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A Unified Proof of the Extended, Generalized, and Grand Riemann Hypothesis

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Abstract

The Extended Riemann Hypothesis (for Dedekind zeta function), the Generalized Riemann Hypothesis (for Dirichlet L-function), and the Grand Riemann Hypothesis (for modular form L-function, automorphic L-function, and etc.) are proved within a unified framework based on the divisibility in the symmetrical functional equations of the completed L-functions represented as Hadamard products.

Keywords: Extended Riemann Hypothesis; Generalized Riemann Hypothesis; Grand Riemann Hypothesis; L-functions; Hadamard product; functional equation

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1. Introduction

The Riemann Hypothesis (RH) is proved in Ref.[1] based on a new expression of the completed zeta function $\xi(s)$, which was obtained through pairing the conjugate zeros ρ_i and $\bar{\rho}_i$ in the Hadamard product expression, with consideration of the multiplicities of zeros, i.e.

$$\xi(s) = \xi(0)\prod_{\rho}(1-\frac{s}{\rho}) = \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i}) = \xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{m_i}$$

where $\xi(0)=\frac{1}{2}$, $\rho_i=\alpha_i+j\beta_i$, $\bar{\rho}_i=\alpha_i-j\beta_i$, with $0<\alpha_i<1$, $\beta_i\neq 0$, $0<|\beta_1|\leq|\beta_2|\leq\cdots$, and $m_i\geq 1$ is the multiplicity of ρ_i.

It should be noted that in this article and Ref.[1], j is used to denote the imaginary unit ($j^2=-1$), while i serves as a natural number index.

Lemma 8 is the key lemma to the proof of the RH in Ref.[1]. The key points include the divisibility contained in a variant of the functional equation $\xi(s)=\xi(1-s)$ and the uniqueness of the multiplicity m_i of zero ρ_i (although it is unknown). According to Lemma 8, we finally obtain

$$\xi(s)=\xi(1-s)\Leftrightarrow \alpha_i=\frac{1}{2}, 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots, i=1,2,3,\ldots$$

Here we give the details of Lemma 8 as the base of subsequent discussions.

Lemma 8 [1]: Given two entire functions represented as absolutely convergent (on the whole complex plane) infinite products of polynomial factors

$$f(s)=\prod_{i=1}^{\infty}\Big(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i} \tag{1}$$

and

$$f(1-s)=\prod_{i=1}^{\infty}\Big(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i} \tag{2}$$

where s is the complex variable, $\rho_i=\alpha_i+j\beta_i$ and $\bar{\rho}_i=\alpha_i-j\beta_i$ are the complex conjugate zeros of the completed zeta function $\xi(s)$, $0<\alpha_i<1$ and $\beta_i\neq 0$ are real numbers, $0<|\beta_1|\leq|\beta_2|\leq|\beta_3|\leq\cdots$, $m_i\geq 1$ is the multiplicity of quadruplets of zeros $(\rho_i,\bar{\rho}_i,1-\rho_i,1-\bar{\rho}_i)$.

Then we have

$$f(s) = f(1-s) \Leftrightarrow \begin{cases} \alpha_i = \frac{1}{2} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \\ i = 1, 2, 3, \cdots \end{cases} \tag{3}$$

The other related lemmas in Ref.[1] are also provided in the following.

Lemma 1 [1]: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 < \alpha < 1$;
4) $\rho, \bar{\rho}, 1-\bar{\rho}, 1-\rho$ are all non-trivial zeroes.

Lemma 2 [1]: The zeros of $\xi(s)$ coincide with the non-trivial zeros of $\zeta(s)$.

Lemma 3 [1]: Let $m(x), g_1(x), ..., g_n(x) \in \mathbb{R}[x], n \geq 2$. If $m(x)$ is irreducible (prime) and divides the product $g_1(x) \cdots g_n(x)$, then $m(x)$ divides one of the polynomials $g_1(x), ..., g_n(x)$.

Lemma 4 [1]: Let $f(x), m(x) \in \mathbb{R}[x]$. If $m(x)$ is irreducible and $f(x)$ is any polynomial, then either $m(x)$ divides $f(x)$ or $gcd(m(x), f(x)) = 1$.

Lemma 5 [1]: Let $f(x) = \prod_{i=1}^{\infty} g_i(x)$ (the infinite product is absolutely convergent in the whole complex plane), $g_i(x) \in \mathbb{R}[x]$, be an entire function. If $g_i(x)$ are irreducible polynomials with the same degree as $m(x)$, $m(x) \in \mathbb{R}[x]$ is irreducible and $m(x) \mid f(x)$, then $m(x)$ divides one of the polynomials $g_1(x), g_2(x), \ldots$.

Lemma 6 [1]: Let $f(s)$ be a non-zero entire function, and let s_0 be a zero of $f(s)$. Then the multiplicity of s_0 is a finite positive integer.

Lemma 7 [1]: Let $f(s)$ be a non-zero entire function, and let s_0 be a zero of $f(s)$. Then the multiplicity of s_0 is unique.

Lemma 9 [1]: The infinite product $\prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} \right)^{m_i}$ converges to a non-zero constant, given the conditions: $0 < \alpha_i < 1, \beta_i \neq 0, \sum_{i=1}^{\infty} \frac{1}{\beta_i^2} < \infty$, and $m_i \geq 1$ is the multiplicity of zero $\alpha_i + j\beta_i$.

Remark: Lemmas 1-4 are the well-known results summarized from related journal papers, or textbooks/monographs. Lemmas 5-9 are proved in Ref.[1].

Motivated by Lemma 8, we have the following theorems 1-4, in which Theorem 4 is a unified basis for the proofs of the Extended Riemann Hypothesis, the Generalized Riemann Hypothesis, and the Grand Riemann Hypothesis.

2. Four Theorems

As pointed in Ref.[2] (on page 57), we can enumerate the nontrivial zeros of the zeta function in order of the increasing absolute value of their imaginary parts, where zeros whose imaginary parts have the same absolute value are arranged arbitrarily. Thus we remove the default ordering of $|\beta_i|$, $|\beta_1| \leq |\beta_2| \leq \cdots$, as condition hereafter for simplicity.

Theorem 1: Given two entire functions represented as absolutely convergent (on the whole complex plane) infinite products of polynomial factors

$$F(s) = \prod_{i=1}^{\infty} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \tag{4}$$

and

$$F(k-s) = \prod_{i=1}^{\infty} \left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \tag{5}$$

where s is a complex variable, $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $F(s)$, $0 < \alpha_i < k$, $k > 0$, and $\beta_i \neq 0$ are real numbers, $m_i \geq 1$ is the multiplicity of quadruplets of zeros $(\rho_i, \bar{\rho}_i, k-\rho_i, k-\bar{\rho}_i)$, $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$.

Then we have

$$F(s) = F(k-s) \Rightarrow \begin{cases} \alpha_i = \frac{k}{2}, i = 1,2,3,\cdots \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \end{cases} \tag{6}$$

Proof. According to the definition of divisibility of entire functions [3,4] (or more specifically the definition that a polynomial divides an entire function expressed as infinite product of polynomial factors [1]), the functional equation $F(s) = F(k-s)$ implies that each polynomial factor on either side divides the infinite product on the opposite side, i.e.,

$$F(s) = F(k-s) \Rightarrow \begin{cases} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) \mid F(k-s) \\ \left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right) \mid F(s) \\ i = 1,2,3,\cdots \end{cases} \tag{7}$$

where "$|$" is the divisible sign.

Since $1 + \frac{(s-\alpha_i)^2}{\beta_i^2}$ (with discriminant $\Delta = -4 \cdot \frac{1}{\beta_i^2} < 0$) and $1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}$ (with discriminant $\Delta = -4 \cdot \frac{1}{\beta_i^2} < 0$) are irreducible over the field $\mathbb{R}$, then by Lemma 5, Eq.(7) yields:

$$\begin{cases} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) \mid \left(1 + \frac{(k-s-\alpha_l)^2}{\beta_l^2}\right) \\ \left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right) \mid \left(1 + \frac{(s-\alpha_l)^2}{\beta_l^2}\right) \\ i = 1,2,3,\cdots; l = 1,2,3,\cdots \end{cases} \tag{8}$$

For the special kind of polynomials in Eq.(8), "divisible" means "equal", which can be verified by comparing the like terms in equation

$$\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) = k'\left(1 + \frac{(k-s-\alpha_l)^2}{\beta_l^2}\right), k' \neq 0 \in \mathbb{R} \tag{9}$$

to get $k' = 1$. Further, due to the uniqueness of the multiplicity m_i, the only solution to Eq.(8) is $i = l$, otherwise, new duplicated zeros with $\alpha_l = k - \alpha_i, k - \alpha_l = \alpha_i, \beta_i^2 = \beta_l^2, l \neq i$ would be generated to change m_i. Therefore we have from Eq.(8):

$$\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right) = \left(1 + \frac{(k-s-\alpha_i)^2}{\beta_i^2}\right), i = 1,2,3,\cdots \tag{10}$$

By comparing the like terms in Eq.(10), we obtain $\alpha_i = \frac{k}{2}$. Further, to ensure the uniqueness of m_i while $\alpha_i = \frac{k}{2}$, we need limit the β_i values to be distinct, i.e., $0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots$.

That completes the proof of Theorem 1. □

Theorem 2: Given two entire functions represented as absolutely convergent (on the whole complex plane) infinite products of polynomial factors

$$\begin{aligned} G(s) &= \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)^{m_i} \\ &= \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right)^{m_i} \left(1 - \frac{s}{\bar{\rho}_i}\right)^{m_i} \\ &= \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right) \end{aligned} \tag{11}$$

and

$$\begin{aligned}G(k-s) &= \prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(k-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{m_i} \\ &= \prod_{i=1}^{\infty}\Big(1-\frac{k-s}{\rho_i}\Big)^{m_i}\Big(1-\frac{k-s}{\bar{\rho}_i}\Big)^{m_i} \\ &= \prod_{i=1}^{\infty}\Big(1-\frac{k-s}{\rho_i}\Big)\end{aligned} \tag{12}$$

where s is a complex variable, $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $G(s)$, $0 < \alpha_i < k$, $k > 0$, and $\beta_i \neq 0$ are real numbers, $m_i \geq 1$ is the multiplicity of quadruplets of zeros $(\rho_i, \bar{\rho}_i, k-\rho_i, k-\bar{\rho}_i)$, $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2} < \infty$.

Then we have

$$G(s) = G(k-s) \Rightarrow \begin{cases} \alpha_i = \frac{k}{2}, i = 1,2,3,\cdots \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \end{cases} \tag{13}$$

Proof. According to Theorem 1 and Lemma 9, we have

$$\begin{aligned}&F(s) = F(k-s) \\ &\Leftrightarrow \\ &\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\Big)^{m_i}F(s) = \prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\Big)^{m_i}F(k-s) \\ &\Leftrightarrow \\ &\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{m_i} = \prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(k-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{m_i} \\ &\Leftrightarrow \\ &G(s) = G(k-s)\end{aligned} \tag{14}$$

Then we know that

$$G(s) = G(k-s) \Rightarrow \begin{cases} \alpha_i = \frac{k}{2}, i = 1,2,3,\cdots \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \end{cases} \tag{15}$$

That completes the proof of Theorem 2. □

Theorem 3: Given two entire functions represented by their Hadamard products:

$$\Lambda(\lambda, s) = e^{A(\lambda)+B(\lambda)s}\prod_{i=1}^{\infty}\Big(1-\frac{s}{\rho_i}\Big)e^{\frac{s}{\rho_i}} \tag{16}$$

and

$$\Lambda(\overline{\lambda}, k-s) = e^{A(\overline{\lambda})+B(\overline{\lambda})(k-s)}\prod_{i=1}^{\infty}\Big(1-\frac{k-s}{\rho_i}\Big)e^{\frac{k-s}{\rho_i}} \tag{17}$$

where s is a complex variable, λ denotes a mathematical object (e.g., Dirichlet character, modular form, automorphic representation), $\overline{\lambda}$ is the dual of λ, $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $\Lambda(\lambda, s)$, $0 < \alpha_i < k$, $k > 0$, and $\beta_i \neq 0$ are real numbers, $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2} < \infty$.

Then we have

$$\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\overline{\lambda}, k-s) \Rightarrow \begin{cases} \alpha_i = \frac{k}{2}, i = 1,2,3,\cdots \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \end{cases} \tag{18}$$

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where $\varepsilon(\lambda)$ is a complex number of absolute value 1, called the "root number" of L-function $L(\lambda, s)$.

Remark: For more details about $\varepsilon(\lambda)$, see Ref.[5] on page 94.

Proof. First, we have

$$\begin{aligned}\Lambda(\lambda,s) &= e^{A(\lambda)+B(\lambda)s}\prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)e^{\frac{s}{\rho_i}} \\ &= e^{A(\lambda)+B(\lambda)s}\prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)\left(1-\frac{s}{\bar{\rho}_i}\right)e^{\frac{2\alpha_i s}{\alpha_i^2+\beta_i^2}} \\ &= e^{A(\lambda)+B(\lambda)s}e^{s\cdot\sum_{i=1}^{\infty}\frac{2\alpha_i}{\alpha_i^2+\beta_i^2}}\prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)\left(1-\frac{s}{\bar{\rho}_i}\right)\end{aligned} \tag{19}$$

Noticing that $\sum_{i=1}^{\infty}\frac{2\alpha_i}{\alpha_i^2+\beta_i^2} < 2k\cdot\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2} < \infty$, then we have $\sum_{i=1}^{\infty}\frac{2\alpha_i}{\alpha_i^2+\beta_i^2} = c, c\in\mathbb{R}, c\neq 0$. With further consideration of the multiplicity $m_i \geq 1$ of each zero, we obtain

$$\Lambda(\lambda,s) = e^{A(\lambda)+[B(\lambda)+c]s}\prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)^{m_i}\left(1-\frac{s}{\bar{\rho}_i}\right)^{m_i} \tag{20}$$

Accordingly

$$\Lambda(\bar{\lambda},k-s) = e^{A(\bar{\lambda})+[B(\bar{\lambda})+c](k-s)}\prod_{i=1}^{\infty}\left(1-\frac{k-s}{\rho_i}\right)^{m_i}\left(1-\frac{k-s}{\bar{\rho}_i}\right)^{m_i}$$

Then we conclude that Theorem 3 is true according to Theorem 2, considering $e^{A(\lambda)+[B(\lambda)+c]s}$ and $\varepsilon(\lambda)e^{A(\bar{\lambda})+[B(\bar{\lambda})+c](k-s)}$ have no zeros, thus both of them have no effect on the complex zeros related divisibility in the functional equation $\Lambda(\lambda,s) = \varepsilon(\lambda)\Lambda(\bar{\lambda},k-s)$.

That completes the proof of Theorem 3. □

In the following Theorem 4, we make further efforts to lay a foundation for the study of completed L-functions that possess both real and complex zeros, denoted by $\rho\in\mathcal{Z}_{\text{real}}$ ($\Im(\rho)=0$) and $\rho\in\mathcal{Z}_{\text{complex}}$ ($\Im(\rho)\neq 0$), respectively. When these two zero sets have no common elements, we express their disjointness by: $\mathcal{Z}_{\text{real}}\cap\mathcal{Z}_{\text{complex}} = \varnothing$.

The reason we need to consider this case is that, so far, we cannot rule out the existence of exceptional zeros (or Landau-Siegel zeros), although their numbers are very limited even if they do exist.

Denote the set of real zeros in the critical strip as

$$\mathcal{Z}_{\text{real}} = \{a_n\in\mathbb{R} \mid 0 < a_n < k,\ n = 1,2,\ldots,N\}$$

where N is a finite natural number. This finiteness follows from the Identity Theorem, which implies that any non-zero entire function cannot have infinitely many zeros in a bounded region.

Theorem 4: Given two entire functions represented by their Hadamard products:

$$\begin{aligned}\Lambda(\lambda,s) &= e^{A(\lambda)+B(\lambda)s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} \\ &= e^{A(\lambda)+B(\lambda)s}\prod_{\rho\in\mathcal{Z}_{\text{real}}}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}\prod_{\rho\in\mathcal{Z}_{\text{complex}}}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} \\ &= e^{A(\lambda)+[B(\lambda)+c]s}\prod_{n=1}^{N}\left(1-\frac{s}{a_n}\right)e^{\frac{s}{a_n}}\prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)\left(1-\frac{s}{\bar{\rho}_i}\right)\end{aligned} \tag{21}$$

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and

$$\begin{aligned}\Lambda(\overline{\lambda},k-s) &= e^{A(\overline{\lambda})+B(\overline{\lambda})(k-s)}\prod_{\rho}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}} \\ &= e^{A(\overline{\lambda})+B(\overline{\lambda})(k-s)}\prod_{\rho\in\mathcal{Z}_{\text{real}}}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}}\prod_{\rho\in\mathcal{Z}_{\text{complex}}}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}} \\ &= e^{A(\overline{\lambda})+[B(\overline{\lambda})+c](k-s)}\prod_{n=1}^{N}\left(1-\frac{k-s}{a_n}\right)e^{\frac{k-s}{a_n}}\prod_{i=1}^{\infty}\left(1-\frac{k-s}{\rho_i}\right)\left(1-\frac{k-s}{\bar{\rho}_i}\right)\end{aligned} \tag{22}$$

where s is a complex variable, $c=\sum_{i=1}^{\infty}\frac{2\alpha_i}{\alpha_i^2+\beta_i^2}$, λ denotes a mathematical object, $\overline{\lambda}$ is the dual of λ, $\rho_i=\alpha_i+j\beta_i$ and $\bar{\rho}_i=\alpha_i-j\beta_i$ are the complex conjugate zeros of $\Lambda(\lambda,s)$, $0<\alpha_i<k$, $0<a_n<k$, $k>0$, and $\beta_i\neq 0$ are real numbers, $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}<\infty$, $\mathcal{Z}_{\text{real}}\cap\mathcal{Z}_{\text{complex}}=\varnothing$.

Then we have

$$\Lambda(\lambda,s)=\varepsilon(\lambda)\Lambda(\overline{\lambda},k-s)\Rightarrow\begin{cases}\alpha_i=\frac{k}{2}, i=1,2,3,\cdots\\ 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots\\ a_n=\frac{k}{2}, n=1\end{cases} \tag{23}$$

i.e., all the zeros (both real and complex) of $\Lambda(\lambda,s)$ lie on the critical line.

Proof. By Theorem 3, to determine the distribution of the complex zeros of $\Lambda(\lambda,s)$, we only need to show that the newly added parts $\prod_{\rho\in\mathcal{Z}_{\text{real}}}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}$ and $\prod_{\rho\in\mathcal{Z}_{\text{real}}}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}}$ do not affect the complex zeros related divisibility in the functional equation $\Lambda(\lambda,s)=\varepsilon(\lambda)\Lambda(\overline{\lambda},k-s)$, which is an obvious fact since the given condition $\mathcal{Z}_{\text{real}}\cap\mathcal{Z}_{\text{complex}}=\varnothing$ means that $\prod_{\rho\in\mathcal{Z}_{\text{real}}}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}$ and $\prod_{\rho\in\mathcal{Z}_{\text{real}}}\left(1-\frac{k-s}{\rho}\right)e^{\frac{k-s}{\rho}}$ are co-prime according to Ref.[3] (on pages 174, 208) and Ref.[4] (see its THEOREM 4).

Thus, we conclude by Theorem 3 that

$$\Lambda(\lambda,s)=\varepsilon(\lambda)\Lambda(\overline{\lambda},k-s)\Rightarrow\begin{cases}\alpha_i=\frac{k}{2}, i=1,2,3,\cdots\\ 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots\end{cases} \tag{24}$$

Next, we consider the real zeros of $\Lambda(\lambda,s)$.

By canceling the complex non-trivial zeros related polynomial factors on both sides of $\Lambda(\lambda,s)=\varepsilon(\lambda)\Lambda(\overline{\lambda},k-s)$, we have

$$e^{A(\lambda)+[B(\lambda)+c]s}\prod_{n=1}^{N}\left(1-\frac{s}{a_n}\right)e^{\frac{s}{a_n}}=\varepsilon(\lambda)e^{A(\overline{\lambda})+[B(\overline{\lambda})+c](k-s)}\prod_{n=1}^{N}\left(1-\frac{k-s}{a_n}\right)e^{\frac{k-s}{a_n}} \tag{25}$$

Further Eq.(25) is equivalent to

$$e^{A(\lambda)+[B(\lambda)+c']s}\prod_{n=1}^{N}\left(s-a_n\right)=\varepsilon(\lambda)e^{A(\overline{\lambda})+[B(\overline{\lambda})+c'](k-s)}\prod_{n=1}^{N}\left(s-(k-a_n)\right) \tag{26}$$

where $c'=c+\sum_{n=1}^{N}\frac{1}{a_n}$.

Suppose the multiplicity of zero $s=a_n$ is m_n ($m_n\geq 1$) that is finite and unique although unknown. Then Eq.(26) becomes

$$e^{A(\lambda)+[B(\lambda)+c']s}\prod_{t=1}^{T}\left(s-a_t\right)^{m_t}=\varepsilon(\lambda)e^{A(\overline{\lambda})+[B(\overline{\lambda})+c'](k-s)}\prod_{t=1}^{T}\left(s-(k-a_t)\right)^{m_t} \tag{27}$$

where $\sum_{t=1}^{T} m_t = N$.

Considering $(s - a_t)$ and $(s - (k - a_t))$ are irreducible over $\mathbb{R}$, then by Lemma 3, Eq.(27) means

$$\begin{cases} (s - a_t) \mid (s - (k - a_m) \\ (s - (k - a_t)) \mid (s - a_m) \\ t = 1, 2, 3, \cdots, T; m = 1, 2, 3, \cdots, T \end{cases} \tag{28}$$

The only solution to Eq.(28) is $t = m, a_t = \frac{k}{2}, t = 1, \cdots, T$, otherwise the uniqueness of m_t would be violated with $a_t + a_m = k, t \neq m$. To avoid changing the multiplicity of m_t while $a_t = \frac{k}{2}$, we need to limit $T = 1$. Thus we get

$$\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\overline{\lambda}, k - s) \Rightarrow a_t = \frac{k}{2}, t = 1 \tag{29}$$

where zero $s = a_1 = \frac{k}{2}$ with multiplicity $m_1 = N$.

Putting Eq.(24) and Eq.(29) together, we proved Eq.(23).

That completes the proof of Theorem 4. □

Remark: As pointed out in Ref.[5] (on page 102), if $\rho_i = \alpha_i + j\beta_i$ is a zero of $\Lambda(\lambda, s)$, then $\bar{\rho}_i = \alpha_i - j\beta_i$ is a zero of $\Lambda(\bar{\lambda}, s)$. Therefore, to use Theorem 4 while $\lambda \neq \bar{\lambda}$, we need to construct a new symmetric functional equation $\Lambda(\bar{\lambda}, s)\Lambda(\lambda, s) = \varepsilon(\lambda)\varepsilon(\overline{\lambda})\Lambda(\lambda, k - s)\Lambda(\bar{\lambda}, k - s)$ to ensure that the conjugate zeros appear together. For more details, see the proofs of Theorem 5 and Theorem 7.

Actually, Theorem 4 provides a unified proof framework for the Extended Riemann Hypothesis, the Generalized Riemann Hypothesis, and the Grand Riemann Hypothesis.

3. The Applications of Theorem 4

We will make use of Theorem 4 to prove the Extended Riemann Hypothesis (for Dedekind zeta function), the Generalized Riemann Hypothesis (for Dirichlet L-function), and the Grand Riemann Hypothesis (for modular form L-Function, automorphic L-function, and etc.).

To facilitate the subsequent discussion, we give the details of the Extended Riemann Hypothesis, the Generalized Riemann Hypothesis, and the Grand Riemann Hypothesis. In the following contents, the critical line means: $\Re(s) = \frac{1}{2}$, or more generally, $\Re(s) = \frac{k}{2}, k > 0$ is a constant real number.

The Generalized Riemann Hypothesis: The nontrivial zeros of Dirichlet L-functions lie on the critical line.

The Extended Riemann Hypothesis: The nontrivial zeros of Dedekind zeta functions lie on the critical line.

The Grand Riemann Hypothesis: The nontrivial zeros of all L-functions lie on the critical line.

Another version of Grand Riemann Hypothesis: The nontrivial zeros of all automorphic L-functions lie on the critical line.

To begin with, we provide a general property of L-functions, which was labeled Lemma 5.5 in Ref.[5] on page 101.

Lemma 5.5 [5]: Let $L(f, s)$ be an L-function. All zeros ρ of $\Lambda(f, s)$ are in the critical strip $0 \leq \sigma \leq 1$. For any $\epsilon > 0$, we have

$$\sum_{\rho \neq 0,1} |\rho|^{-1-\epsilon} < +\infty.$$

where, $\Lambda(f, s)$ is the completed L-function corresponding to $L(f, s)$, σ is the real part of ρ, and f is identical to λ in this paper as a symbol representing a mathematical object (e.g., Dirichlet character, modular form, automorphic representation).

Another general property of L-functions is as follows.

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The zeros of $\Lambda(f,s)$ are precisely the non-trivial zeros of $L(f,s)$, as the trivial zeros of $L(f,s)$ are canceled by the poles of the Gamma factors in the completion process (see Ref.[5] on page 96 for more details).

Thus, we can discuss the non trivial zeros of L-functions based on the zeros of the corresponding completed L-functions.

3.1. Dirichlet L-Function

Definition: The Dirichlet L-function associated with a Dirichlet character χ modulo q is defined for $\Re(s) > 1$ by the series:

$$L(\chi,s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{30}$$

For the principal character χ_0 (where $\chi_0(n) = 1$ if $\gcd(n,q) = 1$ and $\chi_0(n) = 0$ otherwise), the L-function is related to the Riemann zeta function by:

$$L(\chi_0,s) = \zeta(s)\prod_{p|q}\left(1 - \frac{1}{p^s}\right) \tag{31}$$

Completed L-function: The completed Dirichlet L-function is defined as:

$$\Lambda(\chi,s) = \left(\frac{q}{\pi}\right)^{\frac{s+a}{2}}\Gamma\left(\frac{s+a}{2}\right)L(\chi,s) \tag{32}$$

where $a = 0$ if $\chi(-1) = 1$ (even character) and $a = 1$ if $\chi(-1) = -1$ (odd character).

Functional Equation: The completed Dirichlet L-function satisfies the functional equation:

$$\Lambda(\chi,s) = W(\chi)\Lambda(\overline{\chi},1-s) \tag{33}$$

where $W(\chi)$ is the Gauss sum:

$$W(\chi) = \frac{\tau(\chi)}{i^a\sqrt{q}} \tag{34}$$

and $\tau(\chi) = \sum_{n=1}^{q}\chi(n)e^{2\pi i n/q}$ is the Gauss sum associated with χ.

Hadamard Product: For non-principal characters χ, the completed L-function $\Lambda(\chi,s)$ is an entire function and has the Hadamard product:

$$\Lambda(\chi,s) = e^{A(\chi)+B(\chi)s}\prod_{\rho}\left(1 - \frac{s}{\rho}\right)e^{s/\rho} \tag{35}$$

where the product is over all zeros ρ of $\Lambda(\chi,s)$, and $A(\chi)$ and $B(\chi)$ are constants depending on χ.

Next we prove the Generalized Riemann Hypothesis.

Theorem 5: The nontrivial zeros of the above-described Dirichlet L-functions lie on the critical line.

Remark: We only need to prove that all the zeros of the completed Dirichlet L-function $\Lambda(\chi,s)$ have real part $\frac{1}{2}$, i.e., all the zeros of $\Lambda(\chi,s)$ lie on the critical line.

Proof. We conduct the proof in two cases.

CASE 1: $\chi = \bar{\chi}$ (self-dual)

It suffices to verify that the properties of $\Lambda(\chi,s)$ with $\chi = \bar{\chi}$ match the conditions of Theorem 4 with $\lambda = \chi$, $\varepsilon(\lambda) = W(\chi)$, $k = 1$. Eq.(35) is equivalent to Eq.(21) by separating all zeros into two sets $\mathcal{Z}_{\text{real}}$ and $\mathcal{Z}_{\text{complex}}$. Actually, to restrict $\chi = \bar{\chi}$ is to guarantee that the conjugate zeros of $\Lambda(\chi,s)$ appear in pairs, and then the quadruplets of non-trivial zeros $(\rho_i, \bar{\rho}_i, 1-\rho_i, 1-\bar{\rho}_i)$ with their multiplicities appear together according to Eq.(33). The condition $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2} < \infty$ can be assured by Lemma 5.5, considering that $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}$ is a subseries of $\sum_{\rho\neq 0,1}\frac{1}{|\rho|^2}$; The condition $\mathcal{Z}_{\text{real}} \cap \mathcal{Z}_{\text{complex}} = \varnothing$

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holds because $\mathcal{Z}_{\text{real}}$ and $\mathcal{Z}_{\text{complex}}$ are mutually exclusive sets, i.e., if $\rho \in \mathcal{Z}_{\text{real}}$, then $\Im(\rho) = 0$; if $\rho \in \mathcal{Z}_{\text{complex}}$, then $\Im(\rho) \neq 0$.

Therefore, by Theorem 4 with $\lambda = \chi$, $\varepsilon(\lambda) = W(\chi)$, $k = 1$, we know that both the real (if exists) and the complex zeros of $\Lambda(\chi, s)$ with $\chi = \bar{\chi}$ lie on the critical line.

CASE 2: $\chi \neq \bar{\chi}$

In this case, the conjugate non-trivial zeros do not appear together in Eq.(35), because if ρ is a zero of $\Lambda(\chi, s)$, then $\bar{\rho}$ is a zero of $\Lambda(\bar{\chi}, s)$.

Thus, we need to extend Eq.(33) to another form, i.e.,

$$\Lambda(\bar{\chi}, s) = W(\bar{\chi})\Lambda(\chi, 1 - s) \tag{36}$$

Combining (36) with (33), we get a new functional equation

$$\Lambda(\bar{\chi}, s)\Lambda(\chi, s) = W(\bar{\chi})W(\chi)\Lambda(\chi, 1 - s)\Lambda(\bar{\chi}, 1 - s) \tag{37}$$

Both sides of Eq.(37) are the products of entire functions, thus they are still entire functions. And we know that the conjugate zeros of $\Lambda(\bar{\chi}, s)\Lambda(\chi, s)$ appear in pairs, and then the quadruplets of non-trivial zeros $(\rho_i, \bar{\rho}_i, 1 - \rho_i, 1 - \bar{\rho}_i)$ with their multiplicities appear together according to Eq.(37). Further, based on Eq.(35), we have

$$\Lambda(\chi, s)\Lambda(\bar{\chi}, s) = e^{A(\chi) + A(\bar{\chi}) + [B(\chi) + B(\bar{\chi}) + c]s} \prod_{\rho \in \mathcal{Z}_{\text{real}}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right)\left(1 - \frac{s}{\bar{\rho}_i}\right) \tag{38}$$

where $c = \sum_{i=1}^{\infty} \frac{2\alpha_i}{\alpha_i^2 + \beta_i^2}$.

The condition $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2} < \infty$ and condition $\mathcal{Z}_{\text{real}} \cap \mathcal{Z}_{\text{complex}} = \varnothing$ hold for the same reasons as in CASE 1.

Therefore, by Theorem 4, we know that both the real (if exists) and the complex zeros of $\Lambda(\chi, s)\Lambda(\bar{\chi}, s)$ (thus of $\Lambda(\chi, s)$) with $\chi \neq \bar{\chi}$ lie on the critical line.

Combining CASE 1 and CASE 2, we conclude that Theorem 5 holds as a specific case of Theorem 4 with $k = 1$ and $\lambda = \chi$. $\square$

3.2. *Dedekind Zeta Function*

Definition: For a number field K with ring of integers $\mathcal{O}_K$, the Dedekind zeta function is defined for $\Re(s) > 1$ by:

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} \tag{39}$$

where the sum is over all non-zero ideals $\mathfrak{a}$ of $\mathcal{O}_K$, and $N(\mathfrak{a})$ is the norm of the ideal.

Completed Zeta Function: The completed Dedekind zeta function is defined as:

$$\Lambda_K(s) = |D_K|^{s/2}\left(\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\right)^{r_1}\left((2\pi)^{-s}\Gamma(s)\right)^{r_2}\zeta_K(s) \tag{40}$$

where D_K is the discriminant of K, r_1 is the number of real embeddings of K, r_2 is the number of pairs of complex embeddings of K.

Functional Equation: The completed Dedekind zeta function satisfies:

$$\Lambda_K(s) = \varepsilon(K)\Lambda_K(1 - s) \tag{41}$$

where $\varepsilon(K) = 1$ for all number fields K, showing the symmetry of the functional equation.

Hadamard Product: The completed Dedekind zeta function has a simple pole at $s = 1$ with residue $\frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K\sqrt{|D_K|}}$, where h_K is the class number, R_K is the regulator, and w_K is the number of roots of unity in K. The function $s(s - 1)\Lambda_K(s)$ is entire and has the Hadamard product:

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$$s(s-1)\Lambda_K(s) = e^{A_K+B_Ks}\prod_{\rho\neq 0,1}\left(1-\frac{s}{\rho}\right)e^{s/\rho} \tag{42}$$

where the product is over all zeros ρ of $\Lambda_K(s)$ except $\rho = 0$ and $\rho = 1$, and A_K and B_K are constants depending on K.

For more details of the completed Dedekind zeta function, please be referred to Ref.[5] (Chapter 5.10) and Ref.[6] (Section 10.5.1).

Theorem 6: The nontrivial zeros of the above-described Dedekind zeta function lie on the critical line.

Remark: We only need to prove that all the zeros of $\Lambda_K(s)$ have real part $\frac{1}{2}$, i.e., all the zeros of $\Lambda_K(s)$ lie on the critical line.

Proof. It suffices to show that the properties of $\Lambda_K(s)$ match the conditions of Theorem 4 with $\lambda = K$, $\varepsilon(\lambda) = 1$, $k = 1$.

Actually, Eq.(41) and Eq.(42) guarantee that

$$e^{A_K+B_Ks}\prod_{\rho\neq 0,1}\left(1-\frac{s}{\rho}\right)e^{s/\rho} = e^{A_K+B_K(1-s)}\prod_{\rho\neq 0,1}\left(1-\frac{1-s}{\rho}\right)e^{1-s/\rho} \tag{43}$$

where $\prod_{\rho\neq 0,1}\left(1-\frac{s}{\rho}\right)e^{s/\rho} = \prod_{\rho\in\mathcal{Z}_{\text{real}}\setminus\{0,1\}}\left(1-\frac{s}{\rho}\right)e^{s/\rho}\prod_{\rho\in\mathcal{Z}_{\text{complex}}}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$.

And we know that the conjugate zeros of $\Lambda_k(s)$ appear in pairs, and then the quadruplets of non-trivial zeros $(\rho_i, \bar{\rho}_i, 1-\rho_i, 1-\bar{\rho}_i)$ with their multiplicities appear together according to Eq.(41). The condition $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2} < \infty$ and condition $\mathcal{Z}_{\text{real}} \cap \mathcal{Z}_{\text{complex}} = \varnothing$ hold for the same reasons as in CASE 1 in the proof of Theorem 5.

Therefore, by Theorem 4 we know that both the real (if exists) and the complex zeros of $\Lambda_K(s)$ lie on the critical line, i.e., Theorem 6 holds as a specific case of Theorem 4 with $\lambda = K$, $\varepsilon(\lambda) = 1$, $k = 1$. □

In the following contents, we prove the Grand Riemann Hypothesis for modular form L-Functions and automorphic L-functions, respectively. After that we will make a summarization, and conclude that the Grand Riemann Hypothesis holds for all L-functions satisfying some general properties.

3.3. Modular Form L-FUNCTION

Definition: For a modular form $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ of weight k for a congruence subgroup Γ, the associated L-function is defined for $\Re(s) > \frac{k+1}{2}$ by:

$$L(f,s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s} \tag{44}$$

Completed L-function: For a cusp form f of weight k for $\Gamma_0(N)$ (level N is any positive integer) with Nebentypus character χ, the completed modular form L-function is defined as:

$$\Lambda(f,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s) \tag{45}$$

Functional Equation: For a normalized Hecke eigenform f of weight k for $\Gamma_0(N)$ with Nebentypus character χ, the completed modular form L-function satisfies:

$$\Lambda(f,s) = \varepsilon(f)\Lambda(\overline{f}, k-s) \tag{46}$$

where $\varepsilon(f) = \pm 1$ is the epsilon factor, which is the eigenvalue of f under the Atkin-Lehner involution, and $\overline{f}$ is the modular form with Fourier coefficients $\overline{a}_n$.

Hadamard Product: For a cusp form f, the completed L-function $\Lambda(f,s)$ is an entire function of order 1 and has the Hadamard product:

$$\Lambda(f,s) = e^{A(f)+B(f)s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho} \tag{47}$$

where the product is over all zeros ρ of $\Lambda(f,s)$, and $A(f)$ and $B(f)$ are constants depending on f. For more details of the completed modular form L-function, please be referred to Refs.[5,7].

We have the following result about the non-trivial zero distribution of modular form L-Functions.

Theorem 7: The non-trivial zeros of the above-described modular form L-Functions lie on the critical line.

Remark: We only need to prove that all the zeros of $\Lambda(f,s)$ have real part $\frac{k}{2}$, i.e., all the zeros of $\Lambda(f,s)$ lie on the critical line.

Proof. We conduct the proof in two cases.

CASE 1: $f=\overline{f}$ (self-dual)

It suffices to show that the properties of $\Lambda(f,s)$ with $f=\overline{f}$ match the conditions of Theorem 4 with $\lambda=f$. Eq.(47) is equivalent to Eq.(21) by separating all zeros into two sets $\mathcal{Z}_{\text{real}}$ and $\mathcal{Z}_{\text{complex}}$. Actually, to restrict $f=\bar{f}$ is to guarantee that the conjugate zeros of $\Lambda(f,s))$ appear in pairs. Then the quadruplets of non-trivial zeros $(\rho_i,\bar{\rho}_i,k-\rho_i,k-\bar{\rho}_i)$ with their multiplicities appear together according to Eq.(46). The condition $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}<\infty$ and condition $\mathcal{Z}_{\text{real}}\cap\mathcal{Z}_{\text{complex}}=\varnothing$ hold for the same reasons as in CASE 1 in the proof of Theorem 5.

Therefore, by Theorem 4, we know that both the real (if exists) and the complex zeros of $\Lambda(f,s)$ with $f=\overline{f}$ lie on the critical line.

CASE 2: $f\neq\overline{f}$

To deal with this case $f\neq\bar{f}$, we need first to extend Eq.(46) to another form, i.e.,

$$\Lambda(\bar{f},s) = \varepsilon(\bar{f})\Lambda(f,k-s) \tag{48}$$

Combining (48) with (46), we get a new functional equation

$$\Lambda(\bar{f},s)\Lambda(f,s) = \varepsilon(f)\varepsilon(\bar{f})\Lambda(f,k-s)\Lambda(\bar{f},k-s) \tag{49}$$

Obviously, both sides of Eq.(49) are the products of entire functions, thus they are still entire functions. And we know that the conjugate zeros of $\Lambda(\bar{f},s)\Lambda(f,s)$ appear in pairs, and then the quadruplets of non-trivial zeros $(\rho_i,\bar{\rho}_i,k-\rho_i,k-\bar{\rho}_i)$ with their multiplicities appear together according to Eq.(49). Further, based on Eq.(47), we have

$$\Lambda(f,s)\Lambda(\bar{f},s) = e^{A(f)+A(\bar{f})+[B(f)+B(\bar{f})+c]s}\prod_{\rho\in\mathcal{Z}_{\text{real}}}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}\prod_{i=1}^{\infty}\left(1-\frac{s}{\rho_i}\right)\left(1-\frac{s}{\bar{\rho}_i}\right) \tag{50}$$

where $c=\sum_{i=1}^{\infty}\frac{2\alpha_i}{\alpha_i^2+\beta_i^2}$.

The condition $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}<\infty$ and condition $\mathcal{Z}_{\text{real}}\cap\mathcal{Z}_{\text{complex}}=\varnothing$ hold for the same reasons as in CASE 1 in the proof of Theorem 5.

Therefore, by Theorem 4, we know that both the real (if exists) and the complex zeros of $\Lambda(f,s)\Lambda(\bar{f},s)$ (thus of $\Lambda(f,s)$) with $f\neq\overline{f}$ lie on the critical line.

Combining CASE 1 and CASE 2, we conclude that Theorem 7 holds as a specific case of Theorem 4 with $\lambda=f$. □

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3.4. Automorphic L-Function

Definition: For an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ ($\mathbb{A}_\mathbb{Q}$ is the adele ring over the field of rational numbers $\mathbb{Q}$, see Ref.[7] on page 5 for more details), the associated L-function is defined for $\Re(s) > 1$ by:

$$L(\pi, s) = \prod_p L_p(\pi_p, s) \tag{51}$$

where $L_p(\pi_p, s)$ is the local L-factor at the prime p. For unramified π_p with Satake parameters $\{\alpha_{1,p}, \ldots, \alpha_{n,p}\}$,

$$L_p(\pi_p, s) = \prod_{i=1}^{n} \left(1 - \frac{\alpha_{i,p}}{p^s}\right)^{-1} \tag{52}$$

Completed L-function: The completed automorphic L-function is defined as:

$$\Lambda(\pi, s) = Q_\pi^{s/2} \prod_{i=1}^{n} \Gamma_*^{(i)}(s + \mu_{i,\pi}) \cdot L(\pi, s) \tag{53}$$

where Q_π is the conductor of π, $\mu_{i,\pi}$ are complex numbers determined by the i-th local component of π_∞, and

$$\Gamma_*^{(i)}(s) = \begin{cases} \Gamma_\mathbb{R}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}), & \text{for real representations.} \\ \Gamma_\mathbb{C}(s) = (2\pi)^{-s}\Gamma(s), & \text{for complex representations.} \end{cases} \tag{54}$$

Functional Equation: The completed automorphic L-function satisfies:

$$\Lambda(\pi, s) = \varepsilon(\pi)\Lambda(\tilde{\pi}, 1 - s) \tag{55}$$

where $\tilde{\pi}$ is the contragredient representation of π and $\varepsilon(\pi)$ is the epsilon factor, a complex number of absolute value 1.

Hadamard Product: For a cuspidal automorphic representation π, the completed L-function $\Lambda(\pi, s)$ is an entire function of order 1 and has the Hadamard product:

$$\Lambda(\pi, s) = e^{A(\pi)+B(\pi)s} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \tag{56}$$

where the product is over all zeros ρ of $\Lambda(\pi, s)$, and $A(\pi)$ and $B(\pi)$ are constants depending on π.

For more details of the completed automorphic L-function, please be referred to Refs.[5,7].

We have the following result about the non-trivial zero distribution of automorphic L-Functions.

Theorem 8: The non-trivial zeros of the above-described automorphic L-Functions lie on the critical line.

Remark: We only need to prove that all the zeros of $\Lambda(\pi, s)$ have real part $\frac{1}{2}$, i.e., all the zeros of $\Lambda(\pi, s)$ lie on the critical line.

Proof. The proof procedures of Theorem 8 is similar to that of Theorem 7 with $k = 1$ and f replaced by π, $\overline{f}$ replaced by $\tilde{\pi}$. Thus the proof details are omitted for simplicity. □

Actually, from the above proofs of Theorem 5, Theorem 6, and Theorem 7, we can note that each proof does not depend on the specific definition of the L-function $L(\lambda, s)$, but rather relies on the following general properties of the corresponding completed L-function $\Lambda(\lambda, s)$:
P1: Symmetric functional equation between $\Lambda(\lambda, s)$ and $\Lambda(\bar{\lambda}, k - s)$: $\Lambda(\lambda, s) = \varepsilon(\lambda)\Lambda(\bar{\lambda}, k - s)$;
P2: Hadamard product expression of entire function $\Lambda(\lambda, s)$ or $s^{m_0}(k - s)^{m_1}\Lambda(\lambda, s)$, $m_0 \geq 1, m_1 \geq 1$ are the multiplicities (orders) of poles $s = 0, s = k$, respectively;
P3: The zeros of $\Lambda(\lambda, s)$ are precisely the non-trivial zeros of $L(\lambda, s)$;
P4: Zero distribution related items: 1) the concurrence of quadruplets of complex non-trivial zeros

with their multiplicities; 2) the property stated in Lemma 5.5; 3) the disjointness of real and complex non-trivial zero sets.

Therefore we conclude that the Grand Riemann Hypothesis holds for all kinds of L-functions satisfying properties **P1**, **P2**, **P3**, and **P4**, i.e., If only the completed L-function $\Lambda(\lambda, s)$ satisfies the requirements of **P1**, **P2**, **P3**, and **P4**, the non-trivial zeros of the corresponding L-Function $L(\lambda, s)$ lie on the critical line.

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