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[Yu-Ning Huang](#) *

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Article

On Measuring the Rotationality of Turbulence [†]

Yu-Ning Huang

State Key Laboratory for Turbulence and Complex Systems, College of Engineering, Peking University, Beijing 100871, China; yuninghuang@126.com

[†] In Memoriam Professor K. R. Rajagopal.

Abstract

Motivated and inspired by Truesdell’s seminal article [“Two measures of vorticity,” *Journal of Rational Mechanics and Analysis* **2**, 173–217 (1953)], recently the present author has introduced the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K$ to measure the mean rotationality of turbulence [“On the classical Bradshaw–Richardson number: Its generalized form, properties, and application in turbulence,” *Physics of Fluids* **30**, 125110 (2018)]. In this work, first, within the general framework of the Cauchy equation of motion, we derive the general equation of motion for the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K$ in turbulent flows of incompressible non-Newtonian fluids, which depicts the underlying dynamical character of $\tilde{\mathcal{V}}_K$ and in laminar flows reduces to the general equation of motion for the kinematical vorticity number—the Truesdell number $\mathcal{V}_K$. Second, we obtain an inequality which places the relevant dynamical restriction upon the mean Cauchy stress tensor, the Reynolds stress tensor, and the mean body force density vector in the ensemble-averaged Cauchy equation of motion for turbulence modelling. Moreover, we derive the general Reynolds stress transport equation for turbulence modelling of incompressible non-Newtonian fluids based on Cauchy’s laws of motion, which includes as a special case the classical Reynolds stress transport equation for an incompressible Newtonian fluid derived from the Navier–Stokes equation.

Keywords: kinematical vorticity number (Truesdell number); turbulence Truesdell number; Cauchy’s laws of motion; mean rotationality of turbulence; general Reynolds stress transport equation; turbulence modelling

1. Introduction

In keeping with the spirit of Truesdell’s (1953) seminal work on the measure of vorticity, recently Huang (2018) has introduced the *turbulence kinematical vorticity number* $\tilde{\mathcal{V}}_K$ to measure the mean *rotationality* of turbulence, in the definite sense of the mean deformation of turbulence specified in an article of Huang (2004). The turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K$, which is a scalar field, takes the following form

$$\tilde{\mathcal{V}}_K := \tilde{\mathcal{V}}_K(\mathbf{x}, t) = \frac{\|\bar{\mathbf{W}}\|}{\|\bar{\mathbf{D}}\|} = \frac{\bar{\omega}}{\sqrt{2}\|\bar{\mathbf{D}}\|}, \tag{1}$$

where $\|\bar{\mathbf{W}}\|$ is the magnitude of the mean spin tensor $\bar{\mathbf{W}}$, $\|\bar{\mathbf{D}}\|$ is the magnitude of the mean stretching (rate of strain) tensor $\bar{\mathbf{D}}$, and $\bar{\omega}$ denotes $|\bar{\omega}|$, the magnitude of the mean vorticity $\bar{\omega}$. The turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K$ indicates the amount of mean rotation (mean vorticity) relative to the amount of mean deformation (mean stretching) of turbulence.

It is clear that in laminar flows the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K$ reduces to the *kinematical vorticity number* $\mathcal{V}_K$ introduced by Truesdell (1953), which we shall call the Truesdell number, as done in a recent paper of Huang *et al.* (2025).

Here let us briefly recapitulate the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K$ put forth in the article of Huang (2018). Since in the sense of the *mean deformation* of turbulence a *mean rigid rotation* of turbulence is characterized by $\bar{\mathbf{D}} = \mathbf{0}$ and $\bar{\omega} \neq \mathbf{0}$ (thus $\bar{\mathbf{W}} \neq \mathbf{0}$ and vice versa), while a *mean irrotational*

non-rigid motion of turbulence is characterized by $\bar{\omega} = \mathbf{0}$ and $\bar{\mathbf{D}} \neq \mathbf{0}$, from Eq. (1) it follows that at a given point $\mathbf{x}$ and in the sense of the mean deformation of turbulence, a *mean rotational motion* of turbulence is instantaneously mean rigid *if and only if* $\tilde{\mathcal{V}}_K = \infty$, while a *mean non-rigid motion* of turbulence is instantaneously mean irrotational *if and only if* $\tilde{\mathcal{V}}_K = 0$.

Therefore, the *turbulence kinematical vorticity number* $\tilde{\mathcal{V}}_K$ is a dimensionless number that measures the *mean rotationality* of turbulence, ranging from 0 to ∞ . Indeed, for a *mean irrotational non-rigid motion* of turbulence at a point $\mathbf{x}$, $\bar{\omega} = \mathbf{0}$ but $\bar{\mathbf{D}} \neq \mathbf{0}$, we have $\tilde{\mathcal{V}}_K = 0$, while for a *mean rigid rotation* of turbulence at a point $\mathbf{x}$, $\bar{\mathbf{D}} = \mathbf{0}$ but $\bar{\omega} \neq \mathbf{0}$, we get $\tilde{\mathcal{V}}_K = \infty$. Hence, the more the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K$ approaches ∞ , the more the turbulence approaches locally a mean rigid rotation at the point $\mathbf{x}$. Clearly, the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K$ becomes *indeterminate* only when the mean velocity gradient $\text{grad } \bar{\mathbf{v}} = \bar{\mathbf{D}} + \bar{\mathbf{W}} = \mathbf{0}$.

In the following sections, we shall derive the general equation of motion for the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K$, investigate its dynamical character and obtain an inequality which places dynamical restriction upon the mean Cauchy stress tensor, the Reynolds stress tensor, and the mean body force density vector in the ensemble-averaged Cauchy equation of motion for turbulence modelling. Accordingly, we derive the general Reynolds stress transport equation for turbulence modelling of complex fluids, which includes as a special case the classical Reynolds stress transport equation derived from the Navier–Stokes equation.

2. The Dynamical Nature of the Turbulence Kinematical Vorticity Number $\tilde{\mathcal{V}}_K$

We shall focus our attention on the turbulence of an incompressible fluid which obeys Cauchy's laws of motion, and derive the general equation of motion for the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K$.

In an inertial frame of reference, Cauchy's first law of motion, i.e., the Cauchy equation of motion, reads

$$\varrho \frac{d\mathbf{v}}{dt} = \text{div } \mathbf{T} + \varrho \mathbf{B}, \quad (2)$$

where $\mathbf{v}(\mathbf{x}, t)$ is the velocity field, $d(\cdot)/dt$ denotes the material time derivative, $\mathbf{T}$ is the Cauchy stress tensor, ϱ is the mass density, and $\mathbf{B} := \mathbf{B}(\mathbf{x}, t)$ is the body force density vector, which may be, but need not be, a random vector field. And, the incompressibility gives

$$\text{div } \mathbf{v} = 0, \quad (3)$$

which is the continuity equation for an incompressible fluid.

Cauchy's second law of motion asserts that the Cauchy stress tensor is symmetric,

$$\mathbf{T} = \mathbf{T}^T, \quad (4)$$

excluding body couples and couple stresses; see Truesdell and Noll (1965).

2.1. The General Equation of Motion for the Turbulence Kinematical Vorticity Number $\tilde{\mathcal{V}}_K$

Taking the ensemble average of the Cauchy equation of motion yields the following ensemble-averaged Cauchy equation of motion:

$$\varrho \frac{D\bar{\mathbf{v}}}{Dt} = \text{div } (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}, \quad (5)$$

where D/Dt denotes the material time derivative associated with the mean velocity field $\bar{\mathbf{v}}(\mathbf{x}, t)$, $\bar{\mathbf{T}}$ is the mean Cauchy stress tensor, $\boldsymbol{\tau} := -\overline{\varrho \mathbf{v}' \otimes \mathbf{v}'}$ is the Reynolds stress tensor where $\mathbf{v}' = \mathbf{v} - \bar{\mathbf{v}}$ is the fluctuating velocity, and $\bar{\mathbf{B}} := \bar{\mathbf{B}}(\mathbf{x}, t)$ is the mean body force density vector. Also, we shall assume that the mean velocity $\bar{\mathbf{v}}(\mathbf{x}, t)$, the mean Cauchy stress tensor $\bar{\mathbf{T}}$, the Reynolds stress tensor $\boldsymbol{\tau}$, and the mean body force density vector $\bar{\mathbf{B}}$ are sufficiently continuously differentiable in space $\mathbf{x}$ and time t jointly.

And, hereinafter, an overbar ($\bar{\cdot}$) or an overline ($\overline{\cdot}$) denotes the ensemble average of ($\cdot$) and a prime ($\cdot$)' represents the fluctuating part.

From the continuity equation (3) it follows that

$$\operatorname{div} \bar{\mathbf{v}} = 0, \quad (6)$$

and

$$\operatorname{div} \mathbf{v}' = 0. \quad (7)$$

Taking the divergence of the ensemble-averaged Cauchy equation of motion (5) and making use of the incompressibility of the fluid, we obtain

$$\varrho (\|\bar{\mathbf{D}}\|^2 - \|\bar{\mathbf{W}}\|^2) = \operatorname{div} [\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}], \quad (8)$$

where $\|\bar{\mathbf{D}}\|^2 = \operatorname{tr}(\bar{\mathbf{D}}\bar{\mathbf{D}}^T)$ and $\|\bar{\mathbf{W}}\|^2 = \operatorname{tr}(\bar{\mathbf{W}}\bar{\mathbf{W}}^T)$.

Combining Eq. (1) with Eq. (8), we prove the following theorem.

Theorem. *In turbulent flows of an incompressible fluid, if at a point $\mathbf{x}$ the mean stretching tensor $\bar{\mathbf{D}} \neq \mathbf{0}$, then the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K(\mathbf{x}, t)$ obeys its dynamical equation of motion:*

$$\varrho (1 - (\tilde{\mathcal{V}}_K)^2) \|\bar{\mathbf{D}}\|^2 = \operatorname{div} [\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}], \quad (9)$$

while if at a point $\mathbf{x}$ the mean stretching tensor $\bar{\mathbf{D}} = \mathbf{0}$, then the dynamical equation of motion for $\tilde{\mathcal{V}}_K(\mathbf{x}, t)$ reduces to

$$-\varrho \|\bar{\mathbf{W}}\|^2 = \operatorname{div} [\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}],$$

indicating that the turbulent flow at the point $\mathbf{x}$ experiences locally a mean rigid body rotation characterized by the mean vorticity $\bar{\boldsymbol{\omega}}$ with magnitude being $|\bar{\boldsymbol{\omega}}| = \sqrt{2} \|\bar{\mathbf{W}}\| = \sqrt{2} \{-\operatorname{div} [\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}] / \varrho\}^{1/2}$.

Remark 1. The general equation of motion (9) for $\tilde{\mathcal{V}}_K(\mathbf{x}, t) \in [0, \infty)$ provides new insights into the dynamical nature of the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K(\mathbf{x}, t)$ in relation to the mean Cauchy stress tensor $\bar{\mathbf{T}}$, the Reynolds stress tensor $\boldsymbol{\tau}$, and the mean body force density vector $\bar{\mathbf{B}}$.

Example 1. In turbulent flows of an incompressible Newtonian (Navier–Stokes) fluid, the mean Cauchy stress tensor $\bar{\mathbf{T}} = -\bar{p}\mathbf{1} + 2\mu\bar{\mathbf{D}}$, where $\bar{p}$ is the mean pressure and μ is the viscosity, and the general equation of motion (9) for the turbulence vorticity number $\tilde{\mathcal{V}}_K$ reduces to

$$\varrho (1 - (\tilde{\mathcal{V}}_K)^2) \|\bar{\mathbf{D}}\|^2 = \operatorname{div} [-\operatorname{grad} \bar{p} + 2\mu \operatorname{div} \bar{\mathbf{D}} + \operatorname{div} \boldsymbol{\tau} + \varrho \bar{\mathbf{B}}]. \quad (10)$$

In view of Eq. (8), firstly, we obtain the following corollary.

Corollary 1. *In turbulent flows of an incompressible fluid, $[\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}]$ is divergence-free, namely, a solenoidal vector, if and only if $\|\bar{\mathbf{D}}\| = \|\bar{\mathbf{W}}\|$.*

Remark 2. By Eq. (1) we note that, when $\bar{\mathbf{D}} = \bar{\mathbf{W}} = \mathbf{0}$, the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K(\mathbf{x}, t) = \|\bar{\mathbf{W}}\| / \|\bar{\mathbf{D}}\|$ is indeterminate. Nevertheless, Corollary 1 asserts that $\tilde{\mathcal{V}}_K = 0/0$ physically implies that $[\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}]$ is a solenoidal vector, the same as in the case of $\tilde{\mathcal{V}}_K = 1$ with $\|\bar{\mathbf{D}}\| = \|\bar{\mathbf{W}}\| \neq 0$.

Now recalling that in the sense of the mean deformation of turbulence a mean rigid rotation of turbulence is characterized by $\bar{\mathbf{D}} = \mathbf{0}$ and $\bar{\boldsymbol{\omega}} \neq \mathbf{0}$ (thus $\bar{\mathbf{W}} \neq \mathbf{0}$ and vice versa), while a mean irrotational non-rigid motion of turbulence is characterized by $\bar{\mathbf{W}} = \mathbf{0}$ and $\bar{\mathbf{D}} \neq \mathbf{0}$, from Eq. (1) it follows that at a given point $\mathbf{x}$ and in the sense of the mean deformation of turbulence, a mean rotational motion of turbulence is instantaneously mean rigid if and only if $\tilde{\mathcal{V}}_K = \infty$, while a mean non-rigid motion of turbulence is instantaneously mean irrotational if and only if $\tilde{\mathcal{V}}_K = 0$.

Hence, by virtue of Eq. (8), we readily arrive at the following result.

Corollary 2. In turbulent flows of an incompressible fluid, there exists a mean rigid rotation at a point $\mathbf{x}$ with $\bar{\mathbf{D}} = \mathbf{0}$ and $\bar{\mathbf{W}} \neq \mathbf{0}$ if and only if the turbulence vorticity number $\tilde{\mathcal{V}}_K(\mathbf{x}, t) = \infty$, i.e.,

$$-\varrho \|\bar{\mathbf{W}}\|^2 = \operatorname{div} [\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}], \quad (11)$$

while there exists a mean irrotational non-rigid motion at a point $\mathbf{x}$ with $\bar{\mathbf{W}} = \mathbf{0}$ and $\bar{\mathbf{D}} \neq \mathbf{0}$ if and only if the turbulence vorticity number $\tilde{\mathcal{V}}_K = 0$, i.e.,

$$\varrho \|\bar{\mathbf{D}}\|^2 = \operatorname{div} [\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}]. \quad (12)$$

Remark 3. By Corollary 2 it is clear that, in turbulent flows of an incompressible fluid, $\tilde{\mathcal{V}}_K(\mathbf{x}, t) = \infty$ physically implies that the vector field $[\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}]$ forms locally a *sink* at the point $\mathbf{x}$ with its flux intensity being measured by $-\varrho \|\bar{\mathbf{W}}\|^2$, while $\tilde{\mathcal{V}}_K(\mathbf{x}, t) = 0$ physically implies that the vector field $[\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}]$ forms locally a *source* at the point $\mathbf{x}$ with its flux intensity being measured by $\varrho \|\bar{\mathbf{D}}\|^2$.

Example 2. In the case of turbulent flows of an incompressible Newtonian (Navier–Stokes) fluid, when at a point $\mathbf{x}$ the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K(\mathbf{x}, t) = \infty$, i.e., $\bar{\mathbf{D}} = \mathbf{0}$, $\bar{\mathbf{W}} \neq \mathbf{0}$, and in general $\operatorname{div} \bar{\mathbf{D}} \neq \mathbf{0}$, Eq. (11) reduces to

$$-\varrho \|\bar{\mathbf{W}}\|^2 = \operatorname{div} [-\operatorname{grad} \bar{p} + 2\mu \operatorname{div} \bar{\mathbf{D}} + \operatorname{div} \boldsymbol{\tau} + \varrho \bar{\mathbf{B}}], \quad (13)$$

while when at a point $\mathbf{x}$ the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K(\mathbf{x}, t) = 0$, i.e., $\bar{\mathbf{W}} = \mathbf{0}$, $\bar{\mathbf{D}} \neq \mathbf{0}$, Eq. (12) reduces to

$$\varrho \|\bar{\mathbf{D}}\|^2 = \operatorname{div} [-\operatorname{grad} \bar{p} + 2\mu \operatorname{div} \bar{\mathbf{D}} + \operatorname{div} \boldsymbol{\tau} + \varrho \bar{\mathbf{B}}]. \quad (14)$$

Remark 4. In view of the general equation of motion for the turbulence *kinematical* vorticity number $\tilde{\mathcal{V}}_K(\mathbf{x}, t)$ just derived, which is a field equation that depicts the *dynamical* character of $\tilde{\mathcal{V}}_K$ in turbulent flows of an incompressible fluid and gives every $\tilde{\mathcal{V}}_K(\mathbf{x}, t) \in [0, \infty)$ its corresponding physical meaning, we are justified to drop the adjective “kinematical” and simply call $\tilde{\mathcal{V}}_K$ the *turbulence vorticity number*.

2.2. The Dynamical Restriction upon the Mean Cauchy Stress, the Reynolds Stress, and the Mean Body Force in Turbulence Modelling

Noting that in turbulence modelling the mean Cauchy stress tensor $\bar{\mathbf{T}}$, which in general contains *unknown terms* as we shall exemplify below through three typical incompressible non-Newtonian fluids, and the Reynolds stress tensor $\boldsymbol{\tau}$ must be modelled in order to close the ensemble-averaged Cauchy equation of motion (5), from Eq. (8) we read off

Corollary 3. In turbulent flows of an incompressible fluid, the mean Cauchy stress tensor $\bar{\mathbf{T}}$, the Reynolds stress tensor $\boldsymbol{\tau}$, and the mean body force density vector $\bar{\mathbf{B}}$ obey the following inequality:

$$\operatorname{div} [\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}] \leq \varrho \|\bar{\mathbf{D}}\|^2. \quad (15)$$

Remark 5. Corollary 3 asserts that in modelling the turbulence of an incompressible fluid, the mean Cauchy stress tensor $\bar{\mathbf{T}}$, the Reynolds stress tensor $\boldsymbol{\tau}$, and the mean body force density vector $\bar{\mathbf{B}}$ are *dynamically restricted* to such tensors and vector, respectively, as render $\operatorname{div} [\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}] \leq \varrho \|\bar{\mathbf{D}}\|^2$, which therefore gives the *upper bound* for numerical simulations of turbulence.

To illustrate the complexity of the *dynamical restriction* upon the mean Cauchy stress tensor $\bar{\mathbf{T}}$, the Reynolds stress tensor $\boldsymbol{\tau}$, and the mean body force density vector $\bar{\mathbf{B}}$ when modelling the turbulence of an incompressible fluid within the framework of the Cauchy’s laws of motion, let us consider here three typical incompressible non-Newtonian fluids, keeping in mind that the mean Cauchy stress tensor $\bar{\mathbf{T}}$ which contains *unknown terms*, together with the Reynolds stress tensor $\boldsymbol{\tau}$, must be modelled in order to close the ensemble-averaged Cauchy equation of motion (5) for turbulence modelling.

(I). An incompressible Reiner–Rivlin fluid.

In an incompressible Reiner–Rivlin fluid, the Cauchy stress tensor

$$\mathbf{T} = -p\mathbf{1} + \alpha\mathbf{D} + \beta\mathbf{D}^2, \quad (16)$$

where p denotes the pressure, $\mathbf{1}$ is the unit tensor, $\mathbf{D}$ is the stretching (rate of strain) tensor, α and β are material constants.

Taking the ensemble average of Eq. (16), we obtain

$$\bar{\mathbf{T}} = -\bar{p}\mathbf{1} + \alpha\bar{\mathbf{D}} + \beta[\overline{(\bar{\mathbf{D}})^2} + \overline{(\mathbf{D}')^2}], \quad (17)$$

where $\bar{\mathbf{D}}$ is the mean stretching tensor, and the fluctuating stretching tensor $\mathbf{D}' := \mathbf{D} - \bar{\mathbf{D}}$.

And the fluctuating Cauchy stress tensor

$$\mathbf{T}' := \mathbf{T} - \bar{\mathbf{T}} = -p'\mathbf{1} + \alpha\mathbf{D}' + \beta[\bar{\mathbf{D}}\mathbf{D}' + \mathbf{D}'\bar{\mathbf{D}}]. \quad (18)$$

In contrast to the ensemble-averaged Navier–Stokes equation in which the mean Cauchy stress tensor takes the simple form $\bar{\mathbf{T}} = -\bar{p}\mathbf{1} + 2\mu\bar{\mathbf{D}}$, here the *unknown term* $\beta\overline{(\mathbf{D}')^2}$ that appears in Eq. (17) of the mean Cauchy stress tensor $\bar{\mathbf{T}}$, together with the Reynolds stress tensor $\boldsymbol{\tau}$, must be modelled so as to close the corresponding ensemble-averaged Cauchy equation of motion (5) for turbulence modelling. Note that Lumley (1964) was the first to investigate the dissipation of turbulent energy in homogeneous isotropic decaying turbulence of an incompressible Reiner–Rivlin fluid.

(II). An incompressible fluid of grade 2.

In an incompressible fluid of grade 2, the Cauchy stress tensor reads

$$\mathbf{T} = -p\mathbf{1} + \mu_0\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \quad (19)$$

where $\mathbf{A}_1 = 2\mathbf{D}$, $\mathbf{A}_2 = d\mathbf{A}_1/dt + \mathbf{A}_1(\text{grad } \mathbf{v}) + (\text{grad } \mathbf{v})^\top \mathbf{A}_1$, μ_0 , α_1 and α_2 are material constants.

It should be noted that Truesdell (1965) showed that a fluid of the second grade may be regarded as a fluid of *convected elasticity*. And, interestingly, in their experimental study of the elastic turbulence of a polymer solution, Groisman and Steinberg (2000) reported that the turbulence of the polymeric fluid is completely characterized by the Weissenberg number, meanwhile the Reynolds number is negligibly small, in striking contrast to the turbulence of a Newtonian (Navier–Stokes) fluid that is characterized by a sufficiently large Reynolds number.

Taking the ensemble average, Eq. (19) becomes

$$\begin{aligned} \bar{\mathbf{T}} = & -\bar{p}\mathbf{1} + \mu_0\bar{\mathbf{A}}_1 + \alpha_1[\overline{D\bar{\mathbf{A}}_1/Dt} + \bar{\mathbf{A}}_1(\text{grad } \bar{\mathbf{v}}) + (\text{grad } \bar{\mathbf{v}})^\top \bar{\mathbf{A}}_1 \\ & + \overline{(\text{grad } \mathbf{A}'_1)\mathbf{v}'} + \bar{\mathbf{A}}'_1(\text{grad } \bar{\mathbf{v}}) + \overline{(\text{grad } \bar{\mathbf{v}})^\top \mathbf{A}'_1}] + \alpha_2[\overline{(\bar{\mathbf{A}}_1)^2} + \overline{(\mathbf{A}'_1)^2}], \end{aligned} \quad (20)$$

where $\bar{\mathbf{A}}_1 = 2\bar{\mathbf{D}}$ and $\mathbf{A}'_1 = 2\mathbf{D}'$. Notice that, in this case, there are four *unknown terms* in $\bar{\mathbf{T}}$ that must be modelled to close the ensemble-averaged Cauchy equation of motion (5).

And the fluctuating Cauchy stress tensor

$$\begin{aligned} \mathbf{T}' = & -p'\mathbf{1} + \mu_0\mathbf{A}'_1 + \alpha_1[\overline{D\mathbf{A}'_1/Dt} + (\text{grad } \bar{\mathbf{A}}_1)\mathbf{v}' \\ & + (\text{grad } \mathbf{A}'_1)\mathbf{v}' - \overline{(\text{grad } \mathbf{A}'_1)\mathbf{v}'} + \bar{\mathbf{A}}'_1(\text{grad } \bar{\mathbf{v}}) + (\text{grad } \bar{\mathbf{v}})^\top \bar{\mathbf{A}}'_1 \\ & + \bar{\mathbf{A}}'_1(\text{grad } \mathbf{v}') - \overline{\bar{\mathbf{A}}'_1(\text{grad } \mathbf{v}')} + \bar{\mathbf{A}}_1(\text{grad } \mathbf{v}') + (\text{grad } \mathbf{v}')^\top \bar{\mathbf{A}}_1 \\ & + (\text{grad } \mathbf{v}')^\top \bar{\mathbf{A}}'_1 - \overline{(\text{grad } \mathbf{v}')^\top \bar{\mathbf{A}}'_1}] + \alpha_2[\bar{\mathbf{A}}'_1\bar{\mathbf{A}}_1 + \bar{\mathbf{A}}_1\bar{\mathbf{A}}'_1]. \end{aligned} \quad (21)$$

Compared to the case of the turbulent incompressible Reiner–Rivlin fluid, here, the mean Cauchy stress tensor $\bar{\mathbf{T}}$ and the fluctuating Cauchy stress tensor $\mathbf{T}'$ are much more involved and complicated.

(III). An incompressible Maxwell fluid.

In an incompressible Maxwell fluid, the Cauchy stress tensor

$$\mathbf{T} = -p\mathbf{1} + \mathbf{T}_E, \quad (22)$$

where the extra stress tensor $\mathbf{T}_E$ is given by

$$\lambda \overset{\circ}{\mathbf{T}}_E + \mathbf{T}_E = 2\mu\mathbf{D}, \quad (23)$$

in which λ is the relaxation time (constant), μ is the viscosity (constant), and the co-rotational time derivative of the extra stress

$$\overset{\circ}{\mathbf{T}}_E = \frac{d\mathbf{T}_E}{dt} + \mathbf{T}_E\mathbf{W} - \mathbf{W}\mathbf{T}_E = \frac{\partial\mathbf{T}_E}{\partial t} + (\text{grad } \mathbf{T}_E)\mathbf{v} + \mathbf{T}_E\mathbf{W} - \mathbf{W}\mathbf{T}_E. \quad (24)$$

Taking the ensemble average of Eqs. (22) and (23) yields, respectively,

$$\bar{\mathbf{T}} = -\bar{p}\mathbf{1} + \bar{\mathbf{T}}_E, \quad (25)$$

and

$$\lambda \left[\frac{\partial \bar{\mathbf{T}}_E}{\partial t} + (\text{grad } \bar{\mathbf{T}}_E)\bar{\mathbf{v}} + \overline{(\text{grad } \mathbf{T}'_E)\mathbf{v}'} + \bar{\mathbf{T}}_E\bar{\mathbf{W}} - \bar{\mathbf{W}}\bar{\mathbf{T}}_E + \overline{(\mathbf{T}'_E\mathbf{W}' - \mathbf{W}'\mathbf{T}'_E)} \right] + \bar{\mathbf{T}}_E = 2\mu\bar{\mathbf{D}}, \quad (26)$$

which may be written

$$\lambda \left[\frac{D\bar{\mathbf{T}}_E}{Dt} + \overline{(\text{grad } \mathbf{T}'_E)\mathbf{v}'} + \bar{\mathbf{T}}_E\bar{\mathbf{W}} - \bar{\mathbf{W}}\bar{\mathbf{T}}_E + \overline{(\mathbf{T}'_E\mathbf{W}' - \mathbf{W}'\mathbf{T}'_E)} \right] + \bar{\mathbf{T}}_E = 2\mu\bar{\mathbf{D}}. \quad (27)$$

By making use of the mean co-rotational time derivative, i.e., $\overset{\circ}{\bar{\mathbf{T}}}_E = D\bar{\mathbf{T}}_E/Dt + \bar{\mathbf{T}}_E\bar{\mathbf{W}} - \bar{\mathbf{W}}\bar{\mathbf{T}}_E$ that is associated with the mean velocity field $\bar{\mathbf{v}}(\mathbf{x}, t)$, Eq. (27) then reads

$$\lambda \overset{\circ}{\bar{\mathbf{T}}}_E + \lambda \left[\overline{(\text{grad } \mathbf{T}'_E)\mathbf{v}'} + \overline{(\mathbf{T}'_E\mathbf{W}' - \mathbf{W}'\mathbf{T}'_E)} \right] + \bar{\mathbf{T}}_E = 2\mu\bar{\mathbf{D}}. \quad (28)$$

The fluctuating Cauchy stress tensor

$$\mathbf{T}' = -p'\mathbf{1} + \mathbf{T}'_E, \quad (29)$$

where the fluctuating extra stress $\mathbf{T}'_E = \mathbf{T}_E - \bar{\mathbf{T}}_E$ is given by

$$\lambda \left[\frac{D\mathbf{T}'_E}{Dt} + \mathbf{T}'_E\bar{\mathbf{W}} - \bar{\mathbf{W}}\mathbf{T}'_E + (\text{grad } \bar{\mathbf{T}}_E)\mathbf{v}' + \bar{\mathbf{T}}_E\mathbf{W}' - \mathbf{W}'\bar{\mathbf{T}}_E + \overline{(\text{grad } \mathbf{T}'_E)\mathbf{v}'} - \overline{(\text{grad } \bar{\mathbf{T}}_E)\mathbf{v}'} + \mathbf{T}'_E\mathbf{W}' - \mathbf{W}'\mathbf{T}'_E - \overline{(\mathbf{T}'_E\mathbf{W}' - \mathbf{W}'\mathbf{T}'_E)} \right] + \mathbf{T}'_E = 2\mu\mathbf{D}'. \quad (30)$$

These equations, being the expressions for the mean and the fluctuating Cauchy stresses respectively, are sufficient to exemplify the complexities and formidable difficulties in modelling the turbulence of an incompressible non-Newtonian fluid, in comparison to modelling the turbulence of an incompressible Newtonian (Navier–Stokes) fluid such as water, which is merely a very special case of the three typical incompressible non-Newtonian fluids given above.

Even in laminar flows of some typical non-Newtonian fluids, it can be very hard to deal with a boundary-value problem of practical interest and/ or physical importance, not to mention the case of the turbulent flows in which all three normal stress differences play a key role in measuring the Weissenberg effect of turbulence, as recently shown by Huang *et al.* (2019). Indeed, the so-called high-Weissenberg-number problem in combination with the change of type of the governing equations often occurs in the numerical simulation of the laminar flow of a non-Newtonian fluid, as had been studied in detail by Joseph *et al.* (1985). In general, to tackle an initial value problem in modelling

turbulence one must solve a horrid set of partial differential equations with the closure models for the mean Cauchy stress tensor $\bar{\mathbf{T}}$ and the Reynolds stress tensor $\boldsymbol{\tau}$ being used. In addition, the general Weissenberg number GN_{We} put forth by Huang *et al.* (2019) has been applied to measure the Weissenberg effect in laminar flows of a number of complex fluids; see Huang *et al.* (2025).

3. The General Equation of Motion for the Truesdell Number $\mathcal{V}_K$

In 1953 Truesdell put forward the *kinematical vorticity number* $\mathcal{V}_K(\mathbf{x}, t)$ to measure the *rotationality* of fluid motions. The kinematical vorticity number $\mathcal{V}_K$, a scalar field, is defined by

$$\mathcal{V}_K := \mathcal{V}_K(\mathbf{x}, t) = \frac{\|\mathbf{W}\|}{\|\mathbf{D}\|} = \frac{\omega}{\sqrt{2}\|\mathbf{D}\|}, \quad (31)$$

where $\|\mathbf{W}\|$ is the magnitude of the spin tensor $\mathbf{W}$, $\|\mathbf{D}\|$ is the magnitude of the stretching (rate of strain) tensor $\mathbf{D}$, and ω denotes $|\boldsymbol{\omega}|$, the magnitude of the vorticity $\boldsymbol{\omega}$. The kinematical vorticity number $\mathcal{V}_K$, namely, the *Truesdell number*, indicates the amount of rotation (vorticity) relative to the amount of deformation (stretching).

Indeed, in an irrotational non-rigid motion, $\boldsymbol{\omega} = \mathbf{0}$ and $\mathbf{D} \neq \mathbf{0}$, the Truesdell number $\mathcal{V}_K = 0$, while in a rigid rotation, $\mathbf{D} = \mathbf{0}$ but $\boldsymbol{\omega} \neq \mathbf{0}$, thus $\mathcal{V}_K = \infty$. Obviously, only when the velocity gradient vanishes will the Truesdell number $\mathcal{V}_K$ fail to exist, being *indeterminate*. Therefore, all possible motions excluding the rigid translation are measured by a numerical degree of *rotationality* on a scale from 0 to ∞ of $\mathcal{V}_K$, amongst which a rigid rotation is the most rotational motion possible. In fact, at a given point $\mathbf{x}$, a rotational motion is instantaneously rigid *if and only if* $\mathcal{V}_K = \infty$, while a non-rigid motion is instantaneously irrotational *if and only if* $\mathcal{V}_K = 0$. For more details, we refer the reader to the work of Truesdell (1953, 1954) and the article by Serrin (1959).

It is straightforward to obtain the following results with physical significance by letting the flow be laminar in the previous section. First, we arrive at

$$\varrho (\|\mathbf{D}\|^2 - \|\mathbf{W}\|^2) = \operatorname{div} [\operatorname{div} \mathbf{T} + \varrho \mathbf{B}], \quad (32)$$

where $\|\mathbf{D}\|^2 = \operatorname{tr}(\mathbf{D}\mathbf{D}^T)$ and $\|\mathbf{W}\|^2 = \operatorname{tr}(\mathbf{W}\mathbf{W}^T)$. And combining Eqs. (31) and (32), accordingly, there follows

Corollary 4. *In flows of an incompressible fluid, if at a point $\mathbf{x}$ the stretching tensor $\mathbf{D} \neq \mathbf{0}$, then the Truesdell number $\mathcal{V}_K(\mathbf{x}, t)$ obeys its dynamical equation of motion:*

$$\varrho (1 - (\mathcal{V}_K)^2) \|\mathbf{D}\|^2 = \operatorname{div} [\operatorname{div} \mathbf{T} + \varrho \mathbf{B}], \quad (33)$$

while if at a point $\mathbf{x}$ the stretching tensor $\mathbf{D} = \mathbf{0}$, then the dynamical equation of motion for the Truesdell number $\mathcal{V}_K(\mathbf{x}, t)$ reduces to

$$-\varrho \|\mathbf{W}\|^2 = \operatorname{div} [\operatorname{div} \mathbf{T} + \varrho \mathbf{B}],$$

indicating that the flow at the point $\mathbf{x}$ experiences locally a rigid body rotation characterized by the vorticity $\boldsymbol{\omega}$ with magnitude being $|\boldsymbol{\omega}| = \sqrt{2} \|\mathbf{W}\| = \sqrt{2} \{-\operatorname{div} [\operatorname{div} \mathbf{T} + \varrho \mathbf{B}] / \varrho\}^{1/2}$.

Remark 6. Corollary 4 asserts that every Truesdell number $\mathcal{V}_K(\mathbf{x}, t) \in [0, \infty)$ corresponds to an equation of motion (33), reflecting its dynamical nature.

Example 3. In the case of an incompressible Newtonian (Navier–Stokes) fluid in which the Cauchy stress tensor $\mathbf{T} = -p\mathbf{1} + 2\mu\mathbf{D}$, where p is the pressure and μ is the viscosity, the *general equation of motion for the Truesdell number* (33) reduces to

$$\varrho (1 - (\mathcal{V}_K)^2) \|\mathbf{D}\|^2 = \operatorname{div} [-\operatorname{grad} p + 2\mu \operatorname{div} \mathbf{D} + \varrho \mathbf{B}]. \quad (34)$$

Moreover, from Eq. (32) there follows

Corollary 5. *In flows of an incompressible fluid, $[\operatorname{div} \mathbf{T} + \varrho \mathbf{B}]$ is divergence-free, namely, a solenoidal vector, if and only if $\|\mathbf{D}\| = \|\mathbf{W}\|$.*

Remark 7. By definition we know that, when $\mathbf{D} = \mathbf{W} = \mathbf{0}$, the Truesdell number $\mathcal{V}_K(\mathbf{x}, t) = \|\mathbf{W}\| / \|\mathbf{D}\|$ becomes *indeterminate*. However, by *Corollary 5* we see at once that in fact the Truesdell number $\mathcal{V}_K = 0/0$ physically implies that $[\operatorname{div} \mathbf{T} + \varrho \mathbf{B}]$ is a solenoidal vector, the same as in the case of the Truesdell number $\mathcal{V}_K = 1$ with $\|\mathbf{D}\| = \|\mathbf{W}\| \neq 0$.

By virtue of Eqs. (31) and (32), we obtain

Corollary 6. *In flows of an incompressible fluid, there exists a rigid rotation at a point $\mathbf{x}$ with $\mathbf{D} = \mathbf{0}$ and $\mathbf{W} \neq \mathbf{0}$ if and only if the Truesdell number $\mathcal{V}_K(\mathbf{x}, t) = \infty$, i.e.,*

$$-\varrho \|\mathbf{W}\|^2 = \operatorname{div} [\operatorname{div} \mathbf{T} + \varrho \mathbf{B}], \quad (35)$$

while there exists an irrotational non-rigid motion at a point $\mathbf{x}$ with $\mathbf{W} = \mathbf{0}$ and $\mathbf{D} \neq \mathbf{0}$ if and only if the Truesdell number $\mathcal{V}_K = 0$, i.e.,

$$\varrho \|\mathbf{D}\|^2 = \operatorname{div} [\operatorname{div} \mathbf{T} + \varrho \mathbf{B}]. \quad (36)$$

Remark 8. By *Corollary 6* it is clear that, in flows of an incompressible fluid, $\mathcal{V}_K(\mathbf{x}, t) = \infty$ physically implies that the vector field $[\operatorname{div} \mathbf{T} + \varrho \mathbf{B}]$ forms locally a *sink* at the point $\mathbf{x}$ with its flux intensity being measured by $-\varrho \|\mathbf{W}\|^2$, while $\mathcal{V}_K(\mathbf{x}, t) = 0$ physically implies that the vector field $[\operatorname{div} \mathbf{T} + \varrho \mathbf{B}]$ forms locally a *source* at the point $\mathbf{x}$ with its flux intensity being measured by $\varrho \|\mathbf{D}\|^2$.

Example 4. In flows of an incompressible Newtonian (Navier–Stokes) fluid, when at a point $\mathbf{x}$ the Truesdell number $\mathcal{V}_K(\mathbf{x}, t) = \infty$, Eq. (35) reduces to

$$-\varrho \|\mathbf{W}\|^2 = \operatorname{div} [-\operatorname{grad} p + 2\mu \operatorname{div} \mathbf{D} + \varrho \mathbf{B}], \quad (37)$$

while when at a point $\mathbf{x}$ the Truesdell number $\mathcal{V}_K(\mathbf{x}, t) = 0$, Eq. (36) reduces to

$$\varrho \|\mathbf{D}\|^2 = \operatorname{div} [-\operatorname{grad} p + 2\mu \operatorname{div} \mathbf{D} + \varrho \mathbf{B}]. \quad (38)$$

Remark 9. On the basis of the general equation of motion, i.e., Eq. (33), for the Truesdell number $\mathcal{V}_K(\mathbf{x}, t) \in [0, \infty)$, we have shown that its dynamical nature is key to the understanding and interpretation of its physical meaning.

4. The General Reynolds Stress Transport Equation for Turbulence Modelling Based on Cauchy's Laws of Motion

The Reynolds stress tensor $\boldsymbol{\tau}$, which plays a vital role in the general equation of motion (9) for the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K$, obeys its evolution equation, i.e., the so-called Reynolds stress transport equation in the literature.

In fact, Chou (1940) was the first to derive the classical Reynolds stress transport equation for the turbulence of an incompressible Newtonian (Navier–Stokes) fluid from the evolution equation for the fluctuating velocity $\mathbf{v}'$, within the framework of the Navier–Stokes equation; see also Pope (2000). As the starting point of the so-called second-order (moment) closure for modelling the turbulent flows of an incompressible Navier–Stokes fluid, the classical Reynolds stress transport equation takes the following form

$$\begin{aligned} \frac{D\boldsymbol{\tau}}{Dt} + \bar{\mathbf{L}}\boldsymbol{\tau} + \boldsymbol{\tau}\bar{\mathbf{L}}^T = & - \left[\overline{(-\operatorname{grad} p' + 2\mu \operatorname{div} \mathbf{D}') \otimes \mathbf{v}'} + \overline{\mathbf{v}' \otimes (-\operatorname{grad} p' + 2\mu \operatorname{div} \mathbf{D}')} \right] \\ & + \varrho \operatorname{div} (\overline{\mathbf{v}' \otimes \mathbf{v}' \otimes \mathbf{v}'}), \end{aligned} \quad (39)$$

where $\bar{\mathbf{L}} = \operatorname{grad} \bar{\mathbf{v}}$, $p' = p - \bar{p}$ is the fluctuating pressure, ϱ is the mass density, μ is the viscosity of a Newtonian (Navier–Stokes) fluid, and the fluctuating stretching tensor $\mathbf{D}' = [\operatorname{grad} \mathbf{v}' + (\operatorname{grad} \mathbf{v}')^T]/2$, noting that here $\operatorname{div} \bar{\mathbf{v}} = \operatorname{div} \mathbf{v}' = 0$.

Now, within the framework of the Cauchy equation of motion (2), we are ready to derive the *general Reynolds stress transport equation* for turbulence modelling of complex fluids, of which the classical Reynolds stress transport equation (39) is merely a special case.

Subtracting Eq. (5) from Eq. (2) yields the evolution equation for the fluctuating velocity $\mathbf{v}'$:

$$\varrho \left(\frac{D\mathbf{v}'}{Dt} + \bar{\mathbf{L}}\mathbf{v}' \right) = \text{div } \mathbf{T}' + \varrho \text{div} (\overline{\mathbf{v}' \otimes \mathbf{v}'} - \mathbf{v}' \otimes \mathbf{v}') + \varrho \mathbf{B}'. \quad (40)$$

where $\bar{\mathbf{L}} = \text{grad } \bar{\mathbf{v}}$ is the gradient of the mean velocity, $\mathbf{T}' = \mathbf{T} - \bar{\mathbf{T}}$ is the fluctuating Cauchy stress tensor, and $\mathbf{B}' = \mathbf{B} - \bar{\mathbf{B}}$ is the fluctuating body force density vector.

Taking the ensemble average of $(\mathbf{v}' \otimes \text{Eq. (40)} + \text{Eq. (40)} \otimes \mathbf{v}')$ and making use of the incompressibility of the fluid, we then obtain the *general Reynolds stress transport equation* as follows:

$$\begin{aligned} \frac{D\boldsymbol{\tau}}{Dt} + \bar{\mathbf{L}}\boldsymbol{\tau} + \boldsymbol{\tau}\bar{\mathbf{L}}^T = & - [\overline{(\text{div } \mathbf{T}') \otimes \mathbf{v}'} + \overline{\mathbf{v}' \otimes (\text{div } \mathbf{T}')}] - \varrho (\overline{\mathbf{B}' \otimes \mathbf{v}'} + \overline{\mathbf{v}' \otimes \mathbf{B}'}) \\ & + \varrho \text{div} (\overline{\mathbf{v}' \otimes \mathbf{v}' \otimes \mathbf{v}'}), \end{aligned} \quad (41)$$

which includes as a special case the classical Reynolds stress transport equation (39) by letting the fluctuating Cauchy stress tensor $\mathbf{T}' = -p'\mathbf{1} + 2\mu\mathbf{D}'$ and neglecting the fluctuating body force density vector by setting $\mathbf{B}' = \mathbf{0}$.

Example 5. By the general Reynolds stress transport equation (41) and Eq. (18), we find that the Reynolds stress transport equation for the turbulence of an incompressible Reiner–Rivlin fluid (16) takes the form

$$\begin{aligned} \frac{D\boldsymbol{\tau}}{Dt} + \bar{\mathbf{L}}\boldsymbol{\tau} + \boldsymbol{\tau}\bar{\mathbf{L}}^T = & - [\overline{(-\text{grad } p' + \alpha \text{div } \mathbf{D}' + \beta \text{div} [\bar{\mathbf{D}}\mathbf{D}' + \mathbf{D}'\bar{\mathbf{D}}]) \otimes \mathbf{v}'} \\ & + \overline{\mathbf{v}' \otimes (-\text{grad } p' + \alpha \text{div } \mathbf{D}' + \beta \text{div} [\bar{\mathbf{D}}\mathbf{D}' + \mathbf{D}'\bar{\mathbf{D}}])}] \\ & - \varrho (\overline{\mathbf{B}' \otimes \mathbf{v}'} + \overline{\mathbf{v}' \otimes \mathbf{B}'}) + \varrho \text{div} (\overline{\mathbf{v}' \otimes \mathbf{v}' \otimes \mathbf{v}'}). \end{aligned} \quad (42)$$

Example 6. In the case of the turbulence of an incompressible Maxwell fluid with the fluctuating Cauchy stress tensor $\mathbf{T}'$ being given by Eq. (29), the general Reynolds stress transport equation (41) becomes

$$\begin{aligned} \frac{D\boldsymbol{\tau}}{Dt} + \bar{\mathbf{L}}\boldsymbol{\tau} + \boldsymbol{\tau}\bar{\mathbf{L}}^T = & - [\overline{(-\text{grad } p' + \text{div } \mathbf{T}'_E) \otimes \mathbf{v}'} + \overline{\mathbf{v}' \otimes (-\text{grad } p' + \text{div } \mathbf{T}'_E)}] \\ & - \varrho (\overline{\mathbf{B}' \otimes \mathbf{v}'} + \overline{\mathbf{v}' \otimes \mathbf{B}'}) + \varrho \text{div} (\overline{\mathbf{v}' \otimes \mathbf{v}' \otimes \mathbf{v}'}), \end{aligned} \quad (43)$$

where the fluctuating extra stress tensor $\mathbf{T}'_E$ is given by Eq. (30) which contains the mean extra stress tensor $\bar{\mathbf{T}}_E$ that is described by Eq. (28) which, too, involves the fluctuating extra stress tensor $\mathbf{T}'_E$ and its gradient $\text{grad } \mathbf{T}'_E$, making the modelling of the Reynolds stress tensor $\boldsymbol{\tau}$ extremely complicated and difficult.

On the basis of the *general Reynolds stress transport equation*, we can further investigate the *natural viscosity of turbulence* of an incompressible non-Newtonian fluid, using the same approach as that adopted in the article of Huang *et al.* (2003) on the natural viscosity of turbulence of an incompressible Newtonian (Navier–Stokes) fluid. Lately, in the sense of Truesdell's (1964) research on the natural time of a simple fluid, the *natural time of turbulence* of the incompressible Navier–Stokes fluid has been calculated and the Weissenberg effect (normal-stress effect) of turbulence has been investigated by Huang *et al.* (2019). Naturally, further researches may be extended to various kinds of incompressible non-Newtonian fluids.

5. Conclusion

After the classical researches of Truesdell (1953, 1954) on the measure of vorticity, it might seem that little remained to be learned, to be further studied, and to be generalized. But such is not the case. Recently, based on the seminal pioneering work of Truesdell (1953) in which the kinematical vorticity number $\mathcal{V}_K$ was introduced, Huang (2018) put forth the *turbulence kinematical vorticity number* $\tilde{\mathcal{V}}_K(\mathbf{x}, t)$ to measure the *mean rotationality of turbulence* in the definite sense of the *mean deformation of*

turbulence. Not long ago, Huang *et al.* (2025) have shown that the Truesdell number $\mathcal{V}_K(\mathbf{x}, t) > 0$ in a neighborhood of some spatial point $\mathbf{x}$ is a *necessary condition* for the occurrence of the Weissenberg effect in complex fluids.

In the present work, we have further shown that the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K$ actually bears its own dynamical character, not merely a kinematical dimensionless number, as clearly manifested in the *general equation of motion* for $\tilde{\mathcal{V}}_K(\mathbf{x}, t)$:

$$\varrho (1 - (\tilde{\mathcal{V}}_K)^2) \|\bar{\mathbf{D}}\|^2 = \operatorname{div} [\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}], \quad (44)$$

or, equivalently,

$$\tilde{\mathcal{V}}_K(\mathbf{x}, t) = \frac{\|\bar{\mathbf{W}}\|}{\|\bar{\mathbf{D}}\|} = \left(1 - \frac{\operatorname{div} [\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}]}{\varrho \|\bar{\mathbf{D}}\|^2} \right)^{1/2}. \quad (45)$$

Now let us recall the *dynamical vorticity number* $\mathcal{V}_D$ introduced by Truesdell in 1953 (see also his celebrated 1954 treatise entitled **The Kinematics of Vorticity**), which is a proper measure of the dynamical importance of vorticity and within the framework of the Cauchy equation of motion takes the following form

$$\mathcal{V}_D(\mathbf{x}, t) := \frac{|\mathbf{w} \times \mathbf{v}|}{\partial \mathbf{v} / \partial t + \frac{1}{2} \operatorname{grad} (\mathbf{v} \cdot \mathbf{v})} = \frac{\varrho |\mathbf{w} \times \mathbf{v}|}{|\operatorname{div} \mathbf{T} + \varrho \mathbf{B}|} = \frac{\varrho |(\operatorname{curl} \mathbf{v}) \times \mathbf{v}|}{|\operatorname{div} \mathbf{T} + \varrho \mathbf{B}|}. \quad (46)$$

Hence, by comparing Eq. (45) with Eq. (46), we conclude that the *turbulence kinematical vorticity number* $\tilde{\mathcal{V}}_K$, which involves $[\operatorname{div} (\bar{\mathbf{T}} + \boldsymbol{\tau}) + \varrho \bar{\mathbf{B}}]$, is of *dynamical nature* similar to that of the *dynamical vorticity number* $\mathcal{V}_D$ that involves $|\operatorname{div} \mathbf{T} + \varrho \mathbf{B}|$.

Indeed, by virtue of the general equation of motion for $\tilde{\mathcal{V}}_K$, it is evident that the turbulence kinematical vorticity number $\tilde{\mathcal{V}}_K$, which I propose here to call the *turbulence Truesdell number*, not only *kinematically* but all the more importantly, *dynamically* measures the *mean rotationality* of the turbulence of any incompressible fluid that obeys Cauchy's laws of motion, be it a Newtonian (Navier–Stokes) fluid or a non-Newtonian fluid.

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