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Abstract: Celebrated breakthrough sparsity theorem obtained independently by Donoho and Elad *[Proc. Natl. Acad. Sci. USA, 2003]* and Gribonval and Nielson *[IEEE Trans. Inform. Theory, 2003]* and Fuchs *[IEEE Trans. Inform. Theory, 2004]* says that unique sparse solution to NP-Hard ℓ_0-minimization problem can be obtained using unique solution to P-Type ℓ_1-minimization problem. In this paper, we extend their result to abstract Banach spaces using 1-approximate Schauder frames. We notice that the 'normalized' condition for Hilbert spaces can be generalized to a larger extent when we consider Banach spaces.

Keywords: compressed sensing; frame; schauder frame; sparse solution

MSC: 42C15; 94A12; 46B45

1. Introduction

Let $\mathcal{H}$ be a finite dimensional Hilbert space over $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$). Recall that [2,28] a finite collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{H}$ is said to be a **frame** (also known as **dictionary**) for $\mathcal{H}$ if it spans $\mathcal{H}$. A frame $\{\tau_j\}_{j=1}^n$ for $\mathcal{H}$ is said to be **normalized** if $\|\tau_j\| = 1$ for all $1 \le j \le n$. Given a frame $\{\tau_j\}_{j=1}^n$ for $\mathcal{H}$, we define the analysis operator

$$\theta_\tau : \mathcal{H} \ni h \mapsto \theta_\tau h := (\langle h, \tau_j \rangle)_{j=1}^n \in \mathbb{K}^n.$$

Adjoint of the analysis operator is known as the synthesis operator whose equation is

$$\theta_\tau^* : \mathbb{K}^n \ni (a_j)_{j=1}^n \mapsto \theta_\tau^*(a_j)_{j=1}^n := \sum_{j=1}^n a_j \tau_j \in \mathcal{H}.$$

Given $d \in \mathbb{K}^n$, let $\|d\|_0$ be the number of nonzero entries in d. Central problem which occurs in everyday life is the following ℓ_0-minimization problem:

Problem 1.1. *Let $\{\tau_j\}_{j=1}^n$ be a frame for $\mathcal{H}$. Given $h \in \mathcal{H}$, solve*

$$\underset{d \in \mathbb{K}^n}{\text{minimize}} \ \|d\|_0 \quad \textit{subject to} \quad \theta_\tau^* d = h.$$

Recall that $c \in \mathbb{K}^n$ is said to be a unique solution to Problem 1.1 if it satisfies following two conditions.

(i) $\theta_\tau^* c = h$.
(ii)   If $d \in \mathbb{K}^n$ satisfies $\theta_\tau^* d = h$, then

$$\|d\|_0 > \|c\|_0.$$

Unfortunately, in 1995, Natarajan showed that Problem 1.1 is NP-Hard [23,33]. Therefore solution to Problem 1.1 has to be obtained using other means. Entire body of work which is built around Problem

1.1 is known as **sparseland** (term due to Elad [19]) or **compressive sensing** or **compressed sensing** [1,4–10,14–23,27,32,34,37,38]. We note that as the operator θ_τ^* is surjective, for a given $h \in \mathcal{H}$, there is always $d \in \mathbb{K}^n$ such that $\theta_\tau^* d = h$. Thus the central problem is when solution to Problem 1.1 is unique. It is well-known that (see [3,13,18]) following problem is the closest convex relaxation problem to Problem 1.1.

Problem 1.2. *Let $\{\tau_j\}_{j=1}^n$ be a frame for $\mathcal{H}$. Given $h \in \mathcal{H}$, solve*

$$\underset{d \in \mathbb{K}^n}{\text{minimize}} \, \|d\|_1 \quad \text{subject to} \quad \theta_\tau^* d = h.$$

There are several linear programmings available to obtain solution of Problem 1.2 and it is a P-problem [35,36,39].

Most important result which shows by solving Problem 1.2 we also get a solution to Problem 1.1, obtained independently by Donoho and Elad [17] and Gribonval and Nielsen [27] and Fuchs [25,26], is the following.

Theorem 1.3. *[17,19,25–27,31] (**Donoho-Elad-Gribonval-Nielsen-Fuchs Sparsity Theorem**) Let $\{\tau_j\}_{j=1}^n$ be a normalized frame for a Hilbert space $\mathcal{H}$. If $h \in \mathcal{H}$ can be written as $h = \theta_\tau^* c$ for some $c \in \mathbb{K}^n$ satisfying*

$$\|c\|_0 < \frac{1}{2} \left(1 + \frac{1}{\underset{1 \le j,k \le n, j \ne k}{\max} |\langle \tau_j, \tau_k \rangle|} \right),$$

then c is the unique solution to Problem 1.2 and Problem 1.1.

We naturally ask for (both finite and infinite dimensional) Banach space version Theorem 1.3. More than this natural question, many spaces occurring in functional analysis and in applications are Banach and there is no Hilbert space structure associated with them. As frame theory for Hilbert spaces has been successfully extended to Banach spaces which also found applications, we believe that generalization of Theorem 1.3 will have applications. It is interesting to note that a noncommutative version of Theorem 1.3 has been recently derived [29].

2. Functional Donoho-Elad-Gribonval-Nielsen-Fuchs Sparsity Theorem

In the paper, $\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$ and $\mathcal{X}$ denotes a Banach space (need not be finite dimensional) over $\mathbb{K}$. Dual of $\mathcal{X}$ is denoted by $\mathcal{X}^*$. We need the notion of 1-approximate Schauder frames for Banach spaces which is a subclass of Schauder frames [11,12,24].

Definition 2.1. *[30] Let $\mathcal{X}$ be a Banach space over $\mathbb{K}$. Let $\{f_n\}_{n=1}^\infty$ be a sequence in $\mathcal{X}^*$ and $\{\tau_n\}_{n=1}^\infty$ be a sequence in $\mathcal{X}$. The pair $(\{f_n\}_{n=1}^\infty, \{\tau_n\}_{n=1}^\infty)$ is said to be a **1-approximate Schauder frame** (we write 1-ASF) for $\mathcal{X}$ if the following conditions are satisfied.*

*(i) The map (**analysis operator**)*

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x := \{f_n(x)\}_{n=1}^\infty \in \ell^1(\mathbb{N})$$

is a well-defined bounded linear operator.

*(ii) The map (**synthesis operator**)*

$$\theta_\tau : \ell^1(\mathbb{N}) \ni \{a\}_{n=1}^\infty \mapsto \theta_\tau \{a\}_{n=1}^\infty := \sum_{n=1}^\infty a_n \tau_n \in \mathcal{X}$$

is a well-defined bounded linear operator.

1. *The map (**frame operator**)*

$$S_{f,\tau} : \mathcal{X} \ni x \mapsto S_{f,\tau} x := \sum_{n=1}^{\infty} f_n(x) \tau_n \in \mathcal{X}$$

is a well-defined bounded invertible operator.

We the notion of 1-ASF, we generalize Problems 1.1 and 1.2.

Problem 2.2. *Let $(\{f_n\}_{n=1}^{\infty}, \{\tau_n\}_{n=1}^{\infty})$ be an 1-ASF for $\mathcal{X}$. Given $x \in \mathcal{X}$, solve*

$$\underset{d \in \ell^1(\mathbb{N})}{\text{minimize}} \|d\|_0 \quad \text{subject to} \quad \theta_\tau d = x.$$

Problem 2.3. *Let $(\{f_n\}_{n=1}^{\infty}, \{\tau_n\}_{n=1}^{\infty})$ be an 1-ASF for $\mathcal{X}$. Given $x \in \mathcal{X}$, solve*

$$\underset{d \in \ell^1(\mathbb{N})}{\text{minimize}} \|d\|_1 \quad \text{subject to} \quad \theta_\tau d = x.$$

A very important property used to show Theorem 1.3 is the notion of null space property (see [14,31]). We now define the same property for Banach spaces. We use following notations. Let $\{e_n\}_{n=1}^{\infty}$ be the canonical Schauder basis for $\ell^1(\mathbb{N})$. Given $M \subseteq \mathbb{N}$ and $d = \{d_n\}_{n=1}^{\infty} \in \ell^1(\mathbb{N})$, we define

$$d_M := \sum_{n \in M} d_n e_n.$$

Definition 2.4. *An 1-ASF $(\{f_n\}_{n=1}^{\infty}, \{\tau_n\}_{n=1}^{\infty})$ for $\mathcal{X}$ is said to have the **null space property** (we write NSP) of order $k \in \mathbb{N}$ if for every $M \subseteq \mathbb{N}$ with $o(M) \leq k$, we have*

$$\|d_M\|_1 < \frac{1}{2} \|d\|_1, \quad \forall d \in \ker(\theta_\tau), d \neq 0.$$

Following characterization relates NSP with Problem 2.3.

Theorem 2.5. *Let $(\{f_n\}_{n=1}^{\infty}, \{\tau_n\}_{n=1}^{\infty})$ be an 1-ASF for $\mathcal{X}$ and let $k \in \mathbb{N}$. The following are equivalent.*

(i) *If $x \in \mathcal{X}$ can be written as $x = \theta_\tau c$ for some $c \in \ell^1(\mathbb{N})$ satisfying $\|c\|_0 \leq k$, then c is the unique solution to Problem 2.3.*

(ii) *$(\{f_n\}_{n=1}^{\infty}, \{\tau_n\}_{n=1}^{\infty})$ satisfies the NSP of order k.*

Proof. (i) $\implies$ (ii) Let $M \subseteq \mathbb{N}$ with $o(M) \leq k$ and let $d \in \ker(\theta_\tau), d \neq 0$. Then we have

$$0 = \theta_\tau d = \theta_\tau(d_M + d_{M^c}) = \theta_\tau(d_M) + \theta_\tau(d_{M^c})$$

which gives

$$\theta_\tau(d_M) = \theta_\tau(-d_{M^c}).$$

Define $c := d_M \in \ell^1(\mathbb{N})$ and $x := \theta_\tau(d_M)$. Then we have $\|c\|_0 \leq o(M) \leq k$ and

$$x = \theta_\tau c = \theta_\tau(-d_{M^c}).$$

By assumption (i), we then have

$$\|c\|_1 = \|d_M\|_1 < \| -d_{M^c}\|_1 = \|d_{M^c}\|_1.$$

Rewriting previous inequality gives

$$\|d_M\|_1 < \|d\|_1 - \|d_M\|_1 \implies \|d_M\|_1 < \frac{1}{2}\|d\|_1.$$

Hence $(\{f_n\}_{n=1}^\infty, \{\tau_n\}_{n=1}^\infty)$ satisfies the NSP of order k.

(ii) $\implies$ (i) Let $x \in \mathcal{X}$ can be written as $x = \theta_\tau c$ for some $c \in \ell^1(\mathbb{N})$ satisfying $\|c\|_0 \le k$. Define $M := \operatorname{supp}(c)$. Then $o(M) = \|c\|_0 \le k$. By assumption (ii), we then have

$$\|d_M\|_1 < \frac{1}{2}\|d\|_1, \quad \forall d \in \ker(\theta_\tau), d \neq 0. \tag{1}$$

Let $b \in \ell^1(\mathbb{N})$ be such that $x = \theta_\tau b$ and $b \neq c$. Define $a := b - c \in \ell^1(\mathbb{N})$. Then $\theta_\tau a = \theta_\tau b - \theta_\tau c = x - x = 0$ and hence $a \in \ker(\theta_\tau), a \neq 0$. Using Inequality (1), we get

$$\|a_M\|_1 < \frac{1}{2}\|a\|_1 \implies \|a_M\|_1 < \frac{1}{2}(\|a_M\|_1 + \|a_{M^c}\|_1) \implies \|a_M\|_1 < \|a_{M^c}\|_1. \tag{2}$$

Using Inequality (2) and the information that c is supported on M, we get

$$\begin{aligned}
\|b\|_1 - \|c\|_1 &= \|b_M\|_1 + \|b_{M^c}\|_1 - \|c_M\|_1 - \|c_{M^c}\|_1 = \|b_M\|_1 + \|b_{M^c}\|_1 - \|c_M\|_1 \\
&= \|b_M\|_1 + \|(b-c)_{M^c}\|_1 - \|c_M\|_1 = \|b_M\|_1 + \|a_{M^c}\|_1 - \|c_M\|_1 \\
&> \|b_M\|_1 + \|a_M\|_1 - \|c_M\|_1 \ge \|b_M\|_1 + \|(b-c)_M\|_1 - \|c_M\|_1 \\
&\ge \|b_M\|_1 - \|b_M\|_1 + \|c_M\|_1 - \|c_M\|_1 = 0.
\end{aligned}$$

Hence c is the unique solution to Problem 2.3.

$\square$

Using Theorem 2.5 we obtain Banach space version of Theorem 1.3. We do this by relating Problem 2.3 to Theorem 2.5 and then Problem 2.2 to Theorem 2.5.

Theorem 2.6. *Let* $(\{f_n\}_{n=1}^\infty, \{\tau_n\}_{n=1}^\infty)$ *be an 1-ASF for* $\mathcal{X}$ *such that*

$$|f_n(\tau_n)| \ge 1, \quad \forall n \in \mathbb{N}. \tag{3}$$

If $x \in \mathcal{X}$ *can be written as* $x = \theta_\tau c$ *for some* $c \in \ell^1(\mathbb{N})$ *satisfying*

$$\|c\|_0 < \frac{1}{2}\left(1 + \frac{1}{\displaystyle\sup_{n,m \in \mathbb{N}, n \neq m} |f_n(\tau_m)|}\right), \tag{4}$$

then c *is the unique solution to Problem 2.3.*

Proof. We show that $(\{f_n\}_{n=1}^\infty, \{\tau_n\}_{n=1}^\infty)$ satisfies the NSP of order $k := \|c\|_0$. Then Theorem 2.5 says that c is the unique solution to Problem 2.3. Let $x \in \mathcal{X}$ can be written as $x = \theta_\tau c$ for some $c \in \ell^1(\mathbb{N})$ satisfying $\|c\|_0 \le k$. Let $M \subseteq \mathbb{N}$ with $o(M) \le k$ and let $d \in \ker(\theta_\tau), d \neq 0$. Then we have

$$\theta_f \theta_\tau d = 0.$$

By writing $d = \{d_n\}_{n=1}^{\infty} \in \ell^1(\mathbb{N})$, above equation gives

$$0 = \theta_f \theta_\tau \{d_m\}_{m=1}^{\infty} = \theta_f \left(\sum_{m=1}^{\infty} d_m \theta_\tau e_m \right)$$

$$= \theta_f \left(\sum_{m=1}^{\infty} d_m \tau_m \right) = \sum_{m=1}^{\infty} d_m \theta_f(\tau_m) = \sum_{m=1}^{\infty} d_m \sum_{k=1}^{\infty} f_k(\tau_m) e_k.$$

Let $\{\zeta_n\}_{n=1}^{\infty}$ be the coordinate functionals associated with the canonical Schauder basis $\{e_n\}_{n=1}^{\infty}$ for $\ell^1(\mathbb{N})$. Let $n \in \mathbb{N}$. By evaluating previous equation at ζ_n, we get

$$0 = \zeta_n \left(\sum_{m=1}^{\infty} d_m \sum_{k=1}^{\infty} f_k(\tau_m) e_k \right) = \sum_{m=1}^{\infty} d_m \sum_{k=1}^{\infty} f_k(\tau_m) \zeta_n(e_k)$$

$$= \sum_{m=1}^{\infty} d_m f_n(\tau_m) = d_n f_n(\tau_n) + \sum_{m=1, m \neq n}^{\infty} d_m f_n(\tau_m).$$

Therefore

$$d_n f_n(\tau_n) = - \sum_{m=1, m \neq n}^{\infty} d_m f_n(\tau_m), \quad \forall n \in \mathbb{N}.$$

By using Inequality (3),

$$|d_n| \leq |d_n||f_n(\tau_n)| = \left| - \sum_{m=1, m \neq n}^{\infty} d_m f_n(\tau_m) \right|$$

$$\leq \sum_{m=1, m \neq n}^{\infty} |d_m f_n(\tau_m)| \leq \left(\sup_{m \in \mathbb{N}, n \neq m} |f_n(\tau_m)| \right) \sum_{m=1, m \neq n}^{\infty} |d_m|$$

$$\leq \left(\sup_{n,m \in \mathbb{N}, n \neq m} |f_n(\tau_m)| \right) \sum_{m=1, m \neq n}^{\infty} |d_m| = \left(\sup_{n,m \in \mathbb{N}, n \neq m} |f_n(\tau_m)| \right) \left(\sum_{m=1}^{\infty} |d_m| - |d_n| \right)$$

$$= \left(\sup_{n,m \in \mathbb{N}, n \neq m} |f_n(\tau_m)| \right) (\|d\|_1 - |d_n|), \quad \forall n \in \mathbb{N}.$$

By rewriting above inequality we get

$$\left(1 + \frac{1}{\sup\limits_{n,m \in \mathbb{N}, n \neq m} |f_n(\tau_m)|} \right) |d_n| \leq \|d\|_1, \quad \forall n \in \mathbb{N}. \tag{5}$$

Summing Inequality (5) over M leads to

$$\left(1 + \frac{1}{\sup\limits_{n,m \in \mathbb{N}, n \neq m} |f_n(\tau_m)|} \right) \|d_M\|_1 = \left(1 + \frac{1}{\sup\limits_{n,m \in \mathbb{N}, n \neq m} |f_n(\tau_m)|} \right) \sum_{n \in M} |d_n|$$

$$\leq \|d\|_1 \sum_{n \in M} 1 = \|d\|_1 o(M).$$

Finally using Inequality (4)

$$\|d_M\|_1 \leq \left(1 + \frac{1}{\sup\limits_{n,m\in\mathbb{N}, n\neq m}|f_n(\tau_m)|}\right)^{-1} \|d\|_1 o(M) \leq \left(1 + \frac{1}{\sup\limits_{n,m\in\mathbb{N}, n\neq m}|f_n(\tau_m)|}\right)^{-1} \|d\|_1 k$$

$$= \left(1 + \frac{1}{\sup\limits_{n,m\in\mathbb{N}, n\neq m}|f_n(\tau_m)|}\right)^{-1} \|d\|_1 \|c\|_0 < \frac{1}{2}\|d\|_1.$$

Hence $(\{f_n\}_{n=1}^{\infty}, \{\tau_n\}_{n=1}^{\infty})$ satisfies the NSP of order k which completes the proof. $\square$

Theorem 2.7. *(Functional Donoho-Elad-Gribonval-Nielsen-Fuchs Sparsity Theorem)*
Let $(\{f_n\}_{n=1}^{\infty}, \{\tau_n\}_{n=1}^{\infty})$ *be an 1-ASF for* $\mathcal{X}$ *such that*

$$|f_n(\tau_n)| \geq 1, \quad \forall n \in \mathbb{N}.$$

If $x \in \mathcal{X}$ *can be written as* $x = \theta_\tau c$ *for some* $c \in \ell^1(\mathbb{N})$ *satisfying*

$$\|c\|_0 < \frac{1}{2}\left(1 + \frac{1}{\sup\limits_{n,m\in\mathbb{N}, n\neq m}|f_n(\tau_m)|}\right),$$

then c is the unique solution to Problem 2.2.

Proof. Theorem 2.6 says that c is the unique solution to Problem 2.3. Let $d \in \ell^1(\mathbb{N})$ be such that $x = \theta_\tau d$. We claim that $\|d\|_0 > \|c\|_0$. If this fails, we must have $\|d\|_0 \leq \|c\|_0$. We then have

$$\|d\|_0 < \frac{1}{2}\left(1 + \frac{1}{\sup\limits_{n,m\in\mathbb{N}, n\neq m}|f_n(\tau_m)|}\right).$$

Theorem 2.6 again says that d is also the unique solution to Problem 2.3. Therefore we must have $\|c\|_1 < \|d\|_1$ and $\|c\|_1 > \|d\|_1$ which is a contradiction. Therefore claim holds and we have $\|d\|_0 > \|c\|_0$. $\square$

Corollary 2.8. *Theorem 1.3 follows from Theorems 2.6 and 2.7.*

Proof. Let $\{\tau_j\}_{j=1}^{n}$ be a normalized frame for a Hilbert space $\mathcal{H}$. For each $1 \leq j \leq n$, define

$$f_j : \mathcal{H} \ni h \mapsto f_j(h) := \langle h, \tau_j \rangle \in \mathbb{K}.$$

Then

$$|f_j(\tau_j)| = 1, \quad \forall 1 \leq j \leq n$$

and

$$f_k(\tau_j) = \langle \tau_j, \tau_k \rangle, \quad \forall 1 \leq j, k \leq n.$$

$\square$

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