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Control of Time-Delay Systems Based on Symmetric-Injection Virtual Reference Trajectory

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Abstract

An approach for the control of linear control systems with a single time-delay is proposed. The main contribution is the inclusion of a symmetric-injection virtual reference trajectory into the controller to render stability robustness of single-delay linear control systems. The dynamics of the virtual trajectory is included into the closed-loop dynamics allowing theoretical computation of the critical time-delay before losing stability. Moreover, an energy-based symmetry interpretation of the proposed approach is drawn. Numerical simulations considering stable and unstable linear systems are shown, and experiments to control a DC-motor with time-delay measurements validate our proposal.

Keywords: time-delay; control, stability; virtual reference trajectory; symmetry

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1. Introduction

Control systems in daily activities are ubiquitous, [1]. Those systems are affected by several endogenous and exogenous uncertainties and disturbances. In more complex situations, there are also time delays, either in the control input, in the system state, or in the output measurement of the system, [2]. The combined effect of disturbances and time delay is, most of the time, to the detriment of the controller performance. The control of systems with time-delay is challenging, given the different kind of delays arising in practical and theoretical applications, [3–6]. Several approaches to solve this control problem are found in the literature, to mention the most relevant those based on: predictors [7–9], observers [10–12], passivity[13,14], backstepping [15,16], networked control [17,18], and artificial intelligence [19,20]. The study of delay systems extending the Lyapunov approach was first successfully implemented by Krasovskii and contemporarily, Razumikhin proposed an alterntive approach to study those systems [3]; while more recent proposals in this context have been developed, [21].

Undoubtedly, the nonlinear time-delay control problem is highly important, [22]. In some cases, the nonlinear system has a nominal linear part, [23]. In this context, research of linear time-delay systems has led to fruitful developments [24]. As for delay-free control systems, when time-delays are considered, major issues related to the control of time-delay systems are stability and stabilization, [25]. It is well-known that stable linear systems have some degree of robustness to time-delays; therefore, delay dependent conditions have been studied and several results are found in the literature, [26–28]. Nevertheless, the design of stabilizing and robust controllers has been less explored and recently studied in the literature, [29]. In [30], authors propose static-state feedback as a solution to the problem of non-interactive control, for continuous-time, linear time-delay systems. In [31], the normal form of linear systems with non-commensurate state, input and output delays is investigated, contributing to exploring some properties of the system, including invariant zeros, spectral controllability and observability. The design of stabilizing controllers using LMIs has been explored in [32]. The design of a static output feedback stabilization of linear systems with multiple delays using convex

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optimization is presented in [33]. The symmetry properties of time-delay control systems has been much less explored in the literature. Schneider explored the stabilization of Stuart–Landau oscillators using symmetry-breaking equivariant Hopf bifurcation, [34]. The extension of the famous Noether's symmetry Theorem for optimal control problems with delay is presented in [35]. More recently, the stability and stabilization of conformable time-delay systems is approached in [36].

In this work, we propose to use a virtual reference for the control of single time-delay control systems, as a similar approach to that used in force control of robotic systems which is known as impedance/admittance control, [37].In this force control approach, the cartesian robot desired trajectory is on-line modified to render the desired cartesian force. The main idea is to compute an additional virtual reference trajectory which is then added to the desired trajectory in order to render the desired force tracking, [38]. To better describe our proposal, first, we assume the linear system is either stable or stabilized by a feedback control law, [39]; then a state feedback controller with the virtual reference trajectory will ensure time-delay robustness by properly tuning a set of parameters. The on-line computed virtual reference trajectory is generated by an adjoint linear dynamical system perturbed by the time-delay tracking error. The main contribution of this work is the introduction of the mentioned virtual reference trajectory into the state feedback controller, yielding a standard linear single-delay system with closed loop stability that can be analyzed using state of the art stability results. Moreover, an energy-based symmetry interpretation of our approach is provided.

The rest of the paper is organized as follows. Section 2, describe the mathematical models considered, presenting the motivation of our proposal using a second order linear system. In Section 3, the generalization of the proposed approach for linear time-delay control systems, based on the on-line computed reference trajectory is described in detail, while the energy-based symmetry interpretation is provided. Section 4, presents the stability analysis and Section 5, depicts numerical and experimental results to validate our proposal. Finally, conclusions and some research directions are drawn in Section 6.

2. Introductory Mathematical Models

In this section, the motivation of the proposed approach is described using a second order dynamical system as an introductory example. Then, the proposed virtual reference trajectory dynamics is presented to describe the general case.

2.1. Motivation

Consider the linear system:

$$\ddot{x} + a_1\dot{x} + a_0 x = b_0 u_\tau, \tag{1}$$

where $x \in \mathbb{R}$ represents a state position of the system with its corresponding time derivatives; a_i for $i = 1, 2$ and b_0 stand for scalar constant parameters and u_τ the time-delay controller defined as:

$$u_\tau = u(t-\tau) = \frac{1}{b_0}(\ddot{x}_d + a_1\dot{x}_d + a_0 x_d - K_0(x_\tau - x_{d_\tau} - x_r) - K_1(\dot{x}_\tau - \dot{x}_{d_\tau} - \dot{x}_r)), \tag{2}$$

where $x_\tau = x(t-\tau)$ is the delayed output, $x_{d_\tau}(t)$ is the delayed desired trajectory and $x_r(t)$ a reference trajectory to be on-line computed in order to compensate for the delay-time errors. Notice the desired trajectory is always available, so it can be used for direct feedforward or in the delay-time error computation. Considering (1) and (2), the closed-loop equation is:

$$(\ddot{x} - \ddot{x}_d) + a_1(\dot{x} - \dot{x}_d) + a_0(x - x_d) = -K_0(x_\tau - x_d - x_r) - K_1(\dot{x}_\tau - \dot{x}_d - \dot{x}_r). \tag{3}$$

Define the tracking error $e = x - x_d$ and the delayed tracking error $e_\tau = x_\tau - x_{d_\tau}$. Then, equation (3) can be rewritten as:

$$\ddot{e} + a_1\dot{e} + a_0 e = -K_0(e_\tau - x_r) - K_1(\dot{e}_\tau - \dot{x}_r). \tag{4}$$

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From equation (4), to render asymptotic closed-loop stability of the tracking error e, a natural selection for the reference dynamics is chosen as:

$$K_1\dot{x}_r = -K_0x_r + K_0e_\tau + K_1\dot{e}_\tau. \tag{5}$$

If $K_0 > 0$ and $K_1 > 0$ the reference trajectory represents a first order linear stable system for x_r, perturbed by a term dependent of the delayed tracking error and its time derivative. In this ideal case, as it will be shown latter, the proposed controller yields stability independent of delay, [40].

2.2. Reference Trajectory Modification

Let $z(t) = [e, \dot{e}_\tau, x_r]^T$ and $z(t-\tau) = [e_\tau, \dot{e}_\tau, x_r]^T$. Consider the reference trajectory of equation (5) modified with a tunable gain $\delta_0 > 0$ as:

$$K_1\dot{x}_r = -\delta_0K_0x_r + K_0e_\tau + K_1\dot{e}_\tau \tag{6}$$

On the one hand, the closed-loop considering equations (4) and (6) is:

$$\ddot{e} + a_1\dot{e} + a_0e = -K_0e_\tau + K_0x_r - K_1\dot{e}_\tau - \delta_0K_0x_r + K_0e_\tau + K_1\dot{e}_\tau \tag{7}$$

The perfect cancellation of the delayed terms is in this case achieved to get:

$$\ddot{e} + a_1\dot{e} + a_0e = -K_0(1-\delta_0)x_r \tag{8}$$

Clearly, when $\delta_0 = 1$ the delay does not affect the system asymptotic convergence to zero of the tracking error. On the other hand, equations (4) and equation (6) can be written to represent a neutral time-delay difference equation in standard form:

$$\mathbf{E}\dot{z}(t) = \bar{\mathbf{A}}_0z(t) + \bar{\mathbf{A}}_1z(t-\tau) \tag{9}$$

where:

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -K_1 \\ 0 & 0 & K_1 \end{bmatrix}, \quad \bar{\mathbf{A}}_0 = \begin{bmatrix} 0 & 1 & 0 \\ -a_0 & -a_1 & K_0 \\ 0 & 0 & -\delta_0K_0 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{A}}_1 = \begin{bmatrix} 0 & 0 & 0 \\ -K_0 & -K_1 & 0 \\ K_0 & K_1 & 0 \end{bmatrix}$$

Given that $K_1 > 0$ then clearly the matrix $\mathbf{E}$ is invertible and (9) is written as a single-delay equation:

$$\dot{z}(t) = \mathbf{A}_0z(t) + \mathbf{A}_1z(t-\tau) \tag{10}$$

where:

$$\mathbf{A}_0 = \mathbf{E}^{-1}\bar{\mathbf{A}}_0 = \begin{bmatrix} 0 & 1 & 0 \\ -a_0 & -a_1 & K_0(1-\delta_0) \\ 0 & 0 & -\delta_0\frac{K_0}{K_1} \end{bmatrix} \quad \text{and} \quad \mathbf{A}_1 = \mathbf{E}^{-1}\bar{\mathbf{A}}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{K_0}{K_1} & 1 & 0 \end{bmatrix}$$

Remark. *When $\delta_0 = 1$, it can be verified that the delay equation (10) is exponentially stable independent of delay provided that $det(sI - A_0 - A_1e^{-s\tau}) \neq 0$ for all τ, [3].*

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3. Generalization

Consider the linear dynamical system:

$$x^{(n)} + a_{n-1}x^{(n-1)} + ... + \ddot{x} + a_1\dot{x} + a_0 x = b_0 u. \tag{11}$$

In this case the controller is proposed as:

$$u(t-\tau) = \frac{1}{b_0}\left(x_d^{(n)} + \sum_{i=0}^{n-1} a_i x_d^{(i)} + \sum_{i=0}^{n-1} K_i\left(e_\tau^{(i)} - x_r^{(i)}\right)\right), \tag{12}$$

where:

$$e_\tau^{(i)} = \frac{d^{(i)}}{dt^{(i)}}e(t-\tau) = \frac{d^{(i)}}{dt^{(i)}}(x_\tau - x_{d_\tau}). \tag{13}$$

Substituting the controller (12) in the system (11) renders the closed-loop equation:

$$e^{(n)} + \sum_{i=0}^{n-1} a_i e^{(i)} = -\sum_{i=0}^{n-1} K_i(e_\tau^{(i)} - x_r^{(i)}) \tag{14}$$

Now isolate the $(n-1)$-time derivative of the reference trajectory to get:

$$e^{(n)} + \sum_{i=0}^{n-1} a_i e^{(i)} = -\sum_{i=0}^{n-1} K_i e_\tau^{(i)} + \sum_{i=0}^{n-2} K_i x_r^{(i)} + K_{n-1}x_r^{(n-1)}. \tag{15}$$

The dynamics of the reference trajectory is set as:

$$K_{n-1}x_r^{(n-1)} = -\sum_{i=0}^{n-2} \delta_i K_i x_r^{(i)} + \sum_{i=0}^{n-1} K_i e_\tau^{(i)} \tag{16}$$

Define the state $z \in \mathbb{R}^{(2n-1)}$:

$$z = \begin{bmatrix} e \\ \dot{e} \\ \vdots \\ e^{(n-1)} \\ x_r \\ \dot{x}_r \\ \vdots \\ x_r^{(n-2)} \end{bmatrix} \tag{17}$$

The system (15) in closed loop with the reference trajectory (16) can be written as a single-time delays system:

$$\mathbf{E}\dot{z}(t) = \bar{\mathbf{A}}_0 z(t) + \bar{\mathbf{A}}_1 z(t-\tau) \tag{18}$$

where:

$$\mathbf{E} = \begin{bmatrix} I_{(n-1)\times(n-1)} & 0_{(n-1)\times(1)} & 0_{(n-1)\times(n-2)} & 0_{(n-1)\times(1)} \\ 0_{(1)\times(n-1)} & 1 & 0_{(1)\times(n-2)} & -K_{n-1} \\ 0_{(n-2)\times(n-1)} & 0_{(n-2)\times(1)} & I_{(n-2)\times(n-2)} & 0_{(n-2)\times(1)} \\ 0_{(1)\times(n-1)} & 0 & 0_{(1)\times(n-2)} & K_{n-1} \end{bmatrix}$$

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$$\bar{\mathbf{A}}_0 = \begin{bmatrix} 0_{(n-1)\times(1)} & I_{(n-1)\times(n-1)} & 0_{(n-1)\times(1)} & 0_{(n-1)\times(n-2)} \\ -a_0 & -[a_1,...,a_{n-1}] & K_0 & [K_1,...,K_{n-2}] \\ 0_{(n-2)\times(1)} & 0_{(n-2)\times(n-1)} & 0_{(n-2)\times(1)} & I_{(n-2)\times(n-2)} \\ 0 & 0_{(1)\times(n-1)} & -\delta_0 K_0 & -[\delta_1,...,\delta_{n-2}]K_{n-1} \end{bmatrix}$$

$$\bar{\mathbf{A}}_1 = \begin{bmatrix} 0_{(n-1)\times(n)} & 0_{(n-1)\times(n-1)} \\ -[K_0,...,K_{n-1}] & 0_{(1)\times(n-1)} \\ 0_{(n-2)\times(n)} & 0_{(n-2)\times(n-1)} \\ [K_0,...,K_{n-1}] & 0_{(1)\times(n-1)} \end{bmatrix}$$

The above matrices are size $(2n-1)$. Then, by induction, one can compute:

$$\mathbf{E}_n^{-1} = \begin{bmatrix} I_{(n-1)\times(n-1)} & 0_{(n-1)\times(1)} & 0_{(n-1)\times(n-2)} & 0_{(n-1)\times(1)} \\ 0_{(1)\times(n-1)} & 1 & 0_{(1)\times(n-2)} & 1 \\ 0_{(n-2)\times(n-1)} & 0_{(n-2)\times(1)} & I_{(n-2)\times(n-2)} & 0_{(n-2)\times(1)} \\ 0_{(1)\times(n-1)} & 0 & 0_{(1)\times(n-2)} & \frac{1}{K_{n-1}} \end{bmatrix}$$

Therefore:

$$\mathbf{A_0} = \mathbf{E_n^{-1}}\bar{\mathbf{A}}_0 = \begin{bmatrix} 0_{(n-1)\times(1)} & I_{(n-1)\times(n-1)} & 0_{(n-1)\times(1)} & 0_{(n-1)\times(n-2)} \\ -a_0 & -[a_1,\ldots,a_{n-1}] & K_0(1-\delta_0) & [K_1,\ldots,K_{n-2}] - [\delta_1,\ldots,\delta_{n-2}]K_{n-1} \\ 0_{(n-2)\times(1)} & 0_{(n-2)\times(n-1)} & 0_{(n-2)\times(1)} & I_{(n-2)\times(n-2)} \\ 0 & 0_{(1)\times(n-1)} & -\delta_0\frac{K_0}{K_{n-1}} & -[\delta_1,\ldots,\delta_{n-2}] \end{bmatrix}$$

and

$$\mathbf{A_1} = \mathbf{E_n^{-1}}\bar{\mathbf{A}}_1 = \begin{bmatrix} 0_{(n-1)\times(n)} & 0_{(n-1)\times(n-1)} \\ 0_{(1)\times(n)} & 0_{(1)\times(n)} \\ 0_{(n-2)\times(n)} & 0_{(n-2)\times(n-1)} \\ [K_0,\ldots,K_{n-1}]\left(\frac{1}{K_{n-1}}\right) & 0_{(1)\times(n-1)} \end{bmatrix}$$

The closed loop system is finally represented by:

$$\dot{z}(t) = \mathbf{A_0}z(t) + \mathbf{A_1}z(t-\tau) \tag{19}$$

3.1. Energy-based Symmetry Interpretation

Define the delay scalar $\xi(t-\tau) = \mathbf{K}z_p(t-\tau)$, where $\mathbf{K} = [K_0, K_2, ..., K_{n-1}]$ and $z_p(t-\tau) = [e_\tau, \dot{e}_\tau, ..., e_\tau^{(n-1)}]^T$. On the one hand, the closed loop (15) becomes:

$$e^{(n)} + \sum_{i=0}^{n-1} a_i e^{(i)} = -\xi(t-\tau) + \sum_{i=0}^{n-2} K_i x_r^{(i)} + K_{n-1}x_r^{(n-1)} \tag{20}$$

On the other hand, from (16), the dynamics of the reference trajectory can be written as:

$$K_{n-1}x_r^{(n-1)} + \sum_{i=0}^{n-2} \delta_i K_i x_r^{(i)} = \xi(t-\tau) \tag{21}$$

It is important to highlight that, in equations (20) and (21), the scalar $\xi(t-\tau)$ represents the delayed controller without the influence of the virtual reference trajectory x_r, thus it can be interpreted as a delayed symmetry-injection energy input to the control system; for the dynamical system the delayed terms appear with negative sign while for the virtual reference dynamics appear with positive sign. A more physical-oriented interpretation is that while the virtual reference dynamics absorbs energy, the system looses it and vice-versa.

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4. Stability Analysis

In this section, we consider the following theorems for time-delay systems in order to analyze the stability of the proposed approach. First, we present the stability result for $\tau > 0$.

Theorem 1. *[3] The delay difference equation (19) is stable if and only f the spectrum of the system lies in the open left half-plane of the complex plane.*

$$Re(s_0) < 0 \quad \textit{where: } s_0 \textit{ is the solution of } F(s) = det(s\mathbf{I} - \mathbf{A_0} - \mathbf{A_1}e^{-s\tau}) = 0 \tag{22}$$

Theorem 2. *[3] The delay difference equation (19) is stable independent of delay if:*

$$det(s\mathbf{I} - \mathbf{A_0} - \mathbf{A_1}e^{-s\tau}) \neq 0 \tag{23}$$

holds for all $\tau \geq 0, s \in \bar{\mathbb{C}}_+$

Remark. *In the delay-free case, the system (19) is stabilized by the feedback law (12), even if the open-loop plant (11) is unstable, by properly choosing the gains K_i and δ_0. In this case:*

$$\dot{z}(t) = \lim_{\tau \longrightarrow 0} \dot{z}(t) = \lim_{\tau \longrightarrow 0} \left(\mathbf{A_0}z(t) + \mathbf{A_1}z(t-\tau)\right) = \left(\mathbf{A_0} + \mathbf{A_1}\right)z(t) \tag{24}$$

where the matrix $\left(\mathbf{A_0} + \mathbf{A_1}\right)$ *is Hurwitz.*

Notice that the characteristic function $F(s)$ in equation (22) defines a real quasipolynomial of s. Therefore, it is possible to find the critical delay values, i.e., the values of τ^* that makes the system (19) unstable. In this work, we consider the so-called 2D Method, [3].

5. Results

In this section we present simulation and experimental results to validate our proposal. First, a linear second order is considered to illustrate a baseline example. Then, an unstable model of the anti-rolling stabilization of a ship is simulated. The experimental testbed and results related to a delayed control of a DC-motor are described at the end of this section.

5.1. Numerical Simulations

All simulations were carried out using the $dde23$ solver from Matlab © considering $z(t-\tau) = [z_{1_0}, 0, 0]$ for $t \in \{-\tau, 0\}$.

5.1.1. Second Order System

As a nominal case, consider the linear single-delay system as defined in (10), with $a_0 = 20$, $a_1 = 0.5$. If no delay is considered, we can choose a desired stable second order polynomial with $K_0 = \omega_n^2$ and $K_1 = 2\xi\omega_n$ to render asymptotic stability of the tracking error. By choosing $\omega_n = 30$ and $\xi = 1$ it is possible to compute the critical time-delay to get $\tau^* = 0.021623380140660$ $[s]$. The desired trajectory to be followed by the system is set as $x_d = A + Bsin(2\pi f t)$, where $A = 0.5$, $B = 0.5$ and $f = 0.5$. In this case $z_{1_0} = 0$. Figure 1 depicts simulation results for different time delays approaching and overcoming τ^*.

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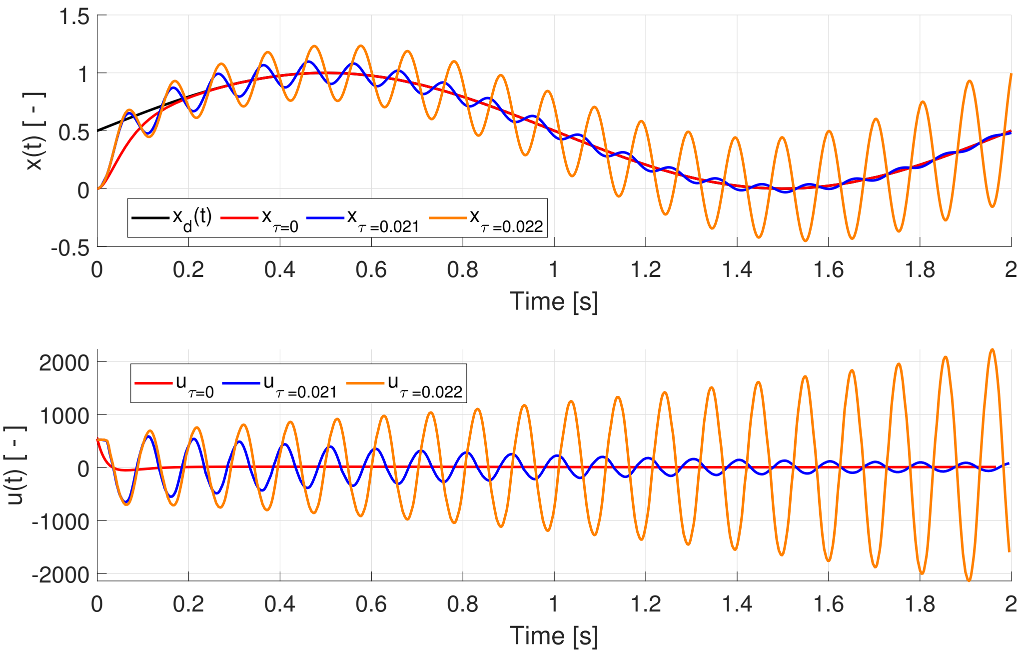

Figure 1. Response of delay system 1 for several values of τ without considering x_r in the controller 2.

If delay is considered, we keep $K_0 = \omega_n^2$ and $K_1 = 2\xi\omega_n$. In this case we choose $\delta_0 = 1.005$ and $z_{1_0} = 0$. By choosing $\omega_n = 30$ and $\xi = 1$ one can compute the critical delay to get $\tau^* = 0.118315659529695$ $[s]$. Figure 2 depicts simulation results for different time delays approaching and overcoming τ^*, notice the reference trajectory $x_d(t)$ is represented in black color and it is tracked when the system is stable.

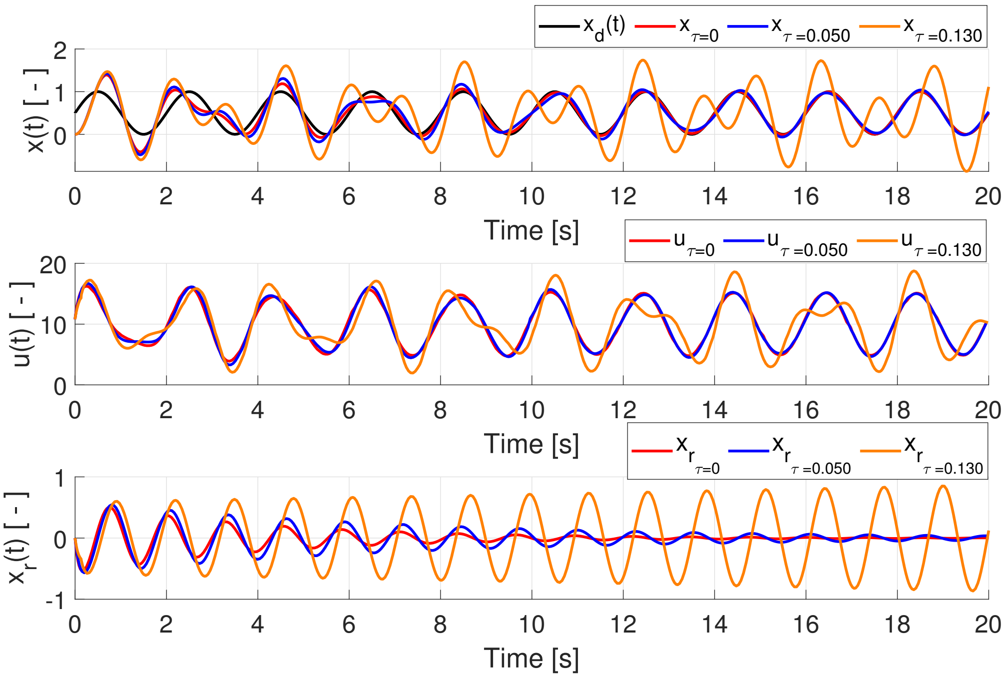

Figure 2. Response of delay system 1 for several values of τ considering the virtual reference trajectory x_r in the controller 2.

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5.1.2. Unstable Second Order System

In this section, we present simulations of the model of a mechanical system with self-excited oscillations with a retarded action from reference [41]. Consider a system with one degree of freedom acted by an external periodic moment $Msin(kt)$. The system is described by:

$$\ddot{\theta} + 2\delta\omega\dot{\theta} + \omega\theta + \rho\dot{\theta}_\tau = Msin(kt) + u_\tau(t), \tag{25}$$

where δ, ω and ρ are known parameters. The anti-rolling stabilization of ships can be modeled using equation (25). It includes a term dependent on the delayed friction $\rho\dot{\theta}_\tau$, with $\dot{\theta}_\tau = \dot{\theta}(t-\tau)$, which is result of the the amplification factor of the electronic control of the anti-rolling stabilization system, [42] . By choosing the input control $u_\tau(t) = 0$ and the set of simulation parameters $\tau = 0.5$, $\delta = 0.006$, $\omega = 3.1321$ and $\rho = 0.9$, $M = 0.1$ and $k = 1$ the system is unstable, as shown in Figure 3.

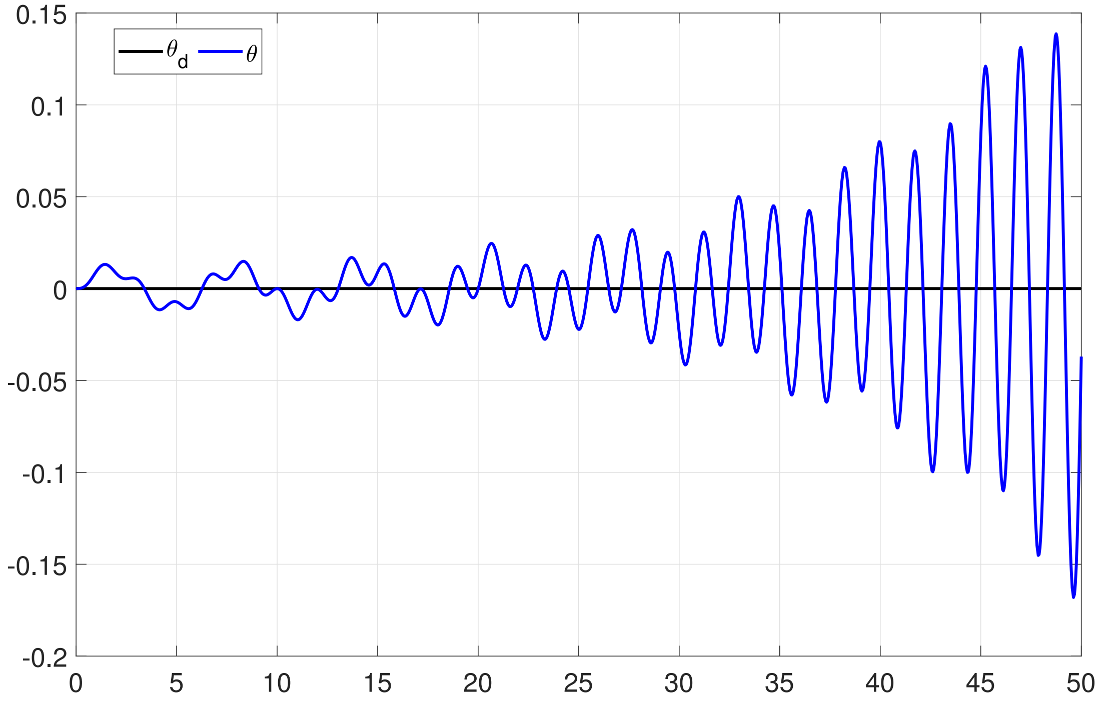

Figure 3. Simulation of anti-rolling stabilization system with $u_\tau = 0$ and $\tau = 0.5$ [s].

Te goal is to keep the oscillation angle θ as close as possible to zero. Thus, $\theta_d = \dot{\theta}_d = \ddot{\theta}_d = 0$. Therefore, consider the controller:

$$u_\tau = u(t-\tau) = -K_0(\theta_\tau - \theta_r) - K_1(\dot{\theta}_\tau - \dot{\theta}_r) \tag{26}$$

and the virtual reference angle dynamics:

$$K_1\dot{\theta}_r = -\delta_0 K_0\theta_r + K_0\theta_\tau + K_1\dot{\theta}_\tau \tag{27}$$

In this we obtain:

$$\mathbf{A}_0 = \begin{bmatrix} 0 & 1 & 0 \\ -\omega & -2\delta\omega & K_0(1-\delta_0) \\ 0 & 0 & -\delta_0\frac{K_0}{K_1} \end{bmatrix} \quad \text{and} \quad \mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\rho & 0 \\ \frac{K_0}{K_1} & 1 & 0 \end{bmatrix}$$

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For the simulation we set the initial condition $\theta(0) = 0.1$, $\delta_0 = 1.001$, $K_0 = \omega_n^2$ and $K_1 = 2\xi\omega_n$ with $\omega_n = 30$ and $\xi = 1$. Therefore, the maximum delay can be computed as $\tau^* = 2.059766210715432$. The results of the simulation considering the feedback controller is depicted in Figure 4. As it is shown, the angle is stabilized for $\tau = 1.7$. Notice that the first oscillation of $\theta(t)$ is bigger than the initial condition, showing the system instability before the controller acts on the system. Then, when the controller starts at $t = 1.7$ [s], the second oscillation is drastically diminished and so the system is stabilized.

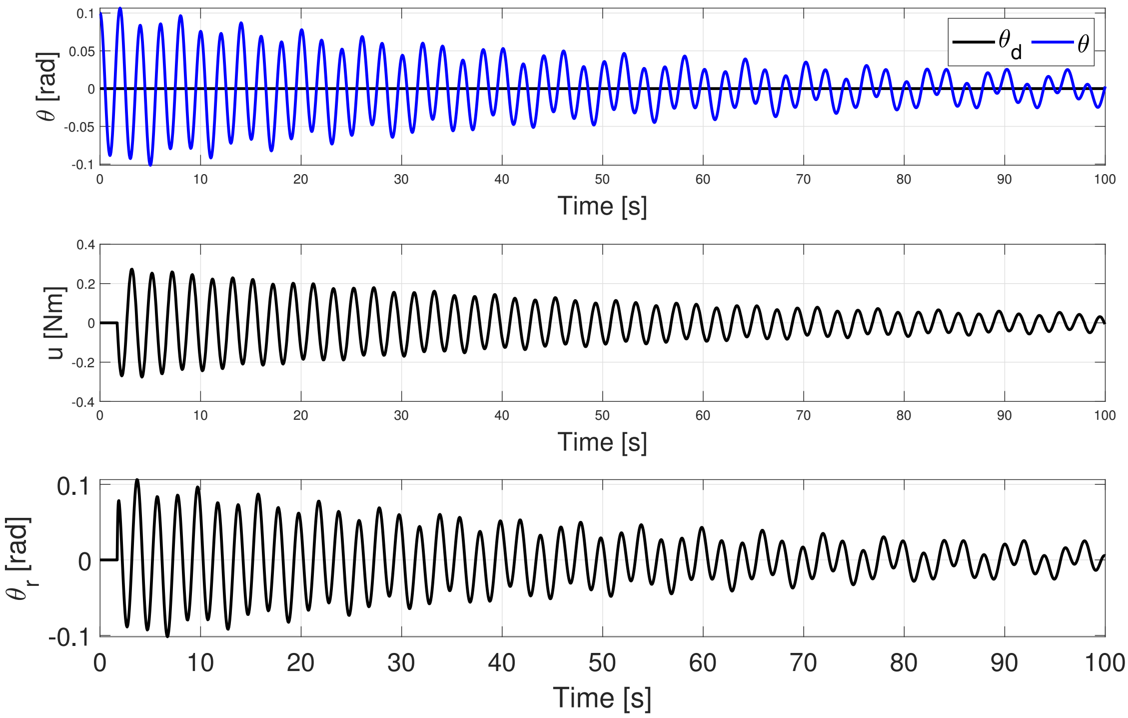

Figure 4. Simulation of controlled anti-rolling system system with the proposed controller and $\tau = 1.7$.

5.2. Experiments

In this section, experiments of the control of a DC-motor with time-delay measurement are depicted to validate our proposal. The experimental testbed is described in detail in appendix A. In this case, the tracking error is defined as $\tilde{\theta} = \theta - \theta_d$. It is important to mention that the available signal to be measured is the motor angular position while the angular velocity is obtained via a Luenberger observer.

Figure 5 shows the response of the DC-motor without considering the reference trajectory θ_r. In this case the maximum delay can be computed as $\tau^* = 0.006534286078178$. Clearly, with $\tau = 0.003$ [s] and $\tau = 0.005$ [s] the system is stable and the tracking error $\theta - \theta_d$ is practically zero. When $\tau = 0.008$ [s], the system becomes unstable even as the trajectory tracking is good, as one can conclude from the saturated control signal V and the increased tracking error.

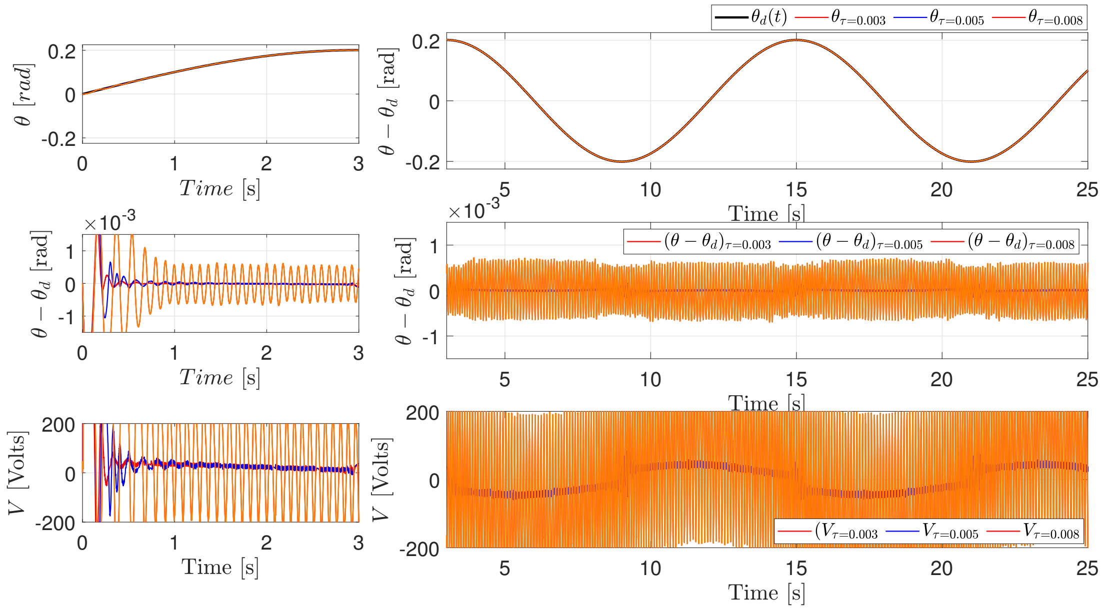

Figure 5. Experiments of controlled DC-Motor with the proposed controller without considering the virtual trajectory θ_r.

Figure 6 shows the response of the DC-motor considering the reference trajectory θ_r with $\delta_0 = 1.01$. In this case the maximum delay can be computed as $\tau^* = 0.026043385697303$. Clearly, with $\tau = 0.010$ [s] and $\tau = 0.040$[s] the system is stable and the tracking error $\theta - \theta_d$ is practically zero. When $\tau = 0.050$ [s], the system becomes unstable, as one can conclude from the control signal V and the increased tracking error. It is observed that the critical delay τ^* is lower than the experimentally observed stability bound of $\tau = 0.040$ [s], but unstable for $\tau = 0.050$ [s]. It is important to mention that in the experiments it was observed that the use of the PWM signal induces some degree of extra stability robustness to the proposed approach, [43]. Its theoretical analysis lies beyond the scope of this work. Additionally, it is important to mention that for larger values of τ the demand of control effort is bigger, thus experimentally we are limited to compensate for certain values of delay, as depicted in the above experiments.

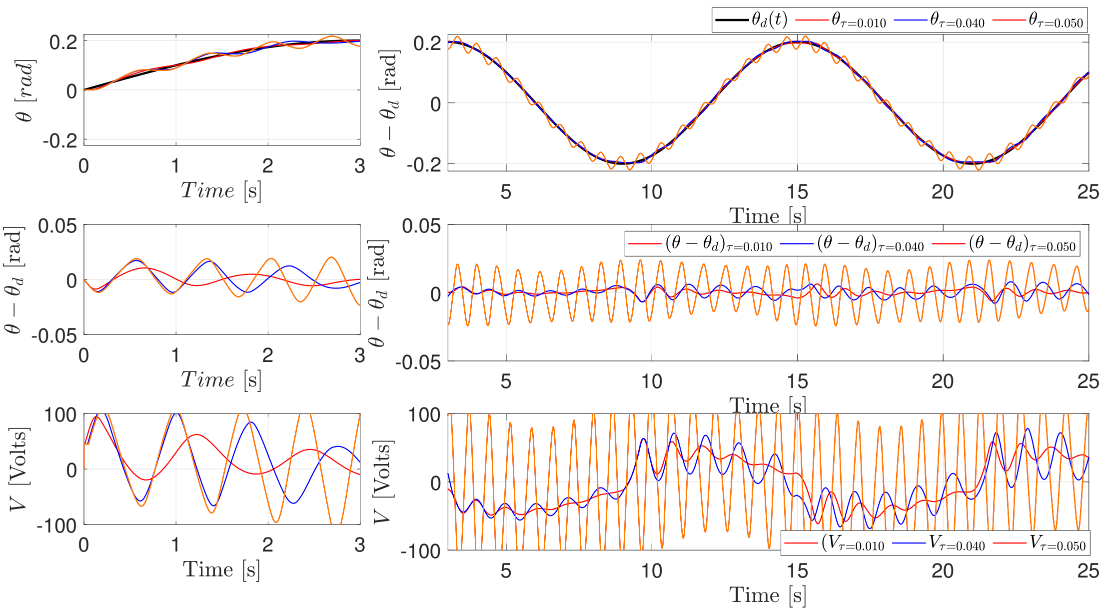

Figure 6. Experiments of controlled DC-Motor with the proposed controller with the virtual reference trajectory θ_r.

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6. Conclusions

In this work, an approach for the control of linear control systems with single time-delay. The paradigm shift is that we consider to use a virtual reference trajectory resulting from natural selection of the standard closed-loop stable or stabilization feedback control law. As shown in this work, the introduction of the virtual reference dynamics into the system model yields a standard single-delay whose stability can be studied using state of the art results. Moreover, an energy-based symmetry interpretation of the proposal has been drawn, paving the way for future developments considering symmetry properties of time-delay control systems. A relevant feature of the proposal is that in the ideal case, the control system can be rendered stable independent of delay. Numerical simulations perfectly match the theoretical predictions of the computed critical time-delay. Nonetheless, it is important to mention, the experimental validation of this case is challenging due to hardware constraints. Future research directions considers extensions to multiple delay systems, its application to nonlinear control systems and the inclusion of bounded control input into the theoretical analysis.

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Appendix A. Experimental Testbed

An experimental platform based on a DC motor drive was developed to validate the control strategies proposed in this work. The setup enables real-time implementation and evaluation of control algorithms under hardware-in-the-loop (HIL) conditions. The actuation system consists of a brushed DC motor coupled to a 70:1 gear reduction stage. The motor shaft position is measured using an integrated Hall-effect sensor providing 16 pulses per revolution. By employing quadrature decoding, the encoder signal yields 64 counts per motor shaft revolution. Considering the gearbox reduction ratio, the resulting angular resolution at the output shaft corresponds to 3200 counts per revolution, allowing precise position and velocity estimation.

Motor actuation is achieved through an H-bridge driver controlled by an STM32f4 Discovery development board [44], which performs both control computation and signal generation. Pulse-width modulation (PWM) is used to regulate the motor voltage with a switching frequency of 3.333 kHz. The control algorithms operate with a sampling period of 1 ms. Control algorithms are designed using a block-based approach in MATLAB/Simulink and automatically translated into C code, which is subsequently deployed onto the STM32 microcontroller for real-time execution.

During experimental operation, the embedded controller executes the control routines independently while transmitting measured variables to a host computer running Simulink for monitoring, saving and data visualization. This configuration enables direct comparison between theoretical predictions and experimental responses while maintaining real-time execution constraints. The described platform provides a flexible environment for rapid prototyping and experimental validation of advanced control strategies applied to electromechanical systems. Figure A1 shows the operating schematic of the testbed and Figure A2 the layout of components.

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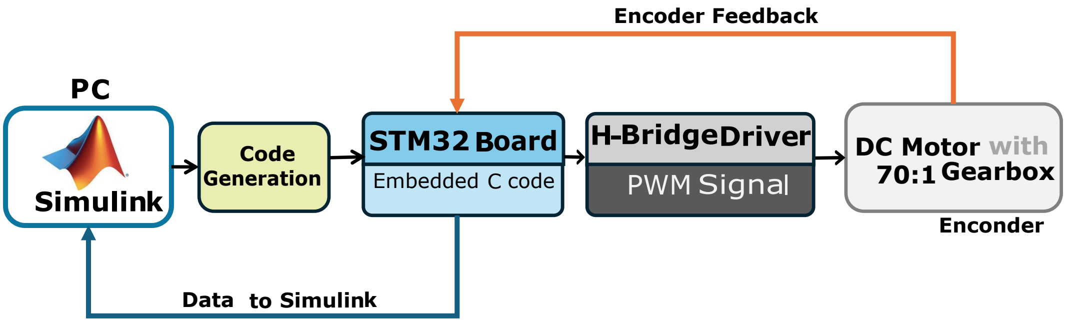

Figure A1. Testbed operation diagram.

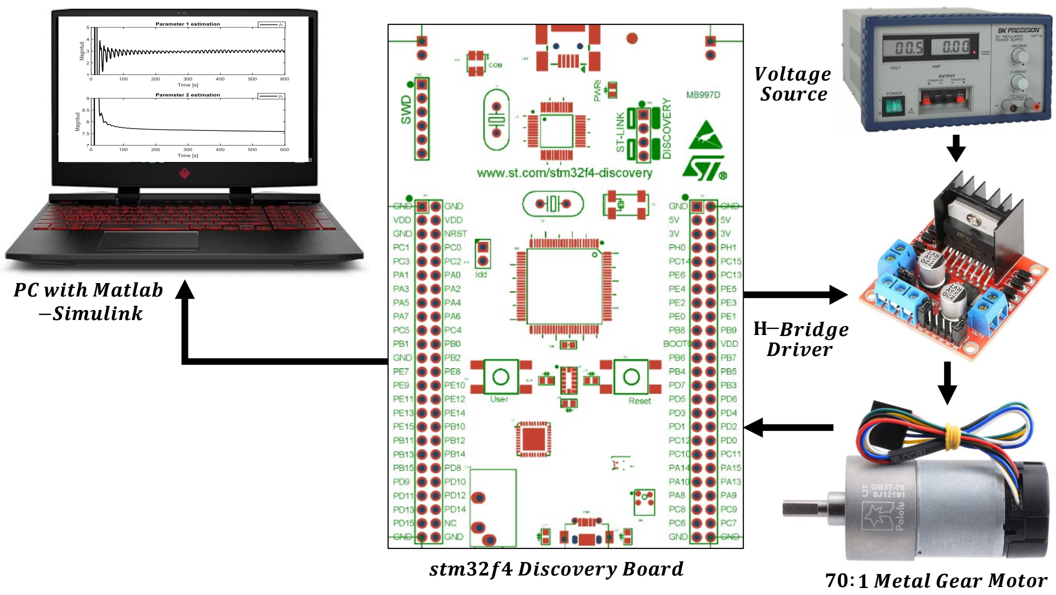

Figure A2. Testbed component layout.

Appendix A.1. Testbed Modeling and Identification

The modeling of the system in the experimental testbed was obtained via the algebraic identification approach, see [45] for details. The problem of identification is described according to equation (A1).

$$A\hat{P} = u \tag{A1}$$

where A is the regressor matrix, $\hat{P}$ are the estimated parameters and u is the system input. The vector $\hat{P}$ was obtained as a solution to the optimization problem (A2).

$$min \int (A\hat{P} - u)^2 dt \tag{A2}$$

Subsequently, by considering the linear system described in (11), the corresponding model to the experimental testbed is expressed as:

$$a_2\ddot{x} + a_1\dot{x} + a_0 x = b_0 u. \tag{A3}$$

For this particular case, the behavior described by Equation (A3) was expressed as a velocity function $v = \dot{x}$ normalized with respect to input parameter b_0 leading to Equation (A4), which is consistent with DC motor model from reference, [46].

$$\frac{a_2}{b_0}\dot{v} + \frac{a_1}{b_0}v = u \tag{A4}$$

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Based on Equation (A4), with $v = \dot{x}$, and the solution to the identification problem, the following algebraic procedure was derived.

$$t(\frac{a_2}{b_0}\dot{v} + \frac{a_1}{b_0}v) = tu \tag{A5}$$

$$\frac{a_2}{b_0}\int t\dot{v}\,\mathrm{d}t + \frac{a_1}{b_0}\int tv\,\mathrm{d}t = \int tu\,\mathrm{d}t \tag{A6}$$

Integrating by parts and rewriting equation (A6) in matrix form as equation (A7).

$$\begin{bmatrix}A\end{bmatrix}\begin{bmatrix}\hat{P}\end{bmatrix} = B, \tag{A7}$$

where:

$$\begin{bmatrix}A\end{bmatrix} = \begin{bmatrix}A_1 & A_2\end{bmatrix} = \begin{bmatrix}tv - \int tv\,\mathrm{d}t & \int tv\,\mathrm{d}t\end{bmatrix} \tag{A8}$$

$$\begin{bmatrix}\hat{P}\end{bmatrix} = \begin{bmatrix}\hat{P}_1 \\ \hat{P}_2\end{bmatrix} = \begin{bmatrix}\frac{a_2}{b_0} \\ \frac{a_1}{b_0}\end{bmatrix} \tag{A9}$$

$$B = \int tu\,\mathrm{d}t \tag{A10}$$

Then, by multiplying both sides of equation (A9) by the transpose of the regressor matrix A and integrating the resulting expression, we obtain:

$$\left[\int A^T A\,\mathrm{d}t\right]\hat{P} = \int B\mathrm{d}t \tag{A11}$$

Finally, equation (A11) can be rewritten to obtain the expression of the estimated parameter vector, $\hat{P}$.

$$\hat{P} = \left[\int A^T A\,\mathrm{d}t\right]^{-1}\int B\,\mathrm{d}t \tag{A12}$$

Appendix A.2. Identification Process

Based on equation (A12) and using MATLAB/Simulink, the offline parameter identification mechanism was implemented according to the diagram presented in figure A3.

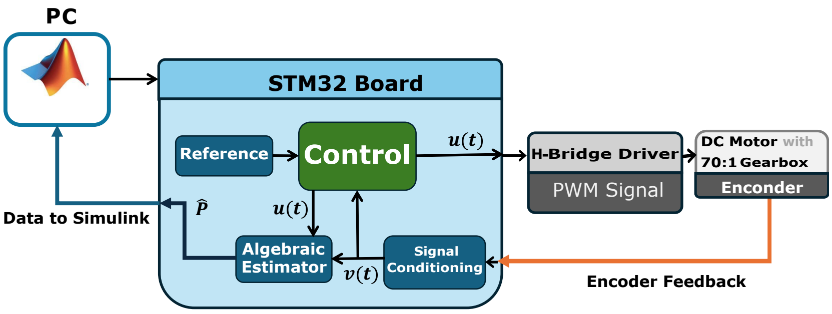

Figure A3. Testbed configuration diagram with algebraic estimator.

Figure A4 shows the estimated values $\hat{P}_1$ and $\hat{P}_2$.

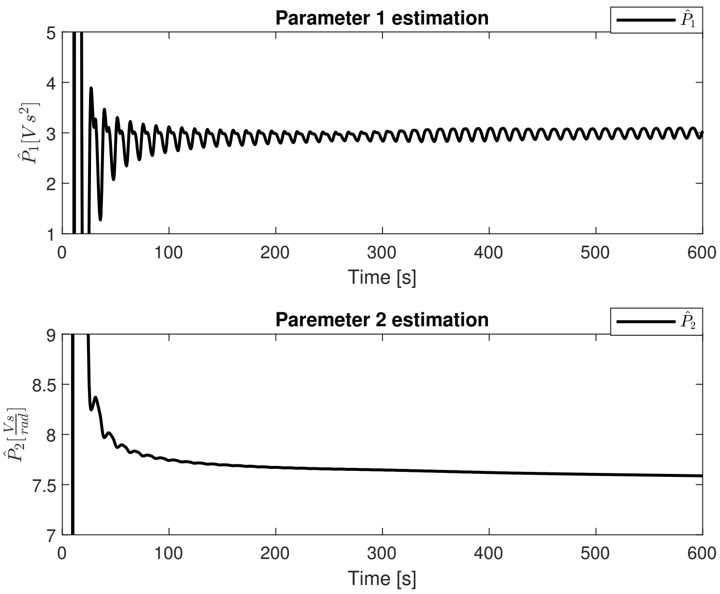

Figure A4. Representation of estimated values $\hat{P}_1$ and $\hat{P}_2$ in a bounded range of magnitude.

The identification procedure provided the parameter estimates $\hat{P}_1 = 3$ and $\hat{P}_2 = 7.6$, which were directly incorporated into equation (A13), as well as into the control law design and delay compensation strategies.

$$\hat{P}_1 \dot{v} + \hat{P}_2 v = u \tag{A13}$$

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