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*Article*

# Analytic Solutions of the KPZ Equation Containing the Kummer's Functions

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**Abstract:** It is well-known that with the Hopf-Cole (HC) transformation the Kardar-Parisi-Zhang (KPZ) non-linear surface growth equation can be transformed to the regular diffusion equation. The reverse operation is also possible. Along this logic we can transform our new-type of self-similar solutions of the regular diffusion equations - which contain the Kummer's functions - to solutions which fulfill the KPZ equation. These solutions give us a glimpse into the true origin of the singularities of the KPZ equation. Due to the free self-similar exponent we discuss if our solutions could explain surface erosion additionally to surface growth. We investigate the KPZ equation without any noise term. At the second part of the manuscript we investigate an analytic source term in the diffusion equation and its transformed form in the KPZ equation.

**Keywords:** KPZ equation; self-similar solutions; diffusion equation; Hopf-Cole transformation

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## 1. Introduction

It is evident that diffusion processes have an extreme importance both in engineering and science. The same is true for surface growing processes. The corresponding literature of both phenomena is enormous, hence we just mention the most important references for diffusion [1–6] and for surface growth [7–10]. There are existing monographs where both physical processes (diffusion and surface growth) are discussed together [11–14]. Instability effects handle surface growth phenomena and diffusion processes on the same level as it was shown by different authors [15,16].

In the following study we will investigate the mathematical properties of the KPZ equation [17] which is one of the simplest non-linear model, based on a non-linear partial differential equation (PDE), and among others it is capable to describe surface growth phenomena. This equation has wide application and can be connected to Burgers turbulence as well [18].

It is also known that with the HC transformation [19,20] the non-linear gradient term of the KPZ equation can be eliminated resulting the regular diffusion equation. There exist some theoretical mathematical studies which exhaustively study how the KPZ equation can be solved beyond the HC transformation [21] or with perturbation theory [22] using the standard field theoretical diagram technique.

Later, Hwa and Frey [23,24] investigated the KPZ model with the help of the renormalization group-theory and the self-coupling method which is a precise and sophisticated method using Green's functions. Various dynamical scaling forms of $C(x,t) = x^{-2\varphi}C(bx, b^z t)$ were considered for the correlation function (where $\varphi, b$ and $z$ are real constants). Kriecherbauer and Krug published a review paper [25], where the KPZ equation was derived from hydrodynamical equations using a general current density relation. Lässig applied the field theoretical approach to derive and investigate the KPZ equation [26].

The Family–Vicsek scaling relation [27] of the roughness gives us the universality class of the KPZ equation in 1+1 dimension with the roughness exponent $\hat{\alpha} = 1/2$ growth exponent $\hat{\beta} = 1/3$ and with the dynamical exponent $z = 3/2$. KPZ universality class is an even-degree polynomial. A family of processes that are conjectured to be universal limits in the (1+1) KPZ universality class, and govern the long time fluctuations are the Airy processes and the KPZ fixed point. (We added an extra 'hat' ˆ on the exponents, not to mix with our self-similar exponents later.)

Gladkov [28] investigated the KPZ equation - with general exponents of the gradient term with the self-similar blow-up solution. In a different study Gladkov *et al.* [29] analyzed the correctness of the KPZ equation with the same Ansatz with unbounded data. As generalization the fractional KPZ equation was also topic of studies [30].

Kersner and Vicsek studied the traveling wave dynamics of the singular interface equation [31], which is closely related to the KPZ equation. Kelling and co-worker intensively investigated the two dimensional KPZ equation with the help of extended dynamical simulations to study the physical aging properties of different systems like polymers or glasses [32].

Additional more recent research results about roughness height and universality can be found in [33].

Numerous models exist, which may lead to similar equations as the KPZ model, i.e., the interface growth of bacterial colonies [34]. More general interface growing models were developed which is so-called Kuramoto-Sivashinsky (KS) [35] equation showing similarities to the KPZ model with and extra fourth order gradient term $-\nabla^4 h(x,t)$.

The KPZ equation was investigated recently in periodic setting, which describes the random growth of an interface in a cylindrical geometry [36].

In our former studies we applied the self-similar Ansatz derived and analyzed new type of solutions which are much beyond the well-known Gaussian and error type of functions [37–39]. In recent years the field of numerical analysis of diffusion equations made a remarkable step [40,41]. In two other studies we investigated the one dimensional KPZ equation with six different additional source terms (like analytic noises) with the self-similar Ansatz [42] and with the traveling wave ansatz [43] and presented highly non-trivial non-continuous analytic solutions containing a large variety of special functions (like, Kummer's, Airy, Heun, modified Bessel or Mathieu functions).

In our third study we developed a model in which the Gaussian solution of the regular diffusion acts as a source term for surface growth phenomena [44].

Beyond KPZ and KS continuous models based on partial differential equations (PDEs), there are numerous number of purely numerical methods available to investigate diverse surface growth phenomena. As a view we mention the Lattice-Boltzmann simulations [45], the kinetic Monte Carlo [46] and the etching model [47].

In the following we show how our solutions of the diffusion equation (containing the Kummer's functions) can be transformed to fulfill the KPZ equation.

All our presented analytic formulas are in complete agreement with the statement A.B. Muravnik [48] giving a rigorous mathematical proof on absence of global positive solutions of elliptic inequalities with KPZ - nonlinearities. All the domains of our results are open sets but never the whole real axis.

## 2. Theory and Results

To write a consistent and self-supporting study we first outline our self-similar solutions. Later we present the transformation which turn the solutions of the KPZ equation to the solutions of the diffusion equation, and inversly. Lastly, we present and analyze the derived solutions in details.

### 2.1. Self-Similar Solutions of the Diffusion Equation

First we give an overview how new-type of self-similar solutions can be derived to the regular diffusion equation:

$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial x^2}, \tag{1}$$

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where $C(x,t)$ is the distribution of the particle concentration in space and time, $D$ is the diffusion coefficient and from physical reasons it is a positive real constant. The function $C(x,t)$ fulfills the necessary smoothness conditions with existing continuous first and second derivatives. Numerous physics textbooks gives us the derivation how the fundamental (the Gaussian) solutions can be obtained e.g. [1]. In the well-known work of Bluman and Cole in 1969 [49] numerous analytic solutions were given for the diffusion equation, and they arrived to a certain level. Regarding microscopic aspects of diffusion an important overview have been realized in [50], and there are recent three-dimensional mathematical modeling and simulation of the impurity diffusion process exit under the given statistics of systems of internal point mass sources [51].

In our analysis we apply the self-similar Ansatz in the form of $C(x,t) = t^{-\alpha} g\left(\frac{x}{t^\beta}\right) = t^{-\alpha} g(\eta)$, where $\alpha$ and $\beta$ are the self-similar exponents being real numbers describing the decay and the spreading of the solution [52] and $g(\eta)$ is the shape function depending on the reduced variable $\eta = \frac{x}{t^\beta}$. After some simple algebraic steps the self-similar Ansatz leads us to the relations of $\alpha =$ arbitrary real number, $\beta = 1/2$, and to the clear-cut time-independent ODE of

$$-\alpha g - \frac{1}{2}\eta g' = Dg''. \tag{2}$$

With the choice of $\alpha = 1/2$ and setting the first integration constant to zero $(c_1 = 0)$ we get back the well-known Gaussian solution.

This is the so-called fundamental solution and sometimes referred to as *source type* solution – by mathematicians – because for $t \to 0$ the $C(x,0) \to \delta(x)$. As far as we know this is the simplest and shortest derivation to obtain the fundamental solution from the original PDE of Equation (1). Therefore this Ansatz is original among others and can help to find physically relevant disperse solutions to other physical systems like the heated boundary layer flow [53]. Using the formula manipulating software package Maple 12 for general real $\alpha$, the solutions for the shape function read as:

$$g(\eta) = e^{-\frac{\eta^2}{4D}} \cdot \eta \cdot \left[c_1 M\left(1-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right) + c_2 U\left(1-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right)\right], \tag{3}$$

where $M(\cdot,\cdot,\cdot)$ and $U(\cdot,\cdot,\cdot)$ are the Kummer's M and Kummer's U functions. For more details consult the NIST Handbook [54]. For completeness we give the final solution for the concentration distribution as well:

$$C(x,t) = \frac{e^{-\frac{x^2}{4Dt}}}{t^\alpha} \left(\frac{x}{t^{\frac{1}{2}}}\right) \left[c_1 M\left(1-\alpha, \frac{3}{2}, \frac{x^2}{4Dt}\right) + c_2 U\left(1-\alpha, \frac{3}{2}, \frac{x^2}{4Dt}\right)\right]. \tag{4}$$

At this point for a better understanding we have to mention that the solutions Equation (4) according to the value of the $\alpha$ parameter, can be divided into four separate groups and each is ordered with essentially different properties:

- $1-\alpha < 0$ the $M$ function is a finite polynomial if $\alpha$ is integer and larger than two,
- $1-\alpha = 0$ the $M$ function is constant, the decay of solutions are determined by the Gaussian term,
- $0 < 1-\alpha \leq 1$ the solutions have a local maxima and a decay to zero at large arguments,
- $1 < 1-\alpha$ the solutions have oscillations proportional to the value of $(1-\alpha)$ and have quicker and quicker decays to zero at larger $(1-\alpha)$ values.

It is clear to see that most of the resulting functions have odd symmetry, however it can be shown that for some $\alpha$ parameters the Kummer's U functions can have even symmetry [38] as well, which has far reaching consequences. An exhaustive analysis of Equation (4) and similar systems was done in previous studies [38,39,55]. There are recent results in case of more driving forces are present or reactions occur [56,57].

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### 2.2. The Hopf-Cole Transformation

The HC transformation [19,20] is defined es follows:

$$C(x,t) = e^{\frac{\lambda}{\mu}h(x,t)}, \tag{5}$$

where $C(x,t)$ is the concentration of the regular diffusion equation of (1) and $h(x,t)$ is the height profile of the local growth in KPZ equation:

$$\frac{\partial h}{\partial t} = +\mu\frac{\partial^2 h}{\partial x^2} + \lambda\left(\frac{\partial h}{\partial x}\right)^2 + \epsilon(x,t). \tag{6}$$

The first term on the right hand side describes relaxation of the interface by a surface tension, which prefers a smooth surface. The second term is the lowest-order nonlinear term that can appear in the surface growth equation justified with the Eden model and originates from the tendency of the surface to locally grow normal to itself and has a non-equilibrium origin.

(Interestingly, we note that the G-equation which describes the front of a flame surface containing the $\sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2}$ type of non-linearity from geometrical reasoning can be analytically solved with the self-similar Ansatz [58]. )

The last third term is a Langevin noise [59,60] to mimic the stochastic nature of any growth process and usually corresponds to a Gaussian distribution.

It is important to emphasize that in the original KPZ equation the non-linear term is "just a gradient squared" and no additional absolute value is considered. The absence of an absolute value is significant because the KPZ equation is derived based on symmetry arguments and physical principles, such as invariance under spatial rotations and translations. Introducing an absolute value would break these symmetries and alter the universality class of the equation. An early summary work by Halpin-Healy and Zhang from 1995 about kinetic roughening phenomena a stochastic Burgers equation is mentioned with the absolute value of the gradient squared term as a kind of generalization of the KPZ equation [61]. We are going to address the question of the absolute value of the gradient squared term at the end of our study.

The literature and the applications of the KPZ equation due its special universality class [62], fix point [63] and scaling [64] have become widely diversified by 2025 which cannot be cited with completeness. As an additional example we just mention that it can be applied to describe late-time correlators and autocorrelators of certain interacting many-body systems like quantum Heisenberg magnets [64].

We should define a self-similar Ansatz as well, like $h(x,t) = t^{-\alpha}f(\eta)$. The $\mu$ and $\lambda$ are two free physical parameters. We neglect the noise term of $\epsilon(x,t)$ in our forthcoming analysis. It will be important in the following that the HC transformation Equation (5) can be inverted and has the form of:

$$h(x,t) = \frac{\mu}{\lambda}\cdot \ln(C[x,t]) = \frac{\mu}{\lambda}\cdot \ln\left(\frac{f[x,t]}{t^{\alpha}}\right). \tag{7}$$

Evidently this is true at the level of the shape function with the form of $f(\eta) = \frac{\mu}{\lambda}ln[g(\eta)]$. Note, that the original HC transformation in continuous on the whole axis, the exponential function is defined on the whole real axis, the logarithmic function is however defined only for positive real numbers. This will be the crucial point in terms of the continuity of the solutions.

(It is worth to mention here that a complex generalization of the Hopf-Cole transformation exists in the form of $\Psi(x,t) = \rho(x,t)e^{iS(x,t)}$) which transforms the Schrödinger equation to Euler type of hydrodynamical equation called the Madelung equation [65]. The real part $\rho(x,t)$ is the density of the Madelung fluid, and the imaginary part $S(x,t)$ is proportional to the velocity field. With the self-similar Ansatz analytic solutions can be derived for the Madelung equation which can be expressed with the help of the Bessel functions of the first and second kind with quadratic arguments. [66].)

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The self-similar aspects regarding the non-linear Schrödinger equation have been also discussed by us in Ref. [67].

For completeness we give here the proof of the transformation. We have to directly calculate the first temporal and the first and the second spatial derivatives of $h(x,t) = \frac{\mu}{\lambda} ln(C[x,t])$, and have to substitute into Equation (15). This gives:

$$\frac{\mu}{\lambda}\frac{1}{C}\frac{\partial C}{\partial t} = \mu\frac{\mu}{\lambda}\left[-\frac{1}{C^2}\left(\frac{\partial C}{\partial x}\right)^2 + \frac{1}{C}\frac{\partial^2 C}{\partial x^2}\right] + \lambda\frac{\mu^2}{\lambda^2}\frac{1}{C^2}\left(\frac{\partial C}{\partial x}\right)^2, \tag{8}$$

after simplifications are completed, and considered that $\frac{\mu}{\lambda} \neq 0, \quad C \neq 0$ and dictate that $\mu = D$ we get

$$\frac{\partial C(x,t)}{\partial t} = \mu\frac{\partial^2 C(x,t)}{\partial x^2}. \tag{9}$$

*2.3. The New Solutions of the KPZ via the Hopf-Cole Transformation*

After this trivial mathematical proof we just substitute Equation (4) into Equation (7) and have the final form of the solutions of the KPZ equation:

$$h(x,t) = \frac{\mu}{\lambda} ln[C(x,t)] = \frac{\mu}{\lambda}\ln\left\{\frac{1}{t^\alpha}\left(\frac{x}{t^{\frac{1}{2}}}\right)e^{-\frac{x^2}{4Dt}}\left(c_1 M\left[1-\alpha,\frac{3}{2},\frac{x^2}{4Dt}\right] + c_2 U\left[1-\alpha,\frac{3}{2},\frac{x^2}{4Dt}\right]\right)\right\}. \tag{10}$$

For the shape functions the equation looks more transparent

$$f(\eta) = \frac{\mu}{\lambda} ln[g(\eta)] = \frac{\mu}{\lambda}\ln\left\{\eta e^{-\frac{\eta^2}{4D}}\left(c_1 M\left[1-\alpha,\frac{3}{2},\frac{\eta^2}{4D}\right] + c_2 U\left[1-\alpha,\frac{3}{2},\frac{\eta^2}{4D}\right]\right)\right\}. \tag{11}$$

In most of the cases, we are interested in the solutions which decay in time and which are usually described with $\alpha > 0$. Self-similar Ansatz is a reliable tool to find disperse solutions which decay to zero. The KPZ equation describes the surface growth as a dynamical process where the solutions should increase in time, therefore now $\alpha < 0$ will be relevant. The $\frac{\mu}{\lambda}$ scaling factor was set to unity from now on.

We know from our decade long experience that for positive $\alpha$ self-similar exponents the solutions have decaying properties. In most cases these are physically relevant solutions for parabolic and dissipative systems. We are going to discuss if such solutions could describe surface degradation which is another relevant phenomena. Surface erosion is a distinct natural effect with its physical models and scientific literature. We do not go into detail, (that is out of our present scope) just mention some references [46,68].

The KPZ equation is however describes surface growth therefore the extending and divergent solutions $\alpha < 0$ are interesting physically. We will analyze our derived solutions within this concept.

The first figure presents shape functions with the Kummer's M functions for seven different $\alpha$s. Note, that all curves with $\alpha > 0$ decay for large arguments $\eta$. Curves with negative exponents are however divergent. Figure 2 shows the Kummer's U type of functions for the same seven different $\alpha$s. Here all solutions show decaying properties for increasing arguments. Due to symmetry reasons only positive even $\alpha$ numbers define solutions for negative spatial coordinates.

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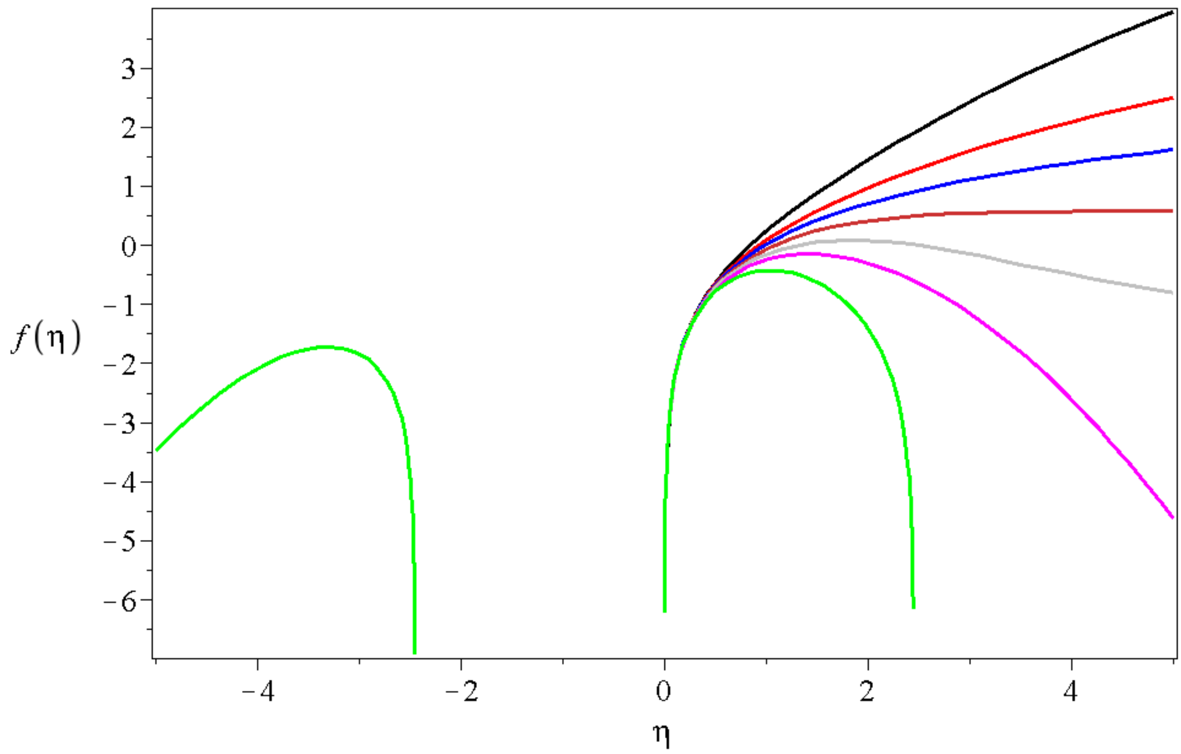


**Figure 1.** The Kummer's M shape functions of Equation (11) for $c_1 = 1, c_2 = 0$ and$D = 1$. The black, red, blue, orange, gray, magenta and green lines are for $\{-2, -1, -1/2, 0, 1/2, 1, 2\}$ $\alpha$ values, respectively.

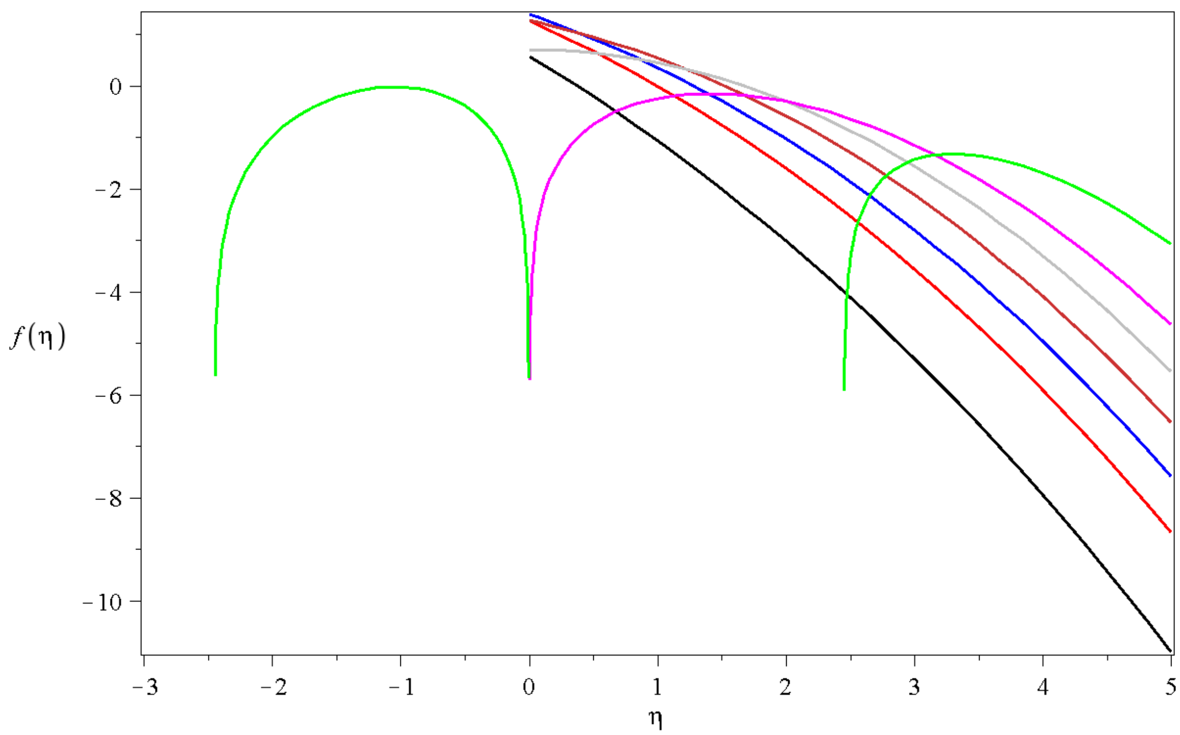


**Figure 2.** The Kummer's U shape functions of Equation (11) for $c_1 = 0, c_2 = 1$ and$D = 1$. The black, red, blue, orange, gray, magenta and green lines are for $\{-2, -1, -1/2, 0, 1/2, 1, 2\}$ $\alpha$ values, respectively.

Figure 3 a shows the $h(x,t)$ surface height for the Kummer's U functions for various $\alpha$s. It is clear to see, that for negative $\alpha$s the solutions are growing for larger and larger times. An island grows up around the origin. Due to the $ln$ function the difference between $\alpha = -5$ and $\alpha = -1$ is not so drastic but visible. For $\alpha = 0$ the solutions only grow up to an asymptotic zero value. For positive

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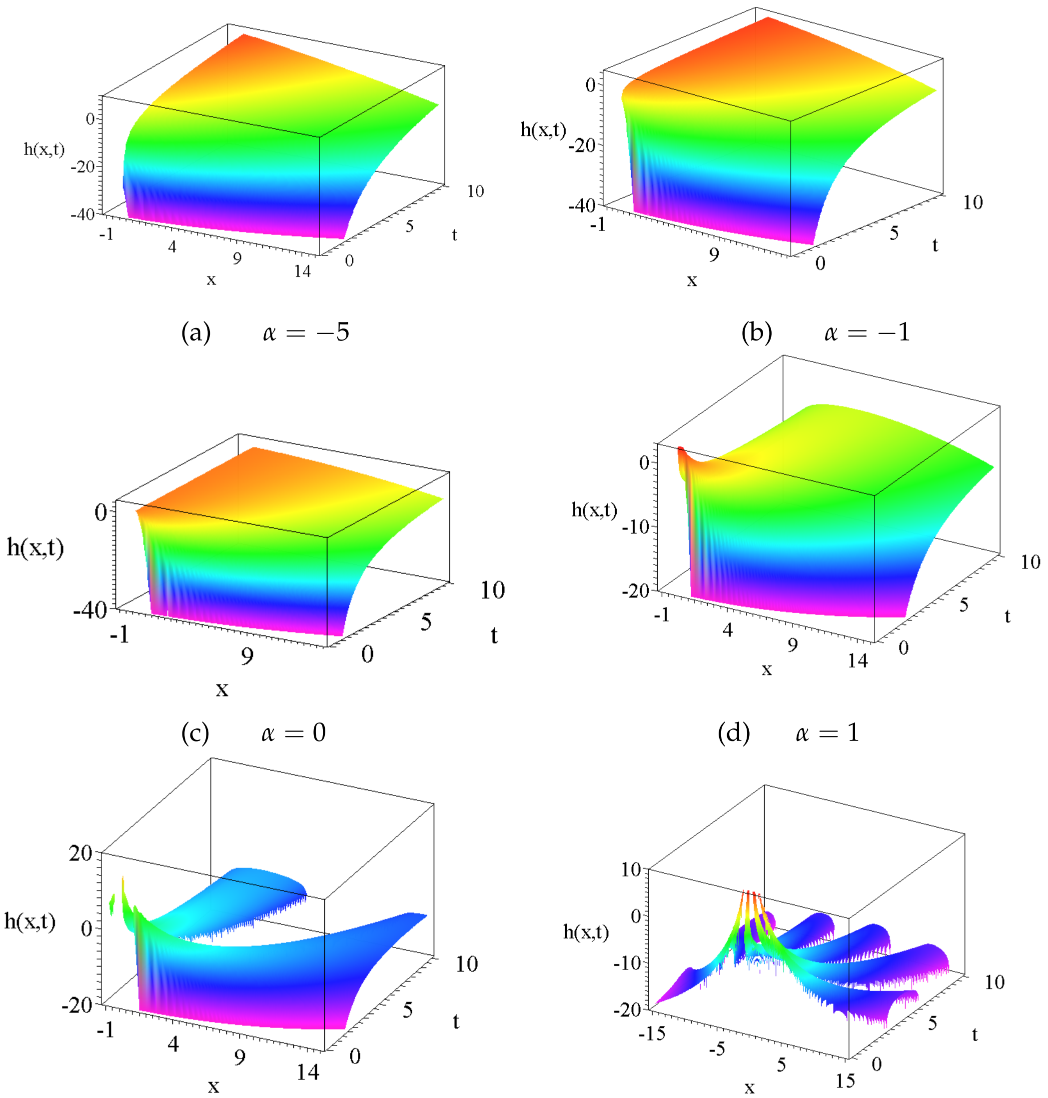
$\alpha$s the solutions become very interesting. Again due to the $ln$ function, only the positive parts of the oscillating Kummer's U function remain solutions. Therefore the solutions are defined only on finite supports. At short times, positive " tinny islands" exist as solutions which sink down to negative heights at larger times. These are not physical solutions in the sense of surface growth physics.




(a) $\alpha = -5$ (b) $\alpha = -1$

(c) $\alpha = 0$ (d) $\alpha = 1$

(e) $\alpha = 3$ (f) $\alpha = 5.5$

**Figure 3.** The surface height $h(x,t)$ for the Kummer's U function for $D = 1$ and for different $\alpha$s, respectively.

Figure 4 a shows the $h(x,t)$ surface height for the Kummer's M functions for distinct $\alpha$s. The trend of the solutions is very similar. Negative $\alpha$s produce surfaces with growing height, now at large spatial coordinates not in the origin. The $\alpha = 0$ gives a constant solution and positive $\alpha$s produce "tinny islands" which disappear in time.

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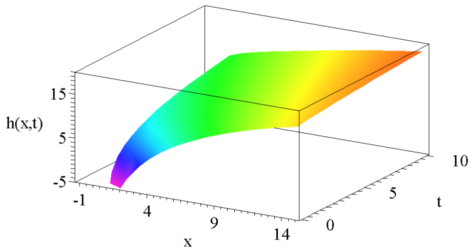
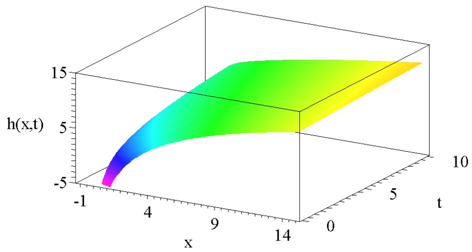
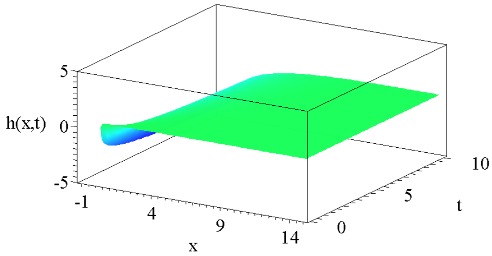


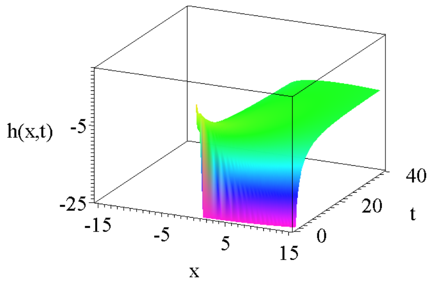

(a) $\alpha = -5$



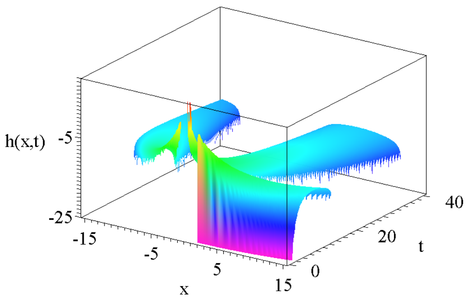

(b) $\alpha = -1$



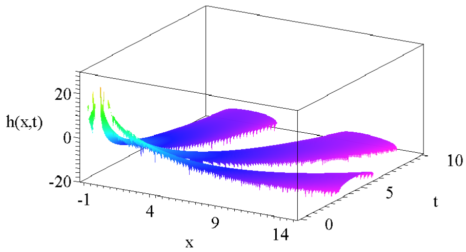

(c) $\alpha = 0$




(d) $\alpha = 1$




(e) $\alpha = 3$




(f) $\alpha = 5.5$

**Figure 4.** The surface height $h(x,t)$ for the Kummer's M function for $D = 1$ and for different $\alpha$s, respectively.

### *2.4. The Question of Additional Source Terms in the Diffusion Equation*

Our personal experience showed that only for the linear source term can we derive analytic self-similar solutions. Therefore the starting PDE is:

$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial x^2} + \frac{aC}{t}, \tag{12}$$

the same self-similar Ansatz reduces to the ODE of

$$-\alpha g - \frac{1}{2}\eta g' = Dg'' + ag. \tag{13}$$

The derived exponents and solutions remain the same just the first parameter of both Kummer's function will be down shifted with the parameter $a$ which is the strength of the source:

$$g(\eta) = e^{-\frac{\eta^2}{4D}} \cdot \eta \cdot \left[c_1 M\left(1-\alpha-a, \frac{3}{2}, \frac{\eta^2}{4D}\right) + c_2 U\left(1-\alpha-a, \frac{3}{2}, \frac{\eta^2}{4D}\right)\right]. \tag{14}$$

(The source strength parameter could have negative real values as well.) Taking into account the transformation formula of Equation (11) after some trivial algebra we arrive to the modified KPZ equation in the next form of:

$$\frac{\partial h}{\partial t} = +\mu\frac{\partial^2 h}{\partial x^2} + \lambda\left(\frac{\partial h}{\partial x}\right)^2 - \frac{\mu a}{\lambda t}. \tag{15}$$

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Note, that the original linear and time dependent diffusion source term is transformed into a constant source term. However, the inverse time dependence remained. At this point we have to add that using the reverse transformation we can produce possible source terms with analytic solutions of the regular diffusion if such terms are known for the KPZ equation. In two of our former studies we investigated the one dimensional KPZ equation with six different additional source terms (like analytic noises) with the self-similar Ansatz [42] and with the traveling wave ansatz [43] and presented highly non-trivial non-continuous analytic solutions containing a large variety of special functions (like, Kummer's, Airy, Heun, modified Bessel or Mathieu functions). The back transformation is possible and the in-depth analysis could be the topic of our future study.

### 2.5. The Question of the Absolute Value

The above givens functions are the solutions of the KPZ equation. Due to the logarithm for negative $\alpha$ values, the solutions are only defined on some finite number of compact support intervals. In real life surfaces are smooth objects. To fix this problem we may consider an extra absolute value function inside the logarithm in the form of:

$$h(x,t) = \frac{\mu}{\lambda} \cdot \ln(\mathrm{abs}[C\{x,t\}]), \tag{16}$$

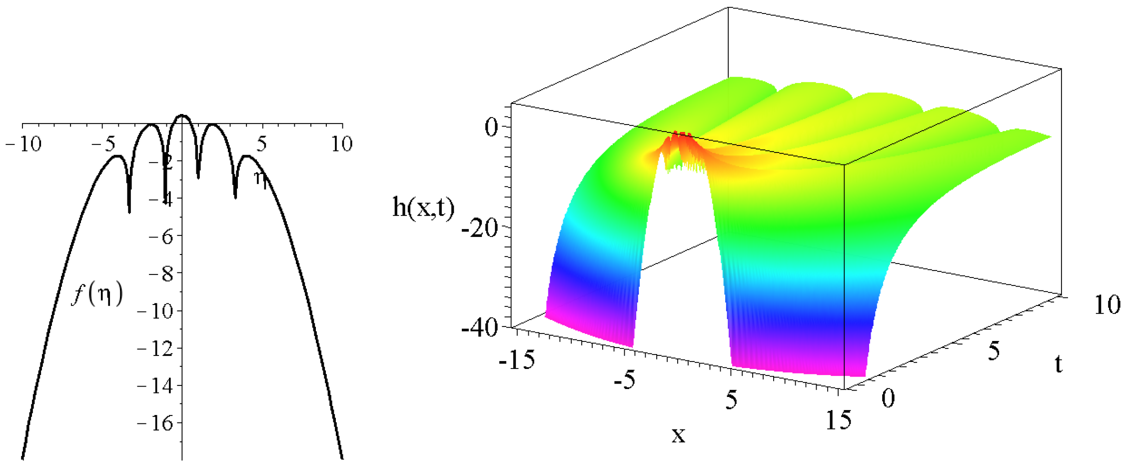
the shape function now reads as:

$$f(\eta) = \frac{\mu}{\lambda} \cdot \ln(\mathrm{abs}[g\{\eta\}]). \tag{17}$$

We do not want to reproduce all the problematic sub cases of Figures 3 and 4 so we just show one example. Considering $c_1 = 0$ and investigating only the Kummer's U function we take $\alpha = 5/2$ which reduces it to a finite fourth order polynomial. The surface is now:

$$h(x,t) = \ln\left| \frac{1}{t^{\frac{5}{2}}} e^{-\frac{x^2}{4Dt}} \left(1 - \frac{x^2}{t} + \frac{x^4}{12t^2}\right) \right|. \tag{18}$$

On Figure 5 one can see the strongly irregular form of the shape function $f(\eta)$ and the surface, $h(x,t)$ being the basic solution of the problem of KPZ equation, without noise. The absolute value function helps to define all kind of solutions for $x < 0$ values as well.




((a) (b)

**Figure 5.** (a) The shape function of the solution Equation (18), (b) the $h(x,t)$ surface. One can clearly see the irregularities of the surface of the basic equation.

As it was mentioned in [37], in case of the diffusion equation, the linear combination of solutions belonging to different $\alpha$ values is also solution of the equation. As a consequence the derived solutions lead to combined possibilities, for instance

$$h(x,t) = \ln\left|\frac{1}{t^{\frac{1}{2}}}e^{-\frac{x^2}{4Dt}} + \frac{1}{t^{\frac{5}{2}}}e^{-\frac{x^2}{4Dt}}\left(1 - \frac{x^2}{t} + \frac{x^4}{12t^2}\right)\right|. \quad (19)$$

### 2.6. The Question of Surface Erosion

As a final idea we may try to construct solutions which describe surface degradation or erosion. We investigated some solutions in the form of

$$h(x,t) = \text{Const} - \frac{\mu}{\lambda} \cdot \ln(\text{abs}[C\{x,t\}]), \quad (20)$$

where $\alpha < 0$ and Const $> 0$. We were looking for solutions which are finite on large part of the spatial coordinate 'x' at small times, and decay as time grows. Unfortunately, we cannot found such kind of solutions for any $\alpha$ neither for Kummer's U nor for Kummer's M functions. We found something which is more interesting. The filing up of a finite depth an finite width hole can be modeled in time. Figure 6 shows such a recharge process for $\alpha = 3$ with Const $= 2$ for the Kummer's M function.

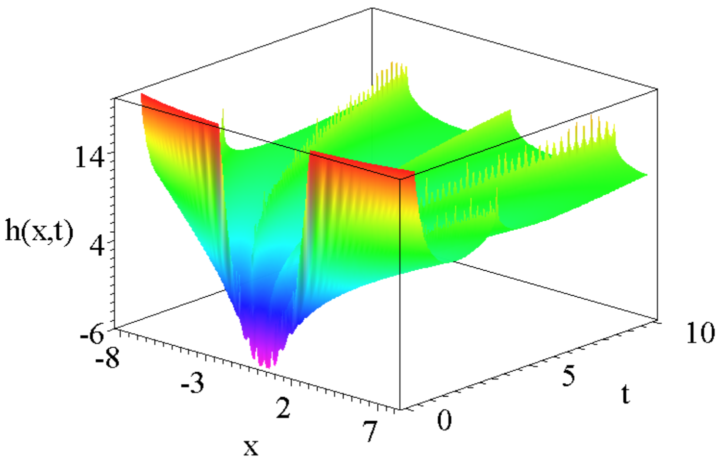



**Figure 6.** Filling up of a finite depth and finite deep hole in time, modeled by Equation (20). Calculated for the Kummer's U functions with the self-similar exponents of $\alpha = 3$.

## 3. Summary and Outlook

In our former studies we derived new type of analytic self-similar solutions for the regular diffusion equations. These are a kind of upper harmonics of the usual Gaussian solution which are defined on the infinite horizon. The solutions contain the regular and irregular Kummer's functions with quadratic argument. With the help of the Hopf-Cole transformation we could easily transform these new type of solutions for the KPZ equation. Due to the *ln* function in the reverse HC transformation we get insight into solution function which have compact support.

In a former publication about regular diffusion equations [38] we found particular solutions with compact support for the $C(x,t) = t^{-\alpha} f(\eta)^2$ Ansatz. To investigate the effect of this trial-function could be the topic of a future study.

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