

Review

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Review

On the Conservation Laws of General Theory of Relativity

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Abstract: This study describes the energy-momentum density tensor of matter. The main focus is on discussing the conservation laws of the general theory of relativity.

Keywords: general theory of relativity; energy-momentum density tensor of the matter; conservation laws

1. Distances and Intervals in General Relativity

In general relativity, there are no constraints in the choice of a coordinate system. For instance, the space coordinates, (x^1, x^2, x^3) , can define the positions of the bodies in space, and time x^0 can be defined by an arbitrary running clock. We will describe how the chart, $x^\mu = (x^0, x^i)$ (for $i = 1, 2, 3$), determines the distance and time intervals.

1.1. Time Intervals

Let τ be the proper time, which is the time measured by the observer at the rest frame. Consider two infinitesimally separated events occurring at the same point in space, then the distance interval of the space-time geometry is

$$ds = c d\tau \tag{1}$$

where $d\tau$ denotes the proper time interval between the two events. Here, we have taken $dx^1 = dx^2 = dx^3 = 0$ since the events occur at the same point in space. In general,

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

Thus, we have

$$c^2 d\tau^2 = g_{00} (dx^0)^2 \tag{3}$$

Or,

$$d\tau = \frac{1}{c} \sqrt{g_{00}} dx^0 \tag{4}$$

Then, the proper time is obtained by the following integral

$$\tau = \frac{1}{c} \int \sqrt{g_{00}} dx^0 \tag{5}$$

These equations indicate that

$$g_{00} > 0 \tag{6}$$

It is important to note that when we say that the signature of the metric is $(+, -, -, -)$, that indicates the sign of three principal values of the three-dimensional space of the metric tensor $g_{\mu\nu}$. A metric without $(-, -, -)$ can not be a metric of a real gravitational field; that is, it does not correspond to the metric of the real space-time. If the condition $g_{00} > 0$ is not fulfilled, it will mean that the

reference frame can not be realised with real bodies. In that case, if $(-, -, -)$ is fulfilled, then a transformation on the coordinates would make $g_{00} > 0$.

1.2. Spatial Distance

In the special theory of relativity, the spatial distance dl was the interval between two infinitesimally separated events occurring at the same time. In general relativity, it is not possible to do the same thing because we can not take $dx^0 = 0$, when ds is defined. That is because in general relativity in the presence of a gravitational field, τ at different points in space has a different dependence on x^0 .

Consider a light signal traveling from A to B and reflected back to A by a mirror located at B , as shown in Figure 1. The signal leaves the point A at $x^0 + (dx^0)_1$, arrives at point B at x^0 , and is reflected by the mirror at that point, and it arrives back in A at $x^0 + (dx^0)_2$. The object is moving in space-time along the geodesic shown in red (see also Figure 1). For an observer at rest at point A , the distance between the points A and B is

$$dl = \frac{cd\tau}{2} \quad (7)$$

where $d\tau$ is the proper time interval that takes for the signal to travel from A to B and back to A . Here, $(dx^0)_{1,2}$ are the roots of the equation $ds = 0$, as described in the following.

First, we can write

$$\begin{aligned} (ds)^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= g_{00}(dx^0)^2 + g_{0i}dx^0 dx^i + g_{i0}dx^i dx^0 + g_{ij}dx^i dx^j \\ &= g_{00}(dx^0)^2 + 2g_{0i}dx^0 dx^i + g_{ij}dx^i dx^j \end{aligned} \quad (8)$$

Requiring that $ds = 0$, and solving it for dx^0 , we get two roots, $(dx^0)_1$ and $(dx^0)_2$, as

$$\begin{aligned} (dx^0)_{1,2} &= \frac{-g_{0i}dx^i \pm \sqrt{(g_{0i}dx^i)^2 - g_{00}g_{ij}dx^i dx^j}}{g_{00}} \\ &= \frac{-g_{0i}dx^i \pm \sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})dx^i dx^j}}{g_{00}} \end{aligned} \quad (9)$$

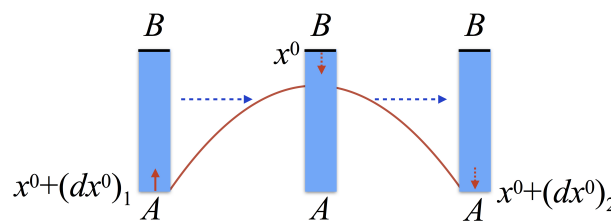


Figure 1. A light signal traveling from A to B and reflected back to A by a mirror located at B . The signal leaves point A at $x^0 + (dx^0)_1$, arrives in point B at x^0 , is reflected back by the mirror, and arrives in A at $x^0 + (dx^0)_2$. The object is moving in the space-time along the geodesic shown in red.

The interval of time passed as the signal travels from A to B and back to A is

$$\begin{aligned} (dx^0)_2 - (dx^0)_1 &= \frac{2\sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})dx^i dx^j}}{g_{00}} \\ &= \frac{2}{g_{00}} \sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})dx^i dx^j} \end{aligned} \quad (10)$$

and the proper time interval is

$$\begin{aligned} d\tau &= \frac{\sqrt{g_{00}}}{c} \frac{2}{g_{00}} \sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})dx^i dx^j} \\ &= \frac{2}{c} \sqrt{\left(\frac{g_{0i}g_{0j}}{g_{00}} - g_{ij}\right)dx^i dx^j} \end{aligned} \quad (11)$$

Substituting Eq. 11 into Eq. 7, we find that

$$\begin{aligned} (dl)^2 &= c^2 \left(\frac{d\tau}{2}\right)^2 \\ &= \left(\frac{g_{0i}g_{0j}}{g_{00}} - g_{ij}\right)dx^i dx^j \end{aligned} \quad (12)$$

If we write the three-dimensional distance dl in the form

$$(dl)^2 = h_{ij}dx^i dx^j \quad (13)$$

then, we will get

$$h_{ij} = \frac{g_{0i}g_{0j}}{g_{00}} - g_{ij} \quad (14)$$

Here, h_{ij} is the three-dimensional metric tensor, which determines the properties of Euclidean space geometry. Note that g_{ij} relates to the real space metric of Riemann geometry, where g_{00} and g_{0i} are the components of the space-time metric. Besides, g_{ij} depends on x^0 , which implies that h_{ij} space metric also depends on x^0 .

Furthermore, $-h_{ij}$ is the inverse of the contravariant three-dimensional g_{ij} metric of Riemann geometry. Using,

$$g^{\mu\sigma}g_{\sigma\nu} = \delta^\mu_\nu \quad (15)$$

we have

$$\begin{aligned} g^{i\sigma}g_{\sigma k} &= g^{ij}g_{jk} + g^{i0}g_{0k} = \delta^i_k \\ g^{i\sigma}g_{\sigma 0} &= g^{ij}g_{j0} + g^{i0}g_{00} = 0 \\ g^{0\sigma}g_{\sigma 0} &= g^{0j}g_{j0} + g^{00}g_{00} = 1 \end{aligned} \quad (16)$$

From Eq. 16, we get

$$g^{i0} = -\frac{g^{ij}g_{j0}}{g_{00}} \quad (17)$$

Substituting Eq. 17 into the first expression of Eq. 16, we find that

$$g^{ij}\left(g_{jk} - \frac{g_{j0}}{g_{00}}g_{0k}\right) = \delta^i_k \quad (18)$$

Or,

$$-g^{ij}h_{jk} = \delta^i_k \quad (19)$$

That is,

$$h_{jk} = -g^{jk} \quad (20)$$

Furthermore,

$$g = -g_{00}h \quad (21)$$

where g is the determinant of $g^{\mu\nu}$ and h is the determinant of h^{ij} .

We introduce a vector $\tilde{\mathbf{h}}$ in the three-dimensional space, with its covariant components defined by

$$\tilde{h}_i = \frac{g_{0i}}{g_{00}}, \quad i = 1, 2, 3 \quad (22)$$

and its contravariant components are

$$\tilde{h}^i = h^{ij}\tilde{h}_j = -g^{0i}, \quad i = 1, 2, 3 \quad (23)$$

Thus,

$$g^{00} = \frac{1}{g_{00}} - \tilde{h}_i\tilde{h}^i \quad (24)$$

Here, the repeating indices up-down indicate summation.

1.3. Synchronisation in General Relativity

Simultaneity in general relativity is related to the synchronisation of the clock readings placed at different points in space. Thus, the goal would be to find a relationship between the readings of those clocks. Consider again the experiment shown in Figure 2. A light signal travels from A to B and is reflected back to A by the mirror placed at point B . The signal leaves the point A at $x^0 + (dx^0)_1$, arrives in point B at x^0 and it is reflected back and it arrives in A at $x^0 + (dx^0)_2$. We consider two objects; one is at rest and the other object is moving in the space-time along the geodesic shown in red. Besides, we assume two clocks are associated with these objects. The clocks will be synchronised, if two signals that are sent simultaneously from A to B from each object are such that when the light signal sent at the object in rest reaches point B , then the signal sent at the moving object reaches point B when the object has moved half-way to the right. Mathematically,

$$x^0 + \Delta x^0 = x^0 + \frac{1}{2}[(dx^0)_1 + (dx^0)_2] \quad (25)$$

Or,

$$\Delta x^0 = \frac{1}{2}[(dx^0)_1 + (dx^0)_2] = -\frac{g_{0i}dx^i}{g_{00}} = \tilde{h}_i dx^i \quad (26)$$

The relation in Eq. 26 is used to synchronise the clocks in any infinitesimal region in space.

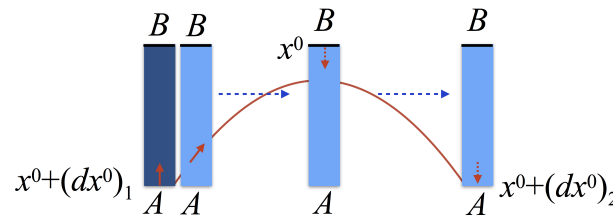


Figure 2. A light signal traveling from A to B and reflected back to A by the mirror placed at B. The signal leaves the point A at $x^0 + (dx^0)_1$, arrives in point B at x^0 and it is reflected back and it arrives in A at $x^0 + (dx^0)_2$. We consider two objects; one is at rest and the object is moving in the space-time along the geodesic shown in red. We associate a clock with each object in space.

2. Energy-Momentum Density Tensor of the Matter

2.1. Generalised Energy-Momentum Density Tensor

In general, the energy-momentum density tensor of a system can be determined from the action function given as

$$I = \int \mathcal{L}(\mathbf{g}^{\mu\nu}, \nabla_\sigma \mathbf{g}^{\mu\nu}) \sqrt{-g} d^4x \quad (27)$$

where $\mathcal{L}(\mathbf{g}^{\mu\nu}, \nabla_\sigma \mathbf{g}^{\mu\nu})$ is Lagrangian density of the system, which is a function of the system's degrees of freedom, namely $\mathbf{g}^{\mu\nu}$ and $\nabla_\sigma \mathbf{g}^{\mu\nu}$. In Eq. 27, the integration is overall three-dimensional space and over the time between any two given times, which forms an infinite region of the Riemann space-time constrained between two hyper-surfaces. Here, $-g$ is the determinant of Riemann space-time metric tensor $g^{\alpha\beta}$.

From Eq. 27, since $\mathcal{L}(\mathbf{g}^{\mu\nu}, \nabla_\sigma \mathbf{g}^{\mu\nu})$ is a function of $\mathbf{g}^{\mu\nu}$ and $\nabla_\sigma \mathbf{g}^{\mu\nu}$, the differential of the action function I is

$$\begin{aligned} \delta I &= \int \delta(\mathcal{L}(\mathbf{g}^{\mu\nu}, \nabla_\sigma \mathbf{g}^{\mu\nu}) \sqrt{-g}) d^4x \\ &= \int \left[\frac{\partial(\mathcal{L} \sqrt{-g})}{\partial \mathbf{g}^{\mu\nu}} \delta \mathbf{g}^{\mu\nu} + \frac{\partial(\mathcal{L} \sqrt{-g})}{\partial (\nabla_\sigma \mathbf{g}^{\mu\nu})} \delta (\nabla_\sigma \mathbf{g}^{\mu\nu}) \right] d^4x \\ &= \int \left[\frac{\partial(\mathcal{L} \sqrt{-g})}{\partial \mathbf{g}^{\mu\nu}} \delta \mathbf{g}^{\mu\nu} + \frac{\partial(\mathcal{L} \sqrt{-g})}{\partial (\nabla_\sigma \mathbf{g}^{\mu\nu})} \nabla_\sigma (\delta \mathbf{g}^{\mu\nu}) \right] d^4x \\ &= \int \left[\frac{\partial(\mathcal{L} \sqrt{-g})}{\partial \mathbf{g}^{\mu\nu}} \delta \mathbf{g}^{\mu\nu} \right. \\ &\quad \left. + \nabla_\sigma \left(\frac{\partial(\mathcal{L} \sqrt{-g})}{\partial (\nabla_\sigma \mathbf{g}^{\mu\nu})} \delta \mathbf{g}^{\mu\nu} \right) - \nabla_\sigma \left(\frac{\partial(\mathcal{L} \sqrt{-g})}{\partial (\nabla_\sigma \mathbf{g}^{\mu\nu})} \right) \delta \mathbf{g}^{\mu\nu} \right] d^4x \\ &= \int \left[\frac{\partial(\mathcal{L} \sqrt{-g})}{\partial \mathbf{g}^{\mu\nu}} \delta \mathbf{g}^{\mu\nu} - \nabla_\sigma \left(\frac{\partial(\mathcal{L} \sqrt{-g})}{\partial (\nabla_\sigma \mathbf{g}^{\mu\nu})} \right) \delta \mathbf{g}^{\mu\nu} \right] d^4x \\ &\quad + \int \nabla_\sigma \left(\frac{\partial(\mathcal{L} \sqrt{-g})}{\partial (\nabla_\sigma \mathbf{g}^{\mu\nu})} \delta \mathbf{g}^{\mu\nu} \right) d^4x \\ &= \int \left[\frac{\partial(\mathcal{L} \sqrt{-g})}{\partial \mathbf{g}^{\mu\nu}} \delta \mathbf{g}^{\mu\nu} - \nabla_\sigma \left(\frac{\partial(\mathcal{L} \sqrt{-g})}{\partial (\nabla_\sigma \mathbf{g}^{\mu\nu})} \right) \delta \mathbf{g}^{\mu\nu} \right] d^4x \end{aligned} \quad (28)$$

where

$$\int \nabla_\sigma \left(\frac{\partial(\mathcal{L} \sqrt{-g})}{\partial (\nabla_\sigma \mathbf{g}^{\mu\nu})} \delta \mathbf{g}^{\mu\nu} \right) d^4x = 0 \quad (29)$$

because the variations of $\delta g^{\mu\nu}$ vanish at the boundary hyper-surfaces; that is, $\delta g^{\mu\nu} = 0$. Therefore, we finally have

$$\delta I = \int \left[\frac{\partial(\mathcal{L}\sqrt{-g})}{\partial g^{\mu\nu}} - \nabla_\sigma \left(\frac{\partial(\mathcal{L}\sqrt{-g})}{\partial(\nabla_\sigma g^{\mu\nu})} \right) \right] \delta g^{\mu\nu} d^4x \quad (30)$$

Now, the following quantity is introduced

$$\frac{1}{2} \sqrt{-g} T_{\mu\nu} = \frac{\partial(\mathcal{L}\sqrt{-g})}{\partial g^{\mu\nu}} - \nabla_\sigma \left(\frac{\partial(\mathcal{L}\sqrt{-g})}{\partial(\nabla_\sigma g^{\mu\nu})} \right) \quad (31)$$

where $T_{\mu\nu}$ is a general form of the so-called energy-momentum density tensor for any physical system (including gravitational and electromagnetic fields) and $\mathcal{L}$ is the Lagrangian density function of that system. Expression in Eq. 31 can further be simplified. Using the Hamilton principal of minimum action $\delta I = 0$ and since $\delta g^{\mu\nu}$ are arbitrary chosen, from Eq. 30, we have

$$\nabla_\sigma \left(\frac{\partial(\mathcal{L}\sqrt{-g})}{\partial(\nabla_\sigma g^{\mu\nu})} \right) - \frac{\partial(\mathcal{L}\sqrt{-g})}{\partial g^{\mu\nu}} = 0 \quad (32)$$

which is the so-called Lagrangian equation.

Furthermore,

$$\begin{aligned} \nabla_\lambda(\sqrt{-g}\mathcal{L}) &= \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} \nabla_\lambda g^{\mu\nu} + \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\nabla_\sigma g^{\mu\nu})} \nabla_\lambda(\nabla_\sigma g^{\mu\nu}) \\ &= \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} \nabla_\lambda g^{\mu\nu} + \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\nabla_\sigma g^{\mu\nu})} \nabla_\sigma(\nabla_\lambda g^{\mu\nu}) \end{aligned} \quad (33)$$

Combining Eq. 32 and Eq. 33, we get

$$\begin{aligned} \nabla_\lambda(\sqrt{-g}\mathcal{L}) &= \nabla_\sigma \left(\frac{\partial(\mathcal{L}\sqrt{-g})}{\partial(\nabla_\sigma g^{\mu\nu})} \right) \nabla_\lambda g^{\mu\nu} + \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\nabla_\sigma g^{\mu\nu})} \nabla_\sigma(\nabla_\lambda g^{\mu\nu}) \\ &= \nabla_\sigma \left(\frac{\partial(\mathcal{L}\sqrt{-g})}{\partial(\nabla_\sigma g^{\mu\nu})} \nabla_\lambda g^{\mu\nu} \right) \end{aligned} \quad (34)$$

Moreover,

$$\nabla_\lambda(\sqrt{-g}\mathcal{L}) = \nabla_\sigma(\sqrt{-g}\mathcal{L}) \nabla_\lambda x^\sigma = \nabla_\sigma(\sqrt{-g}\mathcal{L}) \delta^\sigma_\lambda \quad (35)$$

By equalising both sides of Eq. 34 and Eq. 35, we find another expression for Eq. 31, which gives the generalised energy-momentum density tensor in the following mixed components

$$T^\sigma_\lambda = \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\nabla_\sigma g^{\mu\nu})} \nabla_\lambda g^{\mu\nu} - (\sqrt{-g}\mathcal{L}) \delta^\sigma_\lambda \quad (36)$$

In contravariant form, Eq. 36 can be written as (after multiplying both sides by $g^{\lambda\alpha}$)

$$T^{\sigma\alpha} = \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\nabla_\sigma g^{\mu\nu})} \nabla^\alpha g^{\mu\nu} - g^{\sigma\alpha}(\sqrt{-g}\mathcal{L}) \quad (37)$$

In the covariant form, Eq. 37 is written as

$$T_{\sigma\alpha} = \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\nabla_\sigma g^{\mu\nu})} \nabla_\alpha g^{\mu\nu} - g_{\sigma\alpha}(\sqrt{-g}\mathcal{L}) \quad (38)$$

2.2. Action Function of Gravitational Field

Now, we consider the action function of the gravitational field. For that case,

$$\begin{aligned} \mathfrak{g}^{\mu\nu} &\Rightarrow g^{\mu\nu} \\ \nabla_\alpha \mathfrak{g}^{\mu\nu} &\Rightarrow \nabla_\sigma g^{\mu\nu} \end{aligned} \quad (39)$$

where $g^{\mu\nu}$ is the metric of Riemann space-time.

A coordinate transformation is given as [3]:

$$\eta^\mu = x^\mu + \xi^\mu, \quad \mu = 0, 1, 2, 3 \quad (40)$$

where ξ^μ is a small amount of change. Then, using transformation in Eq. 40, $g^{\mu\nu}$ components transform as

$$\begin{aligned} \tilde{g}^{\mu\nu}(\eta^\tau) &= g^{\sigma\lambda}(x^\tau)(\nabla_\sigma \eta^\mu)(\nabla_\lambda \eta^\nu) \\ &= g^{\sigma\lambda}(x^\tau) \left(\delta_\sigma^\mu + \nabla_\sigma \xi^\mu \right) (\delta_\lambda^\nu + \nabla_\lambda \xi^\nu) \\ &= \left(g^{\mu\lambda}(x^\tau) + g^{\sigma\lambda}(x^\tau) \nabla_\sigma \xi^\mu \right) (\delta_\lambda^\nu + \nabla_\lambda \xi^\nu) \\ &= g^{\mu\nu}(x^\tau) + g^{\mu\lambda}(x^\tau) \nabla_\lambda \xi^\nu + g^{\sigma\nu}(x^\tau) \nabla_\sigma \xi^\mu \\ &\quad + g^{\sigma\lambda}(x^\tau) (\nabla_\sigma \xi^\mu) (\nabla_\lambda \xi^\nu) \end{aligned} \quad (41)$$

where the symmetry of the metric tensor $g^{\sigma\nu} = g^{\nu\sigma}$ is used. Omitting terms of the second order in the derivative of ξ^μ in Eq. 41, we obtain

$$\tilde{g}^{\mu\nu}(\eta^\tau) = g^{\mu\nu}(x^\tau) + g^{\mu\lambda}(x^\tau) \nabla_\lambda \xi^\nu + g^{\sigma\nu}(x^\tau) \nabla_\sigma \xi^\mu \quad (42)$$

Next, a Taylor expansion of $\tilde{g}^{\mu\nu}(\eta^\tau)$ around $\xi^\eta = 0$ is calculated

$$\begin{aligned} \tilde{g}^{\mu\nu}(\eta^\tau) &= \tilde{g}^{\mu\nu}(x^\tau + \xi^\tau) \\ &= \tilde{g}^{\mu\nu}(x^\tau) + (\nabla_\sigma \tilde{g}^{\mu\nu}(x^\tau)) \xi^\sigma \end{aligned} \quad (43)$$

Substituting Eq. 43 into the left-hand side of Eq. 42, and re-arranging it, we will obtain

$$\begin{aligned} \tilde{g}^{\mu\nu}(x^\tau) &= g^{\mu\nu}(x^\tau) - (\nabla_\sigma g^{\mu\nu}(x^\tau)) \xi^\sigma + g^{\mu\lambda}(x^\tau) \nabla_\lambda \xi^\nu \\ &\quad + g^{\sigma\nu}(x^\tau) \nabla_\sigma \xi^\mu \end{aligned} \quad (44)$$

We assume that the coordinates transformation (see Eq. 40) results in a small transformation on the metric tensor; that is, in the contravariant form, we write

$$\tilde{g}^{\mu\nu}(x^\tau) = g^{\mu\nu}(x^\tau) + \delta g^{\mu\nu} \quad (45)$$

and in covariant form as

$$\tilde{g}_{\mu\nu}(x^\tau) = g_{\mu\nu}(x^\tau) - \delta g_{\mu\nu} \quad (46)$$

where $\delta g_{\mu\nu} = -\delta g^{\mu\nu}$ is used. Then, we can obtain that

$$\begin{aligned} \delta g^{\mu\nu}(x^\tau) &= g^{\mu\lambda}(x^\tau) \nabla_\lambda \xi^\nu + g^{\sigma\nu}(x^\tau) \nabla_\sigma \xi^\mu - (\nabla_\sigma g^{\mu\nu}(x^\tau)) \xi^\sigma \\ &= \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu \end{aligned} \quad (47)$$

We re-write the differential of action function in Eq. 30 for the gravitational field as

$$\delta I = \int \left[\frac{\partial(\mathcal{L}\sqrt{-g})}{\partial g^{\mu\nu}} - \nabla_\sigma \left(\frac{\partial(\mathcal{L}\sqrt{-g})}{\partial(\nabla_\sigma g^{\mu\nu})} \right) \right] \delta g^{\mu\nu} d^4x \quad (48)$$

Then, generalised energy-momentum density function of the matter in Riemann space-time of gravitation field is

$$\frac{1}{2} \sqrt{-g} T_{\mu\nu} = \frac{\partial(\mathcal{L}\sqrt{-g})}{\partial g^{\mu\nu}} - \nabla_\sigma \left(\frac{\partial(\mathcal{L}\sqrt{-g})}{\partial(\nabla_\sigma g^{\mu\nu})} \right) \quad (49)$$

Substituting Eq. 49 into Eq. 48, we get

$$\begin{aligned} \delta I &= \frac{1}{2} \int T_{\mu\nu} \sqrt{-g} \delta g^{\mu\nu} d^4x \\ &= -\frac{1}{2} \int T^{\mu\nu} \delta g_{\mu\nu} \sqrt{-g} d^4x \end{aligned} \quad (50)$$

where $\delta g_{\mu\nu} = -\delta g^{\mu\nu}$ is used. Furthermore,

$$\begin{aligned} T^{\mu\nu} \delta g_{\mu\nu} &= g^{\lambda\mu} T_{\lambda\sigma} g^{\nu\sigma} \delta g_{\mu\nu} \\ &= -T_{\lambda\sigma} \delta g^{\lambda\sigma} \\ &= -T_{\lambda\sigma} (\nabla^\lambda \xi^\sigma + \nabla^\sigma \xi^\lambda) \end{aligned} \quad (51)$$

Substituting Eq. 51 into Eq. 50, we get

$$\begin{aligned} \delta I &= \frac{1}{2} \int T_{\lambda\sigma} (\nabla^\lambda \xi^\sigma + \nabla^\sigma \xi^\lambda) \sqrt{-g} d^4x \\ &= \int T_{\lambda\sigma} (\nabla^\lambda \xi^\sigma) \sqrt{-g} d^4x \\ &= \int [\nabla^\lambda (T_{\lambda\sigma} \xi^\sigma) - \xi^\sigma (\nabla^\lambda T_{\lambda\sigma})] \sqrt{-g} d^4x \\ &= \int [\nabla_\mu (g^{\mu\lambda} T_{\lambda\sigma} \xi^\sigma) - \xi^\sigma (\nabla_\mu g^{\mu\lambda} T_{\lambda\sigma})] \sqrt{-g} d^4x \\ &= \int [\nabla_\mu (T^\mu_\sigma \xi^\sigma) - \xi^\sigma (\nabla_\mu T^\mu_\sigma)] \sqrt{-g} d^4x \\ &= - \int \xi^\sigma (\nabla_\mu T^\mu_\sigma) \sqrt{-g} d^4x \end{aligned} \quad (52)$$

where

$$\int \nabla_\mu (T^\mu_\sigma \xi^\sigma) \sqrt{-g} d^4x = 0 \quad (53)$$

since ξ^σ is zero at the boundary hyper-surfaces. Requiring $\delta I = 0$, for the existence of a minimum on I , and knowing that ξ^σ are arbitrary chosen, then

$$\nabla_\mu T^\mu_\sigma = 0 \quad (54)$$

It is important to note that Eq. 54 does not represent a conservation law. In particular, if $T^{\mu\sigma}$ is the energy-momentum density tensor of the matter (including the electromagnetic field), then

$$\nabla_\mu T^{\mu\sigma} \neq 0 \quad (55)$$

since $T^{\mu\sigma}$ does not include the energy-momentum density tensor of the gravitational field, and hence it does not represent a conservation law; only, in the absence of the gravitational field, $\nabla_\mu T^{\mu\sigma} = 0$,

and therefore, it will be a conservation law. In the following sections, we will show how to obtain the conservation law for Einstein's field equations.

Moreover, for a particular choice of Lagrangian density function, corresponding to the physical system of interest, $T_{\mu\nu}$ from Eq. 37 (or Eq. 38) is identical to the energy-momentum density tensor of that system. Note that, depending on the physical system, the choice of $g^{\mu\nu}$ is arbitrary; in general, $g^{\mu\nu}$ and $\nabla_\sigma g^{\mu\nu}$ characterise the degrees of freedom of the physical system. For example, for the electromagnetic field, they are the 4-potential A^μ and $\nabla_\sigma A^\mu$, respectively; on the other hand, for the gravitational field, they describe the metric tensor $g^{\mu\nu}$ and $\nabla_\sigma g^{\mu\nu}$, respectively. In the following discussion, we will describe some of these physical systems.

2.3. System of Discrete Point-like Particles

Consider a system of discrete relativistic non-interacting point-like particles with a rest mass of m_i . The mass density at the position $\mathbf{r}$ can be given as

$$\rho(\mathbf{r}) = \sum_i m_i \delta(\mathbf{r} - \mathbf{r}_i) \quad (56)$$

where $\mathbf{r}_i$ denotes the position of the i -th particle at some reference frame and $\delta(\dots)$ is the delta-function. Furthermore, the 4-momentum density can be defined as

$$p^\mu = \rho u^\mu, \quad \mu = 0, 1, 2, 3 \quad (57)$$

where u^μ is the 4-velocity, $u^\mu = (\gamma c, \gamma \mathbf{v})$; therefore,

$$p^\mu = (\gamma \rho c, \gamma \rho \mathbf{v}) = \left(\frac{\varepsilon}{c}, \mathbf{p} \right) \quad (58)$$

where ε is the relativistic particle energy density and $\mathbf{p}$ is the relativistic three-dimensional momentum density vector.

We propose the following scalar form of the Lagrangian density function:

$$\sqrt{-g} \mathcal{L} = -\frac{1}{\gamma} \rho g_{\alpha\beta} u^\alpha u^\beta = -\frac{1}{\gamma} g_{\alpha\beta} u^\alpha p^\beta \quad (59)$$

where the summation of repeating indices is assumed.

Using Eq. 37, we obtain the energy-momentum density tensor of the system as

$$\mathcal{G}^{\mu\nu} = \frac{1}{\gamma} u^\mu p^\nu \quad (60)$$

The spatial components of the energy-momentum density tensor are given as

$$\begin{aligned} \mathcal{G}^{0i} &= \frac{1}{\gamma} u^0 p^i = \frac{1}{\gamma} \rho u^0 u^i = c p^i = c p_i \\ \mathcal{G}^{ij} &= \frac{1}{\gamma} u^i p^j = \frac{1}{\gamma} \rho u^i u^j \end{aligned} \quad (61)$$

where p_i is the Cartesian component of the relativistic three-dimensional momentum density vector and p^i is the contravariant component.

From Eq. 61, it can be seen that the energy-momentum density tensor is symmetric; that is, $\mathcal{G}^{0i} = \mathcal{G}^{i0}$ and $\mathcal{G}^{ij} = \mathcal{G}^{ji}$. Moreover, the zeroth component (which equals the energy density) is given as

$$\mathcal{G}^{00} = \frac{1}{\gamma} u^0 p^0 = \varepsilon \quad (62)$$

where ε is the relativistic energy density given as

$$\varepsilon = \gamma \rho c^2 = \frac{\rho c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (63)$$

Here,

$$\varepsilon_0 = \rho c^2 \quad (64)$$

is the rest energy density.

Now, consider a system of discrete relativistic non-interacting point-like particles with charges q_i , then the charge density at the position $\mathbf{r}$ will be as

$$\rho_e(\mathbf{r}) = \sum_i q_i \delta(\mathbf{r} - \mathbf{r}_i) \quad (65)$$

The 4-current charge density is defined as

$$J^\mu = (\rho_e c, \rho_e \mathbf{v}), \quad \mu = 0, 1, 2, 3 \quad (66)$$

and its covariant form is

$$J_\mu = (\rho_e c, -\rho_e \mathbf{v}), \quad \mu = 0, 1, 2, 3 \quad (67)$$

We propose the following scalar form of the Lagrangian density function for this system:

$$\sqrt{-g} \mathcal{L} = -\kappa_e \rho_e g_{\alpha\beta} u^\alpha u^\beta = -\kappa_e g_{\alpha\beta} J^\alpha u^\beta \quad (68)$$

Using Eq. 37, we obtain the energy-momentum density tensor as

$$\mathcal{T}^{\mu\nu} = \kappa_e J^\mu u^\nu \quad (69)$$

The spatial components of the energy-momentum density tensor due to the current charges are written as

$$\begin{aligned} \mathcal{T}_q^{0i} &= \kappa_e J^0 u^i = \gamma \kappa_e \rho_e c v^i \\ \mathcal{T}_q^{ij} &= \kappa_e J^i u^j = \gamma \kappa_e \rho_e v^i v^j \end{aligned} \quad (70)$$

where κ_e is a scaling factor [11]

$$\kappa_e = \frac{4\pi\sqrt{G}/s^2}{\kappa} = \frac{c^4}{2\sqrt{G}s^2} \quad (71)$$

where $s = 1$ and its SI unit is m^2/s^2 .

Similarly, see also Eq. 70, the energy-momentum density tensor is symmetric; that is, $\mathcal{T}_q^{0i} = \mathcal{T}_q^{i0}$ and $\mathcal{T}_q^{ij} = \mathcal{T}_q^{ji}$. The zeroth component is given as

$$\mathcal{T}_q^{00} = \gamma \kappa_e \rho_e c^2 \quad (72)$$

Note that in Eq. 72 the term $\mathcal{T}_q^{00}$ equals some relativistic charge energy density, similar to the relativistic energy density of the mass (see also Eq. 62). Therefore, the presence of the charge and current density of the matter in the Riemann space-time deforms the space-time [11], similar to the presence of mass. Here, $\mathcal{T}_q^{\mu\nu}$ are the energy-momentum density tensor components of the deformable charged medium, which depends on the velocity $\mathbf{v}$ and charge density ρ_e .

Besides, if there is an electromagnetic field present in the matter, then we will discuss two different approaches. In the first approach, the energy-momentum density tensor of the electromagnetic field contributes to the total energy-momentum density tensor of the matter, which is given as follows

$$T^{\mu\nu} = \mathcal{G}^{\mu\nu} + \mathcal{T}_q^{\mu\nu} + \mathcal{T}_{\text{EM}}^{\mu\nu} \quad (73)$$

where the space-time curvature of Riemann geometry is determined by the gravitational field.

In Eq. 73, $\mathcal{T}_{\text{EM}}^{\mu\nu}$ is the energy-momentum density tensor of the electromagnetic field, which is calculated by taking Lagrangian density function as

$$\sqrt{-g}\mathcal{L} = -\frac{1}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta} \quad (74)$$

Using Eq. 37, we obtain the energy-momentum density tensor of the electromagnetic field:

$$\begin{aligned} \mathcal{T}^{00} &= \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \\ \mathcal{T}^{0i} &= \frac{1}{c\mu_0}(\mathbf{E} \times \mathbf{B})_i = \frac{1}{c}(\mathbf{E} \times \mathbf{H})_i = \frac{S_i}{c} \\ \mathcal{T}^{ij} &= -\epsilon_0 \left[E_i E_j + c^2 B_i B_j - \frac{1}{2} \delta_{ij} (E^2 + c^2 B^2) \right] \end{aligned} \quad (75)$$

In Eq. 73, $\mathcal{T}_{\text{EM}}^{\mu\nu}$ is the energy-momentum density tensor of electromagnetic field:

$$\mathcal{T}_{\text{EM}}^{\mu\nu} = \frac{1}{\mu_0} g^{\mu\alpha} F_{\alpha\beta} F^{\beta\nu} + \frac{1}{4\mu_0} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \quad (76)$$

In this case, however, Maxwell's laws (such as Gauss's laws for electric and magnetic fields, Faraday's law, and Maxwell-Ampère's law) must be revised due the space-time curvature caused by a gravitational field, which will be described in future work.

In the second approach (see also Refs. [13–15]), one can introduce a *joint* space-time, which is a result of both gravitational and electromagnetic fields, characterised by the joint metric tensor of space-time curvature $\tilde{g}_{\mu\nu}$ and joint manifold connection symbols $\Gamma_{\mu\nu}^\nu$. In this case, the total energy-momentum density tensor of the matter is

$$T^{\mu\nu} = \mathcal{G}^{\mu\nu} + \mathcal{T}_q^{\mu\nu} \quad (77)$$

2.4. Macroscopic Masses

The macroscopic mass is considered a continuous body. The flux of momentum through an element $d\mathbf{A}$ of the surface of the mass equals the force on that surface element. Therefore, the i th component of the force vector acting on the surface element is

$$F_i = \sigma_{ij} dA_j \quad (78)$$

In a reference frame in which a volume element is at rest, using Pascal's law, the pressure P is equal in all directions and it is perpendicular to the surface element, thus

$$\sigma_{ij} dA_j = P dA_i \quad (79)$$

Therefore, the stress tensor is

$$\sigma_{ij} = P \delta_{ij} \quad (80)$$

Thus, in the reference frame in which the macroscopic body is at rest, the Lagrangian density function is suggested as

$$\sqrt{-g}\mathcal{L} = -P \quad (81)$$

and in any arbitrary reference frame as

$$\sqrt{-g}\mathcal{L} = -\left(\rho + \frac{P}{c^2}\right)g_{\alpha\beta}u^\alpha u^\beta - P \quad (82)$$

Using Eq. 37, we find that the energy-momentum density tensor of the macroscopic mass in an arbitrary reference frame is

$$\mathcal{G}^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)u^\mu u^\nu + g^{\mu\nu}P \quad (83)$$

where ρ is the mass density of macroscopic mass distribution and P is the pressure. In Eq. 83, $g^{\mu\nu}$ is the metric tensor of Riemann space-time. The zeroth component is

$$\begin{aligned} \mathcal{G}^{00} &= \left(\rho + \frac{P}{c^2}\right)\gamma^2 c^2 + g^{00}P \\ &= \gamma^2 \rho c^2 + \gamma^2 P + g^{00}P \\ &= \gamma \varepsilon + P(\gamma^2 + g^{00}) \end{aligned} \quad (84)$$

where ε is the relativistic energy density. Eq. 84 indicates that $\mathcal{G}^{00} > 0$, as expected.

The other components are given as

$$\mathcal{G}^{0i} = \left(\rho + \frac{P}{c^2}\right)\gamma^2 c v^i + g^{0i}P = c p^i = \frac{S^i}{c}, \quad i = 1, 2, 3 \quad (85)$$

where p^i is the i th 4-momentum density component

$$p^i = \gamma^2(\varepsilon_0 + P)\frac{v^i}{c^2} + g^{0i}\frac{P}{c} \quad (86)$$

and $\mathbf{S}$ is energy density flow vector:

$$S^i = \gamma^2(\varepsilon_0 + P)v^i + g^{0i}cP \quad (87)$$

where ε_0 is the rest energy density, $\varepsilon_0 = \rho c^2$.

Introducing, the three-dimensional components of the stress density tensor as follows:

$$\sigma^{ij} = \gamma^2(\varepsilon_0 + P)\frac{v^i v^j}{c^2} + P g^{ij} \quad (88)$$

then,

$$\mathcal{G}^{ij} = \gamma^2(\varepsilon_0 + P)\frac{v^i v^j}{c^2} + P g^{ij} = \sigma^{ij} \quad (89)$$

In general, the covariant form of the energy-momentum density tensor of the macroscopic body is

$$\mathcal{G}_{\mu\nu} = \frac{1}{c^2}(\varepsilon_0 + P)u_\mu u_\nu + P g_{\mu\nu} \quad (90)$$

and its contravariant form is

$$\mathcal{G}^{\mu\nu} = \frac{1}{c^2}(\varepsilon_0 + P)u^\mu u^\nu + Pg^{\mu\nu} \quad (91)$$

where the metric of the space-time is the Riemann metric tensor of the curved space-time geometry. In a mixed-tensor form, we can write

$$\mathcal{G}^\mu_\nu = \frac{1}{c^2}(\varepsilon_0 + P)u^\mu u_\nu + P\delta^\mu_\nu \quad (92)$$

For the ideal fluid of non-charged masses, the energy-momentum density tensor is also defined by $\mathcal{G}_{\mu\nu}$ (or $\mathcal{G}^{\mu\nu}$) [4].

3. Conservation Laws in General Relativity

3.1. Conservation Law of Total Energy in General Relativity

The Einstein's field equations are given as [1–3,8,9]

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \quad (93)$$

In Eq. 93, $G_{\mu\nu}$ is Einstein's tensor, given as: [1]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} \quad (94)$$

and

$$\kappa = \frac{8\pi k_G}{c^4} \approx 2.07076412 \times 10^{-43} \frac{\text{s}^2}{\text{kg} \cdot \text{m}} \quad (95)$$

where k_G is the Newton's gravitation constant ($k_G = 6.673 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$) and c speed of light in vacuum. Note that the SI units of κ are

$$[\kappa] = \frac{\text{s}^2}{\text{kg} \cdot \text{m}} \quad (96)$$

The SI units of $T_{\mu\nu}$ (the energy-momentum density tensor of the matter) are

$$[T_{\mu\nu}] = \frac{\text{energy}}{\text{volume}} = \frac{\text{kg}}{\text{s}^2 \cdot \text{m}} \quad (97)$$

and hence, the SI units of $G_{\mu\nu}$ are

$$[G_{\mu\nu}] = \text{m}^{-2} \quad (98)$$

(Note that $g_{\mu\nu}$ is a dimensionless tensor.) Contraction of the Riemann tensor, $R^\sigma_{\alpha\mu\nu}$, produces the so-called Ricci tensor with components defined as [10]

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} \quad (99)$$

By definition, the curvature scalar R is defined as

$$R = R^\mu_\mu \quad (100)$$

The Einstein's field equations can also be written as the mixed-tensor representations:

$$\kappa^{-1}\Lambda\delta^\mu_\nu = T^\mu_\nu - \kappa^{-1}G^\mu_\nu \quad (101)$$

where $T^\mu{}_\nu = g^{\mu\alpha} T_{\alpha\nu}$, $G^\mu{}_\nu = g^{\mu\alpha} G_{\alpha\nu}$, and $\delta^\mu{}_\nu = g^{\mu\alpha} g_{\alpha\nu}$ is the Kronecker tensor:

$$\delta^\mu{}_\nu = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases}$$

In Eq. 93, the cosmological constant Λ is a constant parameter, which depends on the metric ds^2 [7,26]. Furthermore, Λ is proportional to the total energy density concerning the metric under consideration. Therefore, there will be different cosmological constants Λ for different metrics. For instance, Λ is positive concerning Friedmann-Lemaître-Robertson-Walker metric (de Sitter space), see also Ref. [18]; however, Λ is negative for the metric of the celestial body (anti-de Sitter space). Thus, the cosmological constant Λ is connected to the cosmology in the universe under the Friedmann-Lemaître-Robertson-Walker metric. In the case of Friedmann-Lemaître-Robertson-Walker metric, the cosmological constant has a value of $\Lambda = 1.1056 \times 10^{-52} \text{ m}^{-2}$. In contrast, κ is a universal constant, and hence it is the same constant in every reference frame.

Note that these equations, either in the form Eq. 93 or Eq. 101, are valid in any reference frame. For the ideal fluid of non-charged masses, the energy-momentum density tensor is $T_{\mu\nu} = \mathcal{G}_{\mu\nu}$. Here, we can use the reference frame in which the fluid velocity $u^i = 0$ (for $i = 1, 2, 3$), then the energy-momentum mixed-tensor of matter becomes [4]

$$T^\mu{}_\nu = \text{diag}(\gamma^2 \varepsilon_0, P, P, P) \quad (102)$$

Therefore, mixed-tensor $T^\mu{}_\nu$ does not depend on the metric coefficients. Then, the covariant divergences are, as shown in Ref. [6]

$$\begin{aligned} \kappa^{-1} \Lambda \frac{D\delta^\mu{}_\nu}{dx^\mu} &\equiv \kappa^{-1} \Lambda \mathbb{D}_\mu \delta^\mu{}_\nu \\ &= \kappa^{-1} \Lambda \left(\nabla_\mu \delta^\mu{}_\nu + \Gamma^\mu_{\mu\sigma} \delta^\sigma{}_\nu - \Gamma^\sigma_{\mu\nu} \delta^\mu{}_\sigma \right) \\ &= \kappa^{-1} \Lambda \nabla_\mu \delta^\mu{}_\nu = 0 \end{aligned} \quad (103)$$

where ∇_μ is the partial derivative, $\nabla_\mu \delta^\mu{}_\nu = 0$, and $\Gamma^\mu_{\mu\sigma} \delta^\sigma{}_\nu - \Gamma^\sigma_{\mu\nu} \delta^\mu{}_\sigma = 0$. Thus, the conservation law of the total energy is written as:

$$\nabla_\mu \left(T^\mu{}_\nu - \kappa^{-1} G^\mu{}_\nu \right) = 0 \quad (104)$$

Furthermore, the covariant derivatives are

$$\begin{aligned} \kappa^{-1} \Lambda \frac{D\delta^\mu{}_\nu}{dx^\lambda} &\equiv \kappa^{-1} \Lambda \mathbb{D}_\lambda \delta^\mu{}_\nu \\ &= \kappa^{-1} \Lambda \left(\nabla_\lambda \delta^\mu{}_\nu + \Gamma^\mu_{\lambda\sigma} \delta^\sigma{}_\nu - \Gamma^\sigma_{\lambda\nu} \delta^\mu{}_\sigma \right) \\ &= \kappa^{-1} \Lambda \nabla_\lambda \delta^\mu{}_\nu = 0 \end{aligned} \quad (105)$$

because $\nabla_\lambda \delta^\mu{}_\nu = 0$ and $\Gamma^\mu_{\lambda\sigma} \delta^\sigma{}_\nu - \Gamma^\sigma_{\lambda\nu} \delta^\mu{}_\sigma = 0$. Then, the conservation laws of the total energy, momentum, and stress are [6]

$$\nabla_\lambda \left(T^\mu{}_\nu - \kappa^{-1} G^\mu{}_\nu \right) = 0 \quad (106)$$

By integrating Einstein's field equations in a mixed-tensor form (see Eq. 101) over a closed region of volume V in Euclidean space [5,6]:

$$\int_V \kappa^{-1} \Lambda \delta^\mu{}_\nu d^3x = \int_V \left(T^\mu{}_\nu - \kappa^{-1} G^\mu{}_\nu \right) d^3x \quad (107)$$

Then, taking the time derivative of both sides of Eq. 107 gives

$$\frac{\partial}{\partial t} \int_V \kappa^{-1} \Lambda \delta^\mu{}_\nu d^3x = \frac{\partial}{\partial t} \int_V \left(T^\mu{}_\nu - \kappa^{-1} G^\mu{}_\nu \right) d^3x = 0 \quad (108)$$

which are the conservation laws of the total energy, momentum, and stress in a three-dimensional space.

3.2. Other Conservation Laws in General Relativity

Eq. 93 can also be written in the following contravariant form:

$$\kappa^{-1} \Lambda g^{\mu\nu} = T^{\mu\nu} - \kappa^{-1} G^{\mu\nu} \quad (109)$$

which indicates that the left-hand side depends on $g^{\mu\nu}$ and constants.

In Eq. 109, $T^{\mu\nu}$ represents the energy-momentum density tensor of matter. On the other hand,

$$\mathcal{F}^{\mu\nu} = -\kappa^{-1} G^{\mu\nu} \quad (110)$$

is the energy-momentum density tensor of the gravitational field. Moreover,

$$\mathcal{E}^{\mu\nu} = \kappa^{-1} \Lambda g^{\mu\nu} \quad (111)$$

is the total energy-momentum density tensor.

Therefore, the total energy-momentum density tensor $\mathcal{E}^{\mu\nu}$ is equal to the sum of the energy-momentum density tensor of matter $T^{\mu\nu}$ and the energy-momentum density tensor of the gravitational field $\mathcal{F}^{\mu\nu}$:

$$\mathcal{E}^{\mu\nu} = T^{\mu\nu} + \mathcal{F}^{\mu\nu} \quad (112)$$

The covariant derivative of $\mathcal{E}^{\mu\nu}$ is

$$\frac{D\mathcal{E}^{\mu\nu}}{dx^\mu} \equiv \mathbb{D}_\mu \mathcal{E}^{\mu\nu} = \nabla_\mu \mathcal{E}^{\mu\nu} + \Gamma^\mu_{\mu\sigma} \mathcal{E}^{\sigma\nu} - \Gamma^\sigma_{\mu\nu} \mathcal{E}^{\mu\sigma} \quad (113)$$

where $\nabla_\mu \mathcal{E}^{\mu\nu}$ is the partial derivative. Besides,

$$\mathbb{D}_\mu \mathcal{E}^{\mu\nu} = \kappa^{-1} \Lambda \mathbb{D}_\mu g^{\mu\nu} = 0 \quad (114)$$

because the choice of the Riemann metric is such that $\mathbb{D}_\mu g^{\mu\nu} = 0$; κ and Λ are constants. Thus, the conservation law for Riemann curved space-time is defined as

$$\mathbb{D}_\mu \mathcal{E}^{\mu\nu} = 0 \quad (115)$$

On the other hand, in a local reference frame, the covariant derivative of Eq. 113 reduces to [4]

$$\mathbb{D}_\mu \mathcal{E}^{\mu\nu} = \nabla_\mu \mathcal{E}^{\mu\nu} = 0 \quad (116)$$

because the last two terms, which include the Christoffel symbols, vanish in a local reference frame (including the first derivatives of the metric tensor); that is, in a local reference frame, locally, the space-time is flat. Hence, the covariant derivative of the total energy-momentum density tensor equals its partial derivative, and the conservation law is written as

$$\nabla_\mu \mathcal{E}^{\mu\nu} = 0 \quad (117)$$

Eq. 117 shows that the total energy-momentum in a locally flat three-dimensional spatial volume is conserved.

Einstein's field equations in the form (that is, without the cosmological constant Λ) given by

$$G_{\mu\nu} = \frac{8\pi k_G}{c^4} T_{\mu\nu} \quad (118)$$

violate the conservation law in general relativity. However, locally (that is, in short distances), the cosmological constant Λ is very small, $\Lambda \ll 1$, Einstein's field equations given by Eq. 118 are a good approximation. Furthermore, in empty space-time, such as around a star, where there is no matter (and hence $T^{\mu\nu} = 0$), equation $G^{\mu\nu} = 0$ is an excellent approximation locally [5].

It is interesting to note that by combining Eq. 112 and Eq. 115, we will obtain, for the Riemann curved space-time, the following:

$$0 = \mathbb{D}_\mu \mathcal{E}^{\mu\nu} = \mathbb{D}_\mu T^{\mu\nu} + \mathbb{D}_\mu \mathcal{F}^{\mu\nu} \quad (119)$$

That is,

$$\mathbb{D}_\mu T^{\mu\nu} + \mathbb{D}_\mu \mathcal{F}^{\mu\nu} = 0 \quad (120)$$

However, $\mathbb{D}_\mu T^{\mu\nu}$, in general, does not vanish, and hence the conservation law is given in the form shown by Eq. 120. Here, we focus on finding an equivalent quantity of the gravitational field $t^{\mu\nu}$ such that $\mathbb{D}_\mu (T^{\mu\nu} + t^{\mu\nu}) = 0$. Here, as it will be shown in the following discussion, $t^{\mu\nu}$ characterises the energy-momentum density pseudo-tensor of the gravitational field, and hence it will not be zero if the gravitational field exists only.

From Eq. 120, we write

$$\mathbb{D}_\mu \left(T^{\mu\nu} - \kappa^{-1} \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) \right) = 0 \quad (121)$$

where

$$R^{\mu\nu} = \frac{1}{2} g^{\mu\sigma} g^{\nu\lambda} g^{\gamma\tau} \times [\mathbb{D}_\sigma \mathbb{D}_\tau g_{\gamma\lambda} + \mathbb{D}_\gamma \mathbb{D}_\lambda g_{\sigma\tau} - \mathbb{D}_\sigma \mathbb{D}_\tau g_{\gamma\lambda} - \mathbb{D}_\gamma \mathbb{D}_\tau g_{\sigma\lambda}] \quad (122)$$

Using the relation $R = g_{\mu\nu} R^{\mu\nu}$, we obtain

$$\begin{aligned} T^{\mu\nu} + \mathcal{F}^{\mu\nu} &= T^{\mu\nu} - \frac{\kappa^{-1}}{2} \mathbb{D}_\sigma \left[\frac{1}{(-g)} \mathbb{D}_\lambda \left((-g) (g^{\mu\nu} g^{\sigma\lambda} - g^{\mu\sigma} g^{\nu\lambda}) \right) \right] \\ &= T^{\mu\nu} - \frac{1}{(-g)} \mathbb{D}_\sigma h^{\mu\nu\sigma} \end{aligned} \quad (123)$$

where

$$h^{\mu\nu\sigma} = \frac{\kappa^{-1}}{2} \mathbb{D}_\lambda \left((-g) (g^{\mu\nu} g^{\sigma\lambda} - g^{\mu\sigma} g^{\nu\lambda}) \right) \quad (124)$$

and $\mathbb{D}_\lambda g_{\mu\nu} = 0$ is used. Denoting

$$\mathbb{D}_\sigma h^{\mu\nu\sigma} - (-g) T^{\mu\nu} \equiv (-g) t^{\mu\nu} \quad (125)$$

where $t^{\mu\nu} = t^{\nu\mu}$ (symmetric property of $t^{\mu\nu}$), we obtain

$$\mathbb{D}_\sigma h^{\mu\nu\sigma} = (-g) (T^{\mu\nu} + t^{\mu\nu}) \quad (126)$$

where Einstein's equation $G^{\mu\nu} = \kappa T^{\mu\nu}$ is also used. Here,

$$\begin{aligned} (-g)t^{\mu\nu} &= \mathbb{D}_\sigma h^{\mu\nu\sigma} - (-g)T^{\mu\nu} \\ &= (-g)\kappa^{-1} \\ &\times \left[\left(2\Gamma_{\sigma\lambda}^\tau \Gamma_{\tau\delta}^\delta - \Gamma_{\sigma\delta}^\tau \Gamma_{\lambda\tau}^\delta - \Gamma_{\sigma\tau}^\tau \Gamma_{\lambda\delta}^\delta \right) (g^{\mu\sigma} g^{\nu\lambda} - g^{\mu\nu} g^{\sigma\lambda}) \right. \\ &+ g^{\mu\sigma} g^{\lambda\tau} \left(\Gamma_{\sigma\delta}^\nu \Gamma_{\lambda\tau}^\delta + \Gamma_{\lambda\tau}^\nu \Gamma_{\sigma\delta}^\delta - \Gamma_{\tau\delta}^\nu \Gamma_{\sigma\lambda}^\delta - \Gamma_{\sigma\lambda}^\nu \Gamma_{\tau\delta}^\delta \right) \\ &+ g^{\nu\sigma} g^{\lambda\tau} \left(\Gamma_{\sigma\delta}^\mu \Gamma_{\lambda\tau}^\delta + \Gamma_{\lambda\tau}^\mu \Gamma_{\sigma\delta}^\delta - \Gamma_{\tau\delta}^\mu \Gamma_{\sigma\lambda}^\delta - \Gamma_{\sigma\lambda}^\mu \Gamma_{\tau\delta}^\delta \right) \\ &\left. + g^{\sigma\lambda} g^{\tau\delta} \left(\Gamma_{\sigma\tau}^\mu \Gamma_{\lambda\delta}^\nu - \Gamma_{\sigma\lambda}^\mu \Gamma_{\tau\delta}^\nu \right) \right] \end{aligned} \quad (127)$$

Finally, the conservation law is written as

$$\mathbb{D}_\mu [(-g)(T^{\mu\nu} + t^{\mu\nu})] = 0 \quad (128)$$

Here, $t^{\mu\nu}$ is the energy-momentum density pseudo-tensor of the gravitational field and $T^{\mu\nu}$ is the energy-momentum density tensor of the matter. Note that Eq. 128 can also be written in the following form

$$P_{\text{total}}^\mu = \frac{1}{c} \int_S (-g)(T^{\mu\nu} + t^{\mu\nu}) dS_\nu \quad (129)$$

which is the conservation law of the total momentum of the matter and field. If there is no gravitational field, $t^{\mu\nu} = 0$, and we have

$$P_{\text{total}}^\mu = \frac{1}{c} \int_S T^{\mu\nu} dS_\nu \quad (130)$$

which is the 4-momentum of the matter, and it is constant. In Eq. 129 and Eq. 130, the integrations are over an infinite hyper-surface of the three-dimensional space. For constant x^0 , then these integrations are over a three-dimensional volume in space:

$$P_{\text{total}}^\mu = \frac{1}{c} \int_V (-g)(T^{\mu 0} + t^{\mu 0}) dV \quad (131)$$

Here, we can also express the equivalence between the gravitational mass and inertia mass. In other words, the gravitational mass is the mass that determines the gravitational field produced by a body; that means, this is the mass that enters in the expressions of metric tensor $g_{\mu\nu}$ and interval $(ds)^2$ in presence of the gravitational field, or in Newton's law. On the other hand, the inertia mass is responsible for determining the relationship of the momentum and energy of a body (in particular, if m is the rest mass, then $P^0 = mc$; $\varepsilon_0 = mc^2$). Besides, they are equal.

The total angular momentum is defined as:

$$\begin{aligned} \mathcal{M}^{\mu\nu} &= \int_S [x^\mu dP^\nu + x^\nu dP^\mu] \\ &= \frac{1}{c} \int_S (-g) [x^\mu (T^{\nu\sigma} + t^{\nu\sigma}) + x^\nu (T^{\mu\sigma} + t^{\mu\sigma})] dS_\sigma \end{aligned} \quad (132)$$

Since $T^{\mu\sigma}$ and $t^{\mu\sigma}$ are symmetric, then the angular momentum satisfies

$$\mathbb{D}_\mu \mathcal{M}^{\mu\nu} = 0 \quad (133)$$

which indicates that, for a closed system of gravitational bodies, the total angular momentum is conserved in the general theory of relativity.

4. Conclusions

We described the energy-momentum density tensor of the matter. Besides, we focused on the conservation laws of the general theory of relativity. In particular, we discussed conservation laws of total momentum, total angular momentum, and energy-momentum density tensor.

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