}{3^{j-k}} = -\sum_{j=k+1}^{\infty} \left(\frac{2^{\frac{a_j - a_{k+1}}{j-k}}}{3}\right)^{j-k} \\
&> -\sum_{j=k+1}^{\infty} \left(\frac{2^{\frac{\ln(3)}{\ln(2)} - \varepsilon}}{3}\right)^{j-k} = -\sum_{j=k+1}^{\infty} 2^{-\varepsilon(j-k)} = -2^{\varepsilon k}\sum_{j=k+1}^{\infty} 2^{-\varepsilon j} \\
&= -2^{\varepsilon k}\left\{\frac{2^{-\varepsilon(k+1)}}{1-2^{-\varepsilon}}\right\} = -\frac{2^{-\varepsilon}}{1-2^{-\varepsilon}}
\end{aligned}$$

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Let $\Omega = \max\left\{\frac{2^{-\varepsilon}}{1-2^{-\varepsilon}}, Col^{a_{k+1}}\left(-\pi^1(\xi)\right) \text{ with } k < J\right\}$, so $Col^{a_{k+1}}\left(\frac{p}{q}\right) > -\Omega$ for all $k \in \mathbb{N}$. By Proposition 9 we have

$$Col^{a_{k+1}}\left(\frac{p}{q}\right) = \frac{1}{q}Col_q^{a_{k+1}}(p) \text{ then } Col_q^{a_{k+1}}(p) > -q\Omega \text{ since } q > 0$$

Since the orbits of $-\pi^1(\xi)$ are always negative, we have to $Col_q^{a_{k+1}}(p) < 0$. Since the sub-orbit of p is bounded, and Col_q is defined on the integers, we have to necessarily have the orbit fall into some cycle.

Now suppose that $\xi \in G_0$, then by Theorem 9 we have the orbit is bounded, Without loss of generality we can assume that its orbit is positive, so its orbit must necessarily fall in some cycle. □

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