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Article

Non-Local Conformable Differential Inclusions Generated by Semigroups of Linear Bounded Operators or by Sectorial Operators with Impulses in Banach Spaces

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Abstract: This paper aims to explore the sufficient conditions for assuring that, the set of mild solutions to two types of non-local semilinear fractional differential inclusions involving the conformable derivative, in the existence of non-instantaneous impulses, is not empty and compact. We will consider the case when the linear part in the studied problem is the infinitesimal generator of a C_0 - semigroup or a sectorial operator. We give the definition of mild solutions, and then, by using appropriate fixed point theorems for multi-valued functions and the properties of both the conformable derivative, and the measure of noncompactness, we achieve to our findings. Since the most of the known fractional derivatives do not satisfy many basic properties that usual derivatives have, the conformable derivative is introduced in a previous paper, and it is shows that it is the most natural definition. Therefore, many works have been done on differential equation with the conformable. But, works on semilinear differential inclusions are not reported until now. We will do not assume that the semigroup generated by the linear term is not compact,also, we will examine the case when the values of the multi-valued function are convex, also nonconvex. So, our work is novel, and interested. We give examples of the application of our theoretical results.

Keywords: differential inclusions; infinitesimal generator of a C_0 - semigroup; sectorial operator, conformable fractional derivative; instantaneous and non-instantaneous impulses; mild solutions; measure of noncompactness

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1. Introduction

Differential inclusions take the form:

$$\zeta(\delta) \in G(\delta, \zeta(\delta)), a.e.,$$

where G is a multi-valued function, and is therefore a generalization of differential equations. Many authors have been studied numerous of differential inclusions [1-4]. Impulsive differential problems are proper models for tell of processes which at certain time change their situation speedily. This processes can't be described by the classical differential equations. If the effect of this change is instantaneous, it is termed instantaneous impulses, but if it remains stable over a period of time, it is called non-instantaneous impulses, for example to non-instantaneous impulses, the consequence of institute medications into the bloodstream and their soaking up the body. To equip the reader with the application of non-instantaneous impulses in physics, biology, population dynamics, ecology and pharmacokinetics, we refer to [5-7]. In [8-10], there are many studies on differential inclusions with non-instantaneous impulses.

Differential equations and inclusions containing fractional derivatives have many applications in various branches of science, engineering and medicine [11-14], which indicates the importance

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of fractional derivatives. Therefore, many researchers pay attention to giving different concepts to fractional derivatives, such as Riemann-Liouville, Caputo, Hilfer, Katugampola, Hadamard and Atangana–Baleanu. All known fractional derivatives, except the conformable fractional derivative was introduced by Khalil et al. [15], do not satisfy many basic properties of the usual derivative, such as the product rule, quotient rule, mean value theorem, chain rule and Taylor power series expansion. Therefore, the conformable fractional derivative is the most natural fractional derivative. For this reason, many researchers have shown interest in exploring more properties of the conformable fractional derivative and studying differential equations involving it. In [16-19], the conformable fractional derivative properties are given, while in [20-22] some of its applications are given. Nonlocal telegraph equations with the conformable fractional derivative are considered in [23]. Meng et al. [24] looked for the existence of external iteration solutions to conformable fractional differential equations. Tajadodi et al. [25] treated with the exact solution to a nonlinear differential equation involving the conformable derivative.

In [26-28], there are other findings on differential equations with conformable derivative.

Let $\alpha \in (0,1]$, $\mho$ be a Banach space, $\Upsilon = [0,b]$, A is the infinitesimal generator of a C_0 - semigroup, $\{T(\delta) : \delta \geq 0\}$, on $\mho$, $f : \Upsilon \times \mho \to \mho$ is a single-valued function, $g : \mho \to \mho$ and $\zeta_0 \in \mho$ be a fixed point. Without assuming the compactness of the family $\{T(\delta) : \delta \geq 0\}$, Bouaouid et al.[29] proved, under the condition that g is continuous and compact, the existence of mild solutions to the non-local conformable fractional semilinear differential equations:

$$\begin{cases} \frac{d^\alpha}{d\zeta^\alpha}\zeta(\delta) = A\zeta(\delta) + f(\delta,\zeta(\delta)),\ \delta \in \Upsilon, \\ \zeta(0) = \zeta_0 + g(\zeta), \end{cases} \tag{1}$$

where $\frac{d^\alpha}{d\zeta^\alpha}\zeta(\delta)$ is the conformable derivative of the function ζ at the point δ.

Bouaouid et al.[30] studied the existence of mild solutions to the following non-local conformable fractional semilinear differential equations:

$$\begin{cases} \frac{d^\alpha}{d\zeta^\alpha}\zeta(\delta) = B\zeta(\delta) + f(\delta,\zeta(\delta)), \delta \in \Upsilon, \\ \zeta(0) = \zeta_0 + g(\zeta), \end{cases} \tag{2}$$

where B is a sectorial operator on $\mho$ that generates a strongly analytic semigroup $\{K(\delta) : \delta \geq 0\}$.To achieve their goal, Bouaouid and associates authors imposed the compactness of both $K(\delta)$, $\forall\delta > 0$ and g.

Motivated by the above works, especially that done in [29,30], we will present in this paper, six existence results for mild solutions to two types of non-local semilinear differential inclusions containing the conformable derivative in the presence of non-instantaneous impulses in Banach spaces. In fact, we will generalize the work in [29,30] and prove three existence results for mild solutions to both problems (1) and (2), when f is replaced with a multi-valued function Θ and there are non-instantaneous impulses in the system. Unlike [30], we will not assume the compactness of $K(\delta)$, $\forall\delta > 0$. In order to formulate the problems that we study, let $0 = s_0 < b_1 \leq s_1 < b_2 \leq s_2 < .... < s_r < b_{r+1} = b$, $\Theta : \Upsilon \to 2^{\mho} - \{\phi\}$ is a multi-valued function, $g_i : [b_i, s_i] \times \mho \to \mho$ and $\Lambda_{1,r} = \{1, 2, ..r\}$.

Consider the following two non-local semilinear fractional differential inclusions involving the conformable derivative, in the presences of non-instantaneous impulses:

$$\begin{cases} \frac{d^\alpha}{d\zeta^\alpha}\zeta(\delta) \in A\zeta(\delta) + \Theta(\delta,\zeta(\delta)), a.e.\ \delta \in \cup_{i=0}^{i=r}(s_i, b_{i+1}], \\ \zeta(\delta) = g_i(\delta,\zeta(b_i^-)), \delta \in (b_i, s_i];\ i \in \Lambda_{1,r}, \\ \zeta(0) = \zeta_0 + g(\zeta), \end{cases} \tag{3}$$

and

$$\begin{cases} \frac{d^\alpha}{d\zeta^\alpha}\zeta(\delta) \in B\zeta(\delta) + \Theta(\delta,\zeta(\delta)), a.e.\ \delta \in \cup_{i=0}^{i=r}(s_i, b_{i+1}], \\ \zeta(\delta) = g_i(\delta,\zeta(b_i^-)), \delta \in (b_i, s_i];\ i \in \Lambda_{1,r}, \\ \zeta(0) = \zeta_0 + g(\zeta), \end{cases} \tag{4}$$

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We will explore the sufficient conditions for assuring that, $S_3(A,\Theta)$ and $S_4(B,\Theta)$ are not empty and compact in $PC(\Upsilon,\mho)$, where $S_3(A,\Theta)$ and $S_4(B,\Theta)$ are the set of mild solutions to problems (3) and (4) respectively.

Remark 1. *1-This work is novel because this is the first time to consider non-local semilinear fractional differential inclusions involving the conformable derivative, in the presences of non-instantaneous impulses in infinite dimensional Banach spaces, where the linear part is the infinitesimal generator of a C_0- semigroup (not necessary compact) or a sectorial operator generates an analytic semigroup(not necessary compact) .Moreover, we will consider the case when the values of Θ are convex as well as nonconvex.*

2-This work is interesting because our studied problems involving the conformable derivative which possess many properties like the usual derivative and that is not verified for all the other known fractional derivatives.

The significant contributions are the following:

1-The representation of mild solutions to Problems (3) and (4) are formulated (Definitions 4 and 5).

2- We have extended the problem (1), studied by Bouaouid et al. [29], to the case when the single-valued f is replaced with a multivalued function Θ and in the presence of non-instantaneous impulsive effects (Problem 3). In fact, three existence results of mild solution to Problems (3) are given (Theorems 1,2 and 3). In Theorem1, the values of Θ are not empty, convex and compact and Θ satisfies a compactness condition containing a measure of noncompactness. In Theorem 2, the values of Θ are not empty, convex and compact and Θ satisfies a Lipschitz condition. In Theorem3, the values of Θ are not empty, compact (not necessary convex) and Θ satisfies a compactness condition containing a measure of noncompactness.

3- We have extended the problem (2), studied by [30] to the case when the single-valued f is replaced with a multivalued function Θ and in the attendance of non-instantaneous impulsive effects.(Problem4). Moreover, we do not suppose that the semi-group generated by the operator B is compact like in [30]. In fact, three existence results of mild solution to Problems (4) are given (Theorems 4,5 and 6).

In the third section of this paper, we demonstrate three existence results of mild solutions to Problem (3). Section four is concerning with three existence of mild solutions to Problem (4). In section 5, we provide examples sections 5

2. Preliminaries and Notations

We use the following notations:

1-$\Upsilon_i = (b_i, b_{i+1}], i \in L_{0,r} = \{0,1,...,r\}$.

2-$P_b(\mho) := \{\Delta \subseteq \mho : \Delta$ is not empty, and bounded$\}$

3- $P_{cc}(\mho) := \{\Delta \subseteq \mho : \Delta$ is not empty, convex and closed$\}$.

4-$P_{bc}(\mho) := \{\Delta \subseteq \mho : \Delta$ is not empty, bounded and closed$\}$.

5-$P_{ck}(\mho) := \{\Delta \subseteq \mho : \Delta$ is not empty, convex and compact$\}$.

6-$S_3(A,\Theta)$ and $S_4(B,\Theta)$ are the set of mild solutions to problems (3) and (4) respectively.

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$$PC(\Upsilon,\mho) \quad : \quad = \{z : \Upsilon \to \mho,\ z \text{ is continuous on } [0,b_1] \text{ and on } (b_i,b_{i+1}]$$
and $\lim_{\delta \to b_i^+} z(b_i^+)$ exists for any $i \in \Lambda_{1,r}\}$.

Note that the space $PC(\Upsilon,\mho)$ is a Banach spaces where the norm is given by:

$$||f||_{PC(\Upsilon,\mho)} := \max\{||z(\delta)|| : \delta \in \Upsilon\}.$$

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The Hausdorff measure of noncompactness on $PC(\Upsilon,\mho)$ is given by $\chi_{PC(\Upsilon,\mho)} : P_b(PC(\Upsilon,\mho)) \to [0,\infty]$,

$$\chi_{PC(\Upsilon,\mho)}(D) := \max_{i\in L_{0,r}} \chi_{C(\overline{\Upsilon_i},\mho)}(D_{|\overline{\Upsilon}_i}),$$

where $\chi_{C(\overline{\Upsilon_i},\mho)}$is the Hausdorff measure of noncompactness on the Banach space $C(\overline{\Upsilon}_i,\mho)$.
As in [15], we give the following definitions:

Definition 1. *The conformable integral of order α for a function $f \in L^1(\Upsilon,\mho)$ is given by:*

$$I^\alpha f(\delta) := \int_0^\delta s^{\alpha-1} f(s)ds.$$

Definition 2. *The conformable fractional derivative of order α for a function $f : [0,\infty) \to \mho$ at a point $\delta \in (0,\infty)$ is defined by*

$$D^\alpha f(\delta) = \lim_{\varepsilon\to 0} \frac{f(\delta + \varepsilon\delta^{1-\alpha}) - f(\delta)}{\varepsilon}, \delta \in (0,\infty).$$

The proof of the following Lemmas is exactly the same as in the scalar case treated in [15]

Lemma 1. *Suppose that $f, g : [0,\infty) \to \mho$ are conformable fractional differentiable of order α at a point $\delta \in (0,\infty)$. Then,*

1-$D^\alpha(f(\delta) + g(\delta)) = D^\alpha f(\delta) + D^\alpha g(\delta)$.
2-$D^\alpha(fg)(\delta) = gD^\alpha f(\delta) + fD^\alpha g(\delta)$.
3-$D^\alpha(\frac{f}{g})(\delta) = \frac{gD^\alpha f(\delta) - fD^\alpha g(\delta)}{g(\delta)^2}$.

Lemma 2. *If $f : [0,\infty) \to \mho$ is differentiable at a point $\delta \in (0,\infty)$, then it is conformable fractional differentiable of order α at δ and $D^\alpha f(\delta) = \delta^{1-\alpha} \frac{df}{d\delta}$.*

3. The Compactness of the Solution Set to Problem (3)

In this section, we will present three theorems, in each one we explore the conditions that make the set of mild solutions to problem (3) is not empty and compact in the Banach space $PC([0,T],\mho)$ or not empty. In Theorem1, the multivalued function Θ has convex values and satisfies a compactness condition involving a measure of non-compactness. In Theorem2, the multivalued function Θ has convex values and verifies a Lipschitz condition, and in Theorem 3, the values of Θ are not necessarily convex and verifies a compactness condition involving a measure of non-compactness.

Definition 3. *([29], Definition 6) Let A be the infinitesimal generator of a C_0- semigroup $\{T(\delta) : \delta \geq 0\}$ and $f : [0,T] \times \mho \to \mho$ be continuous.A continuous function $\zeta : [0,T] \to \mho$ is called a mild solution to the problem (1)*

if

$$\zeta(\delta) := T(\frac{\delta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\delta s^{1-\alpha}\, T(\frac{\delta^\alpha - s^\alpha}{\alpha})\Theta(s,\zeta(s)ds.$$

Based on this definition, we present the definition of mild solutions to Problem (3).

Definition 4. *A function $\zeta \in PC(\Upsilon,\mho)$ is called a mild solution to the problem(3),if*

$$\zeta(\delta) = \begin{cases} T(\frac{\delta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) f(s)ds, \delta \in [0,b_1], \\ g_i(\delta,\zeta(b_i^-)), \delta \in (b_i,s_i], i \in \Lambda_{1,r}, \\ g_i(s_i,\zeta(b_i^-)) - \int_0^{s_i} s^{1-\alpha} T(\frac{s_i^\alpha - s^\alpha}{\alpha}) f(s)ds \\ + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) f(s)ds, \delta \in (s_i,b_{i+1}], i \in \Lambda_{1,r}. \end{cases} \quad (5)$$

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where $f \in S^1_{\Theta(.,\zeta(.))} = \{z \in L^1([0,T],\mho) : f(\delta) \in \Theta(\delta,\zeta(\delta)), a.e.\}$.

Remark 2. *The solution function given by (5) is continuous at the points s_i, and hence it is continuous on $(b_i, b_{i+1}], i \in \{0,1,..,r\}$*

We consider the following assumptions:
(HA) $A : D(A) \subseteq \mho \to \mho$ is the infinitesimal generator of a C_0-semigroup $\{T(\delta) : \delta \geq 0\}$ in $\mho$.
$(H\Theta)_1$ $\Theta : \Upsilon \times \mho \to P_{ck}(\mho)$ with:
(i) For every $z \in \mho$, there is a strongly measurable function $\xi : \Upsilon \to \mho$ satisfying $\xi(\delta) \in \Theta(\delta, z)$, $a.e.$and for almost $\delta \in \Upsilon$, $\zeta \to \Theta(\delta,\zeta)$ is upper semicontinuous from $\mho$ to $\mho$.
(ii)There is $\varphi \in L^p(\Upsilon,\mathbb{R}^+)$, $p > 1$ such that

$$\sup_{u\in\Theta(\delta,z)} ||u|| \leq \varphi(\delta)(1+||z||),\ for\ a.e.\delta \in \Upsilon \text{ and } z \in \mho. \tag{6}$$

(iii) There is $\beta \in L^1(\Upsilon,\mathbb{R}^+)$ such that for any bounded set $D \subseteq \mho$,

$$\chi\Theta(\delta,D) \leq \beta(\delta)\chi(D), \tag{7}$$

where χ is the Hausdorff measure of noncompactness on $\mho$.
$(Hg)_1$ The function $g : PC(\Upsilon,\mho) \to \mho$ is a compact, continuous and there are two positive real numbers a, d such that

$$||g(\zeta)| \leq a||\zeta|| + d, \forall \zeta \in PC(\Upsilon,\mho).$$

$(H)_1$ For every $i = 1,2,..,r$, the function $g_i : [t_i\ s_i] \times \mho \to \mho$ is uniformly continuous on bounded sets, $g_i(\delta,.)$ is compact and there is $h_i > 0$

$$||g_i(\delta,z)|| \leq h_i||z||, \delta \in [b_i,s_i], z \in \mho. \tag{8}$$

Lemma 3. *[([2],Lemma 5.1.1] Let $\Theta : \Upsilon \times \mho \to P_{ck}(\mho)$ be a multifunction satisfying (i) and (ii) in $(H\Theta)_1$, then for any $u \in C(\Upsilon,\mho)$,the set $S^1_{\Theta(.,u(.))} = \{z \in L^1(\Upsilon,\mho) : z(\delta) = \Theta(\delta,u(\delta)), a.e.\}$ is not empty and weakly closed.*

The following Lemma is a version for the previous lemma when $u \in PC(\Upsilon,\mho)$ and its proof can be found in [31]

Lemma 4. *Under the assumptions of the previous lemma, then for any $u \in PC(\Upsilon,\mho)$, the set $S^1_{\Theta(.,u(.))}$ is not empty weakly closed.*

We need to the following Lemmas.
Let $\Re \in P_{cc}(\mho)$, χ be a non-singular measure of noncompactness defined on subsets of $\mho$, $\aleph : \Re \to P_{ck}(\mho)$ be a closed multifunction and $Fix(\aleph) = \{z \in \mho : z \in \aleph(z)\}$.

Lemma 5. *(Kakutani-Glicksberg-Fan theorem)([2], Corollary 3.3.1) If $\aleph : \Re \to P_{ck}(\Re)$ is $v-$condensing, then $Fix(\aleph) \neq \phi$.*

Lemma 6. *([2] ,Proposition.3.5.1)). In supplement of hypothesis of Lemma (10), if χ is a monotone measure of noncompactness defined on $\mho$ and $Fix(\aleph)$ is bounded, then it is compact.*

In the following theorem, we are going to obtain conditions that make $S_3(A,\Theta)$ is not empty and compact.

Theorem 1. *Assume that (HA), $(H\Theta)_1$, $(Hg)_1$ and $(H)_1$ are satisfied*

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Then, $S_3(A,\Theta)$ is not empty and compact provided that

$$Ma + 2Mb^{1-\alpha}||\varphi||_{L^p(J,\,\mathbb{R}^+)} + h < 1, \tag{9}$$

and

$$4Mb^{1-\alpha}||\beta||_{L^1(\Upsilon,\mathbb{R}^+)} < 1, \tag{10}$$

where $M = \sup\{||T(\delta)||, \delta \geq 0\}$, and $h = \max\{h_i : i = 1,2,2,4\}$.

Proof. Let $\zeta \in PC(\Upsilon,\mho)$. In view of Lemma (4), there is $f \in S^1_{\Theta(.,\zeta(.))}$. Then, we can define a multi-valued function $\aleph : PC(\Upsilon,\mho) \to 2^{PC(\Upsilon,\mho)} - \{\phi\}$, where ϕ is the empty set, as follows: $y \in \aleph(\zeta)$ if and only if

$$y(\delta) = \begin{cases} T(\frac{\delta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\delta s^{1-\alpha}T(\frac{\delta^\alpha - s^\alpha}{\alpha})f(s)ds, \delta \in [0,b_1], \\ g_i(\delta,\zeta(b_i^-)), \delta \in (b_i,s_i], i \in \Lambda_{1,r}, \\ g_i(s_i,\zeta(b_i^-)) - \int_0^{s_i} s^{1-\alpha}T(\frac{s_i^\alpha - s^\alpha}{\alpha})f(s)ds \\ + \int_0^\delta s^{1-\alpha}T(\frac{\delta^\alpha - s^\alpha}{\alpha})f(s)ds, \delta \in (s_i,b_{i+1}], i \in \Lambda_{1,r}, \end{cases} \tag{11}$$

where $f \in S^1_{\Theta(.,\zeta(.))}$. Obviously, $Fix\ (\aleph) \subseteq S_3(A,\Theta)$. So, by applying Lemma (5), we show that $Fix\ (\aleph) \neq \phi$. This will proceed in the following steps:

Step 1. There is a natural number $\wp$ such that $\aleph(D_\wp) \subseteq D_\wp$,where $D_\wp = \{\zeta \in PC(\Upsilon,\mho) : ||\zeta|| \leq \wp$.

Assume the contrary. Then, for any $n \in \mathbb{N}$, there are $\zeta_n, y_n \in PC(\Upsilon,\mho)$ with $||y_n||_{PC(\Upsilon,\mho)} > n$, $||\zeta_n||_{PC(\Upsilon,\mho)} \leq n$ and $y_n \in \aleph(\zeta_n)$ such that

$$y_n(\delta) = \begin{cases} T(\frac{\delta^\alpha}{\alpha})(\zeta_0 + g(\zeta_n)) + \int_0^\delta s^{1-\alpha}T(\frac{\delta^\alpha - s^\alpha}{\alpha})f_n(s)ds, \delta \in [0,b_1], \\ g_i(\delta,\zeta_n(b_i^-)), \delta \in (b_i,s_i], i \in \Lambda_{1,r}, \\ g_i(s_i,\zeta_n(b_i^-)) - \int_0^{s_i} s^{1-\alpha}T(\frac{s_i - s^\alpha}{\alpha})f_n(s)ds \\ + \int_0^\delta s^{1-\alpha}T(\frac{\delta^\alpha - s^\alpha}{\alpha})f_n(s)ds, t \in (s_i,b_{i+1}], i \in \Lambda_{1,r}, \end{cases} \tag{12}$$

where $f_n \in S^1_{\Theta(.,\zeta_n(.))}$.Using (6) to get

$$||f_n(\delta)|| \leq \varphi(\delta)(1+n), a.e. \tag{13}$$

Let $\delta \in [0,b_1]$.From(6), $(H_g)_1$, (12) and (13), it results

$$\begin{aligned} ||y_n(\delta)|| &\leq M(||\zeta_0|| + an + d) + Mb^{1-\alpha}\int_0^\delta f_n(s)ds \\ &\leq M(||\zeta_0|| + an + d) + Mb^{1-\alpha}(1+n)||\varphi||_{L^p(J,\,\mathbb{R}^+)}, \end{aligned} \tag{14}$$

where $M = \sup\{||T(\delta)|| : \delta \geq 0\}$. Let $\delta \in [b_1,s_1]$. From (8), we get,

$$||g_i(\delta,\zeta_n(b_i^-))|| \leq nh_i, \forall n \geq 1. \tag{15}$$

Let $\delta \in (s_i\ b_{i+1}], i \in L_{1,r}$. Then, as above, we obtain

$$||y_n(\delta)|| \leq nh_i + 2Mb^{1-\alpha}(1+n)||\varphi||_{L^p(J,\,\mathbb{R}^+)}. \tag{16}$$

Inequalities (14 - 16) lead to

$$n < M(||\zeta_0|| + an + d) + 2Mb^{1-\alpha}(1+n)||\varphi||_{L^p(J,\,\mathbb{R}^+)} + nh_i$$

Dividing this inequality by n and letting $n \to \infty$, it yields from (6),

$$1 < Ma + 2Mb^{1-\alpha}||\varphi||_{L^p(J,\,\mathbb{R}^+)} + h$$

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which contradicts (9).

Step 2. We demonstrate that, if $y_n, \zeta_n \in D_\wp$ such that $\zeta_n \to \zeta$ and $y_n \in \aleph(\zeta_n), \forall n \in \mathbb{N}$, then $y_n \to y$ and $y \in \aleph(\zeta)$.

Because $y_n \in \aleph(\zeta_n), \forall n \in \mathbb{N}$, (12) is satisfied.

Since $p > 1$, then from (13), the set $\{f_n : n \geq 1\}$ is weakly compact in $L^p(\Upsilon, \mho)$. Application of Mazur's lemma, there is, without loss of generality, a subsequence $(f_n^*), n \in \mathbb{N}$ of convex combinations of (f_n) and converging almost everywhere to a function $f \in L^p(\Upsilon, \mho) \subseteq L^1(\Upsilon, \mho)$. By the continuity of both g and $g_i(\delta, .)$, it follows by letting $n \to \infty$ in (12), $y_n \to y$, where

$$y(\delta) = \begin{cases} T(\frac{\delta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) f(s) ds, \delta \in [0, b_1], \\ g_i(\delta, \zeta(b_i^-)), \delta \in (b_i, s_i], i \in \Lambda_{1,r}, \\ g_i(s_i, \zeta(b_i^-)) - \int_0^{s_i} s^{1-\alpha} T(\frac{s_i^\alpha - s^\alpha}{\alpha}) f(s) ds \\ + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) f(s) ds, \delta \in (s_i, b_{i+1}], i \in \Lambda_{1,r}. \end{cases}$$

But, the assumption $\Theta(\delta, .)$ is upper semicontinuous implies that $f(\delta) \in \Theta(\delta, \zeta(\delta)), a.e.$Therefore, $y \in \aleph(\mu)$

Step 3. For any $\zeta \in D_\wp$, $\aleph(\zeta)$ is compact in $PC(\Upsilon, \mho)$.

Let $y_n \in \aleph(\zeta); n \in \mathbb{N}$. Using the same arguments as in Step 2, one can show that there is a subsequence of (y_{n_k}) that converging to $y \in \aleph(\zeta)$. This shows that $\aleph(\zeta)$ is relatively compact, but Step(2) leads to the $\aleph(\zeta)$, is closed and consequently $\aleph(\zeta)$ is compact in $PC(\Upsilon, \mho)$.

Step 4. The family of functions

$$\aleph(D_\wp)|_{\overline{\Upsilon_i}} = \{\omega^* \in C(\overline{\Upsilon_i}, \mho) : \omega^*(\delta) = \omega(\delta), \delta \in (b_i, b_{i+1}],$$
$$\omega^*(b_i) = \lim_{\delta \to b_i^+} \omega(\delta), \omega \in \aleph(D_\wp)\}, i \in \{0, 1, .., r\},$$

are equicontinuous in $C([s_i, b_{i+1}], \mho)$. Assume that $\omega^* \in \aleph(D_\wp)|_{[s_i, b_{i+1}]}; i \in \{0, 1, .., r\}$. Then, there is $\zeta \in D_\wp$ with

$$\omega^*(\delta) = \begin{cases} T(\frac{\delta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) f(s) ds, \delta \in [0, b_1], \\ g_i(\delta, \zeta(b_i^-)), \delta \in (b_i, s_i], i \in \Lambda_{1,r} \\ g_i(s_i, \zeta(b_i^-)) - \int_0^{s_i} s^{1-\alpha} T(\frac{s_i^\alpha - s^\alpha}{\alpha}) f(s) ds \\ + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) f(s) ds, \delta \in (s_i, b_{i+1}], i \in \Lambda_{1,r}. \end{cases} \tag{17}$$

and $\omega^*(b_i) = \lim_{\delta \to b_i^+} \omega(\delta)$. We consider the following cases:

Case1. Let $\delta, \delta + d \in [0, b_1], d > 0$. Using (17), it yields

$$\begin{aligned} & \lim_{d \to 0} ||\omega^*(\delta + d) - \omega^*(\delta)|| \\ \leq \; & \lim_{d \to 0} ||T(\frac{(\delta + d)^\alpha}{\alpha}) - T(\frac{\delta^\alpha}{\alpha})|| \, ||\zeta_0 + g(\zeta)|| \\ & + \lim_{d \to 0} || \int_0^{\delta + d} s^{1-\alpha} T(\frac{(\delta + d)^\alpha - s^\alpha}{\alpha}) f(s) ds - \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) f(s) ds|| \\ \leq \; & \lim_{d \to 0} ||T(\frac{(\delta + d)^\alpha}{\alpha}) - T(\frac{\delta^\alpha}{\alpha})|| \, ||\zeta_0 + g(\zeta)|| \\ & + \lim_{d \to 0} \int_\delta^{\delta + d} || s^{1-\alpha} T(\frac{(\delta + d)^\alpha - s^\alpha}{\alpha}) f(s)|| ds \\ & + \lim_{d \to 0} || \int_0^\delta s^{1-\alpha} T(\frac{(\delta + d)^\alpha - s^\alpha}{\alpha}) f(s) - s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) f(s) ds|| \\ = \; & I_1 + I_1 + I_3. \end{aligned}$$

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In view of (HA), $\lim_{d\to 0} I_1 = 0$. Making use (13), we get

$$\lim_{d\to 0} I_2 \leq (1+\wp)Mb^{1-\alpha} \lim_{d\to 0} \int_{\delta}^{\delta+d} \varphi(s)ds = 0,$$

Next, the strong continuity of $\{T(\delta) : \delta \geq 0\}$ leads to

$$\lim_{d\to 0} I_3 \leq (1+\wp)b^{1-\alpha} \lim_{d\to 0} \int_0^{\delta} ||T(\frac{(\delta+d)^{\alpha}-s^{\alpha}}{\alpha}) - T(\frac{\delta^{\alpha}-s^{\alpha}}{\alpha})||\, \varphi(s)ds = 0,$$

Case2. Let $\delta, \delta+d \in (b_1, s_1]$. By the uniform continuity on bounded sets of g_1, it results

$$\begin{aligned} & \lim_{d\to 0} ||\omega^*(\delta+d) - \omega^*(\delta)|| \\ = \; & \lim_{d\to 0} ||g_1(\delta+d, \zeta(b_1^-)) - g_1(\delta, \zeta(b_1^-))|| = 0, \end{aligned}$$

independently of ζ. Moreover,

$$\begin{aligned} & \lim_{d\to 0} ||\omega^*(\delta+d) - \omega^*(b_i)|| \\ = \; & \lim_{d\to 0} \lim_{\delta\to b_1^+} ||g_1(\delta+d, \zeta(b_1^-)) - g_1(\delta, \zeta(b_1^-))|| \\ = \; & 0, \end{aligned}$$

independently of ζ. Similarly, one can show that $\aleph(D_{\wp})|_{\overline{\Upsilon}_i}, i \in L_{1,r}$ is equicontinuous.
Step 5. The set $\pounds = \cap_{n=1} D_n$ is compact, where $D_1 = \aleph(B_{\wp})$ and $D_{n+1} = \aleph(D_n), n \geq 1$.
Because $D_n \subseteq D_{n+1}$,then, as stated by Cantor's intersection property [32], it be enough to manifest that,

$$\lim_{n\to\infty} \varkappa_{PC(\Upsilon,\mho)}(D_n) = 0. \tag{18}$$

Let $\kappa > 0$, and $n \geq 1$ be fixed. In view of [[33], lemma 2.9], there is a sequence (y_m) in D_n with

$$\begin{aligned} \chi_{PC(\Upsilon,\mho)}(D_n) & \leq 2\chi_{PC(\Upsilon,\mho)}\{y_m : m \geq 1\} + \kappa \\ & = 2 \max_{i\in L_{0,r}} \chi_{C(\overline{\Upsilon}_i,\mho)}(D_{|\overline{\Upsilon}_i}). \end{aligned}$$

As a result of Step4, the sets $D_{|\overline{\Upsilon}_i}$ are equicontinuous, and hence, the last inequality becomes

$$\chi_{PC(\Upsilon,\mho)}(D_n) \leq 2 \max_{\delta\in\Upsilon} \chi(y_m(\delta) : m \geq 1\} + \kappa. \tag{19}$$

Now, since $D_n = \aleph(D_{n-1})$,there is $\zeta_m \in D_{n-1}$ with $y_m \in \aleph(\zeta_m)$,which means that

$$y_m(\delta) = \begin{cases} T(\frac{\delta^{\alpha}}{\alpha})(\zeta_0 + g(\zeta_m)) + \int_0^{\delta} s^{1-\alpha} T(\frac{\delta^{\alpha}-s^{\alpha}}{\alpha}) f_m(s)ds, \delta \in [0, b_1], \\ g_i(\delta, \zeta_m(b_i^-)), \delta \in (b_i, s_i], i \in \Lambda_{1,r}, \\ g_i(s_i, \zeta_m(b_i^-)) - \int_0^{s_i} s^{1-\alpha} T(\frac{s_i^{\alpha}-s^{\alpha}}{\alpha}) f_m(s)ds \\ + \int_0^{\delta} s^{1-\alpha} T(\frac{\delta^{\alpha}-s^{\alpha}}{\alpha}) f_m(s)ds, \delta \in (s_i, b_{i+1}], i \in \Lambda_{1,r}, \end{cases}$$

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where $f_m \in S^1_{\Theta(.,\zeta_m)}$. If $\delta \in [0, b_1]$, then due to the continuity of $T(\frac{\delta^\alpha}{\alpha})$,the compactness of g and (7) we get

$$\begin{aligned} \chi\{y_m(\delta) &: m \geq 1\} \\ &\leq \chi\{\int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) f_m(s) ds : m \geq 1\} \\ &\leq 2Mb^{1-\alpha} \int_0^\delta \chi\{f_m(s) : m \geq 1\} ds \\ &\leq 2Mb^{1-\alpha} \int_0^\delta \chi\{\zeta_m(s)) : m \geq 1\} \beta(s) ds \\ &\leq 2Mb^{1-\alpha} \chi_{PC(\Upsilon,\mho)}(D_{n-1}) ||\beta||_{L^1(\Upsilon,\mathbb{R}^+)}. \end{aligned} \tag{20}$$

If $\delta \in (b_i, s_i]; i \in L_{1,r}$, then the compactness of $g_i(\delta, .)$ implies to

$$\chi\{y_m(\delta) : m \geq 1\} = 0. \tag{21}$$

Finally, if $\delta \in (s_i, b_{i+1}], i \in L_{1,r}$,then as in (20)

$$\begin{aligned} \chi\{y_m(\delta) &: m \geq 1\} \\ &\leq 4Mb^{1-\alpha} \chi_{PC(\Upsilon,\mho)}(D_{n-1}) ||\beta||_{L^1(\Upsilon,\mathbb{R}^+)}. \end{aligned} \tag{22}$$

It results from (19 - 22) that

$$\chi_{PC(\Upsilon,\mho)}(D_n) \leq 4Mb^{1-\alpha} ||\beta||_{L^1(\Upsilon,\mathbb{R}^+)} \chi_{PC(\Upsilon,\mho)}(D_{n-1}) + \kappa.$$

Since κ is arbitrary and this relation is true for any n, it follows

$$\chi_{PC(\Upsilon,\mho)}(D_n) \leq (4Mb^{1-\alpha} ||\beta||_{L^1(\Upsilon,\mathbb{R}^+)})^n \chi_{PC(\Upsilon,\mho)}(D_1).$$

This inequality along (10), we get (18) and then, D is compact.

Step 6. Applying Lemma (5), the multi-valued function $\aleph : D \to P_{ck}(PC(\Upsilon, \mho))$ has a fixed point, which is the solution to Problem (3). Moreover, using the same arguments in step 1, we can show $Fix(\aleph)$ is bounded, and hence by Lemma (6), $S_3(A, \Theta)$ is compact. □

In the following theorem, we offer another result of existence of mild solutions to Problem(3).

Lemma 7. *[34]. If $\Phi : \Omega \to P_{bc}(\Omega)$ is a contraction, then $Fix(\Phi)$ is not empty, where Ω is a complete metric space.*

Theorem 2. *In addition of (HA),suppose the following assumptions:*

$(H\Theta)_2$ $\Theta : \Upsilon \times \mho \to P_{ck}(\mho)$ such that:
(i) For every $u \in \mho$, the multifunction $\delta \to \Theta(\delta, u)$ has a strongly measurable selection.
(ii)There is a function $\varphi \in L^1(J, \mathbb{R}^+)$ satisfying

$$h(\Theta(\delta, u), \Theta(\delta, v)) \leq \varphi(\delta)\|u - v\|, \forall u, v \in \mho \text{ and for } a.e.\ \delta \in \Upsilon, \tag{23}$$

and

$$Sup\{\ ||u||\ :\ u \in \Theta(\delta, 0)\} \leq\ \varphi(\delta),\ \text{for } a.e.\ \delta \in \Upsilon, \tag{24}$$

where h is the Hausdorff distance between two closed convex bounded sets.
$(Hg)_2$ There is a $a > 0$ with

$$\|g(\zeta_1) - g(\zeta_2)\| \leq a\|\zeta_1 - \zeta_2\|,\ \forall \zeta_1, \zeta_2 \in PC(\Upsilon, \mho). \tag{25}$$

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$(H)_2$ For each $i = 1, 2, ...r,$ there is $h_i > 0$ such that for any $\delta \in Y$,

$$||g_i(\delta, u) - g_i(\delta, v)|| \leq h_i ||u - v||, \text{ for all } u, v \in \mho. \tag{26}$$

Then, $S_3(A, \Theta)$ is not empty if inequality (10) is verified.

Proof. From (i) and (ii) of $(H\Theta)_2$,we get

$$\begin{aligned} h(\Theta(\delta, z), \{0\}) &\leq h(\Theta(\delta, z), \Theta(\delta, 0)) + h(\Theta(\delta, 0), \{0\}) \\ &\leq \varphi(\delta)(||z|| + 1), \forall z \in \mho \text{ and for } a.e\ \delta \in Y. \end{aligned}$$

Then, (i) and (ii) of $(H\Theta)_1$ are satisfied. According to Lemma(4), $S^1_{\Theta(.,\zeta(.))}$ is not empty, and hence we can define a multi-valued function $\aleph : PC(Y, \mho) \to 2^{PC(Y,\mho)} - \{\phi\}$,which is defined by (11). We will use Lemma (6) to demonstrate that $\aleph$ has a fixed point. So, we will show that $\aleph$ is a contraction. In order to do this, let $\zeta_1, \zeta_2 \in PC(Y, \mho)$ and $y_1 \in \aleph(\zeta_1)$.In view of the definition of $\aleph$, we have

$$y_1(\delta) = \begin{cases} T(\frac{\delta^\alpha}{\alpha})(\zeta_0 + g(\zeta_1)) + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) f_1(s) ds, \delta \in [0, b_1], \\ g_i(\delta, \zeta_1(b_i^-)), \delta \in (b_i, s_i], i \in \Lambda_{1,r}, \\ g_i(s_i, \zeta_1(b_i^-)) - \int_0^{s_i} s^{1-\alpha} T(\frac{s_i^\alpha - s^\alpha}{\alpha}) f_1(s) ds \\ + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) f_1(s) ds, \delta \in (s_i, b_{i+1}], i \in \Lambda_{1,r}, \end{cases} \tag{27}$$

where $f_1 \in L^1(Y, \mho)$ satisfying $f_1(\delta) \in \Theta(\delta, \zeta_1(\delta)).a.e.$
Next, we consider the multi-valued function:$\Gamma : Y \to 2^{\mho}$, defined by

$$\Gamma(\delta) = \{z \in \Theta(\delta, \zeta_2(\delta)) : ||f_1(\delta) - z|| = d(f_1(\delta), \Theta(\delta, \zeta_2(\delta))\}.$$

Since the values of Θ are in $P_{ck}(\mho)$, then the values of $\Gamma(\delta)$ are not empty. Moreover, $(H\Theta)_2(i)$ implies the measurability of Γ. Thanks to Theorem III-41 in [35], there is a measurable function $f_2 : Y \to \mho$ with $f_2 \in \Gamma(\delta), a.e.$, and consequently,

$$\begin{aligned} ||f_1(\delta) - f_2(\delta)|| &= d(f_1(\delta), \Theta(\delta, \zeta_2(\delta)) \\ &\leq h(\Theta(\delta, \zeta_1(\delta)), \Theta(\delta, \zeta_2(\delta))) \\ &\leq \varphi(\delta)\|\zeta_1(\delta) - \zeta_2(\delta)\| \leq \varphi(\delta)||\zeta_1 - \zeta_2||_{PC(Y,\mho)}, a.e. \end{aligned} \tag{28}$$

Next, define $y_2 : Y \to \mho$ by

$$y_2(\delta) = \begin{cases} T(\frac{\delta^\alpha}{\alpha})(\zeta_0 + g(\zeta_2)) + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) f_2(s) ds, \delta \in [0, b_1], \\ g_i(\delta, \zeta_2(b_i^-)), \delta \in (b_i, s_i], i \in \Lambda_{1,r}, \\ g_i(s_i, \zeta_2(b_i^-)) - \int_0^{s_i} s^{1-\alpha} T(\frac{s_i^\alpha - s^\alpha}{\alpha}) f_2(s) ds \\ + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) f_2(s) ds, \delta \in (s_i, b_{i+1}], i \in \Lambda_{1,r}. \end{cases} \tag{29}$$

Obviously, $y_2 \in \aleph(\zeta_2)$. Now, we compute $||y_1 - y_2||_{PC(Y.\mho)}$. If $\delta \in [0, b_1]$, then from (27) and (29), we get

$$\begin{aligned} & ||y_1(\delta) - y_2(\delta)|| \\ \leq\ & M\, ||g(\zeta_1) - g(\zeta_2)|| + Mb^{1-\alpha} \int_0^\delta ||f_1(s) - f_2(s)|| ds \\ \leq\ & M\, a\, ||\zeta_1 - \zeta_2||_{PC(Y,\mho)} + Mb^{1-\alpha} ||\zeta_1 - \zeta_2||_{PC(Y,\mho)}\, ||\varphi||_{L^1(Y,\mho)}. \end{aligned} \tag{30}$$

If $\delta \in (b_i, s_i], i \in L_{1,r}$,then

$$\begin{aligned} & ||y_1(\delta) - y_2(\delta)|| \\ \leq\ & ||g_i(\delta, \zeta_1(b_i^-)) - g_i(\delta, \zeta_2(b_i^-))|| \\ \leq\ & h_i\, ||\zeta_1(b_i^-) - \zeta_2(b_i^-)\,|| \\ \leq\ & h\, ||\zeta_1 - \zeta_2||_{PC(\Upsilon,\mho)}. \end{aligned} \tag{31}$$

If $\delta \in (s_i, b_{i+1}], i \in L_{1,r}$, then as in the pervious cases

$$\begin{aligned} & ||y_1(\delta) - y_2(\delta)|| \\ \leq\ & h\, ||\zeta_1 - \zeta_2||_{PC(\Upsilon,\mho)} + 2Mb^{1-\alpha}||\zeta_1 - \zeta_2||_{PC(\Upsilon,\mho)}\, ||\varphi||_{L^1(\Upsilon,\mho)}. \end{aligned} \tag{32}$$

By combining relations (30 -32), we get

$$||y_1 - y_2|| \leq ||\zeta_1 - \zeta_2||_{PC(\Upsilon,\mho)}[h + a + 2Mb^{1-\alpha}||\varphi||_{L^1(\Upsilon,\mho)}].$$

By interchanging the rules of y_1 and y_2, it results

$$h(\aleph(\zeta_1), \aleph(\zeta_2)) \leq ||\zeta_1 - \zeta_2||_{PC(\Upsilon,\mho)}[h + a + 2Mb^{1-\alpha}||\varphi||_{L^1(\Upsilon,\mho)}].$$

This inequality along condition (11), leads to $\aleph$ is a contraction, and therefore, by Lemma (7), $\aleph$ has a fixed point which is a solution to Problem(3). □

Now, we offer another set of conditions that make $S_3(A, \Theta)$ is not empty when the values of Θ are not necessarily convex.

Theorem 3. *In addition of $(Hg)_1$, $(H)_1$ suppose that the following condition hold:*

$(H\Theta)_3$ $\Theta : \Upsilon \times \mho \to P_{bc}\,(\mho)$ is a multifunction such that
(i) Θ has a measurable graph and for any $\delta \in \Upsilon$, $z \to \Theta(\delta, z)$ is lower semicontinuous.
(ii)There exists a function $\varphi \in L^1(T, \mathbb{R}^+)$, such that for any $z \in \mho$

$$\|\Theta(\delta, z)\| \leq \varphi(\delta)\ , \ \ a.e.\ \delta \in T. \tag{33}$$

(iii)There is $\beta \in L^1(\Upsilon, \mathbb{R}^+)$ such that for any bounded set $D \subseteq \mho$,

$$\chi\Theta(\delta, D) \leq \beta(\delta)\chi(D),$$

Then, $S_3(A, \Theta)$ is not empty if condition (10) and the following condition are satisfied.

$$Ma + h < 1. \tag{34}$$

Proof. First, by using Theorem 3 in [36], we show the existence of a continuous selection to the multivalued Nemitsky operator $\Re : PC(\Upsilon, \mho) \to 2^{L^1(T,\mho)}$

$$\Re(\zeta) = S^1_{\Theta(.,\zeta(.))} = \{f \in L^1(T, \mho) : f(\delta) \in \Theta(\delta, \zeta(\delta)), a.e.\ \delta \in \Upsilon\}.$$

Obviously, $\Re(\zeta); \zeta \in PC(\Upsilon, \mho)$ is decomposable. Since Θ has a measurable graph and satisfies (33), then by Theorem 3.2 in [37], $\Re(\zeta)$ is not empty. Because $\Theta(\zeta)$ is closed, $\Re(\zeta)$ is closed. To prove the lower semicontinuity of $\Re$, it is sufficient to show that, for every $v \in L^1(T, \mho)$, $\zeta \to d(v, \Re(\zeta))$ is upper semicontinuous (Proposition 1.2.26 in [38]). This is equivalent to show that for any $\theta \geq 0$, the set

$$v_\theta = \{\zeta \in PC(\Upsilon, \mho) : d(v, \Re(\zeta)) \geq \theta\}.$$

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is closed. Let $\{\zeta_n\} \subseteq v_\theta$ and $\zeta_n \to \zeta$ in $PC(\Upsilon, \mho)$. Then, for all $\delta \in \Upsilon$, $\zeta_n(\delta) \to \zeta(\delta)$ in $\mho$. By Theorem 2.2 in [37],

$$\inf_{z\in\Re(\zeta_n)} \int_0^b \|v(\delta) - z(\delta)\| d\delta = \int_0^b \inf_{z\in\Re(\zeta_n)} \|v(\delta) - z(\delta)\| d\delta.$$

Then, for any $n \in \mathbb{N}$,

$$\begin{aligned} d(v, \Re(\zeta_n)) &= \inf_{z\in\Re(\zeta_n)} \|v - z\|_{L^1(\Upsilon,\mho)} = \inf_{v\in\Re(\zeta_n)} \int_0^b \|v(\delta) - z(\delta)\| d\delta \\ &= \int_0^b \inf_{z\in\Re(\zeta_n)} \|v(\delta) - z(\delta)\| d\delta = \int_0^b d(v(\delta), \Theta(\delta, \zeta_n(\delta))) d\delta. \end{aligned}$$

This equality along with Fatou's Lemma, yields

$$\begin{aligned} \theta &\leq \lim_{n\to\infty} \sup d(v, \Re(\zeta_n)) = \lim_{n\to\infty} \sup \int_0^b d(v(\delta), \Theta(\delta, \zeta_n(\delta)) d\delta \\ &\leq \int_0^b \lim_{n\to\infty} \sup d(v(\delta), \Theta(\delta, \zeta_n(\delta)) d\delta. \end{aligned} \tag{35}$$

Since for any $\delta \in \Upsilon$, $\Theta(\delta, z)$ is lower semi continuous, the function, $z \to d(v(\delta), \Theta(\delta, z))$ is upper semicontinuous [38], and hence $\lim_{n\to\infty} \sup d(v(\delta), \Theta(\delta, \zeta_n(\delta)) = d(v(\delta), \Theta(\delta, \zeta(\delta))$. Therefore, inequality (35) implies

$$\theta \leq \int_0^b d(v(\delta), \Theta(\delta, \zeta(\delta)) d\delta = d(v, \Re(\zeta)),$$

and this proves that v_λ is closed, and so, $\Re$ is lower semicontinuous. By applying Theorem 3 in [36], $\Re$ has a continuous selection, that is, there is a continuous function $\eta : PC(\Upsilon, \mho) \to L^1(\Upsilon, \mho), \eta(\zeta) \in \Re(\zeta)$; $\zeta \in PC(T, \mho)$. Then, $\eta(\zeta)(s) \in \Theta(s, \zeta(s)), a.e.$ Now, let $\aleph^* : PC(\Upsilon, \mho) \to PC(\Upsilon, \mho)$ defined by

$$(\aleph^*\zeta)(t) = \begin{cases} T(\frac{\delta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) Z(\zeta)(s), \delta \in [0, b_1], \\ g_i(\delta, \zeta_1(b_i^-)), \delta \in (b_i, s_i], i \in L_{1,r}, \\ g_i(s_i, \zeta_1(b_i^-)) - \int_0^{s_i} s^{1-\alpha} T(\frac{s_i^\alpha - s^\alpha}{\alpha}) Z(\zeta)(s) ds \\ + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) Z(\zeta)(s), \delta \in (s_i, b_{i+1}], i \in L_{1,r}. \end{cases}$$

Notice that assumption (ii) in $(H\Theta)_3$ implies to $\sup_{u\in\Theta(\delta,z)} ||u|| \leq \varphi(\delta)$ for $a.e.\delta \in \Upsilon$ and $z \in \mho$. Then, By following what we did in step 1 in Theorem 1, we can show that relation (34) leads to the existence of $\varrho > 0$ such that $\aleph^*(D_\varrho) \subseteq D_\varrho$. Next, as in steps 2,3,4 and 5 in the proof of Theorem1, the set $D^* = \cap_{n=1} \aleph^*(D_n), D_1 = \aleph^*(D_\varrho), D_{n+1} = \aleph^*(D_n)$ is not empty, convex and compact in $PC(\Upsilon, \mho)$. By applying Schauder's fixed point theorem to the function $\aleph^* : D^* \to D^*$, there is a point $\zeta \in D^*$ with

$$\zeta(\delta) = \begin{cases} T(\frac{\delta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) Z(\zeta)(s) ds, \delta \in [0, b_1], \\ g_i(\delta, \zeta_1(b_i^-)), \delta \in (b_i, s_i], i \in \Lambda_{1,r}, \\ g_i(s_i, \zeta_1(b_i^-)) - \int_0^{s_i} s^{1-\alpha} \delta(\frac{s_i^\alpha - s^\alpha}{\alpha}) Z(\zeta)(s) ds \\ + \int_0^\delta s^{1-\alpha} T(\frac{\delta^\alpha - s^\alpha}{\alpha}) Z(\zeta)(s) ds, \delta \in (s_i, b_{i+1}], i \in \Lambda_{1,r}. \end{cases}$$

Since $Z(\zeta)(s) \in \Theta(s, \zeta(s)), a.e., S_3(A, \Theta)$ is not empty. □

4. Existence of Solutions to Problem(4)

This section concerned to give two existence results of solutions to Problem(4).We start by presenting the concepts and facts that we need.

Definition 5. *[39] A linear closed densely defined operator $B : D(B) \to \mho$ is said to be sectorial of type (M^*, τ, σ),where $M^* > 0, \tau \in \mathbb{R}, \sigma \in (0, \frac{\pi}{2})$ if*

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(1)$\mathbb{C}-(\tau+S_\theta)\subseteq\rho(B)$
(2) For any $\lambda\notin\tau+S_\theta$, $\|R(\lambda,B)\|\leq\frac{M^*}{|\lambda-\omega|}$,
where $\tau+S_\sigma=\{\tau+\lambda\in\mathbb{C}:0<|\arg(-\lambda)|<\sigma\}$, $\rho(B)=\{\lambda\in\mathbb{C}:(\lambda-B)^{-1}$ exists$\}$ is the resolvent set and $R(\lambda,B)=(\lambda-B)^{-1}$ is the $\lambda-$resolvent operator of B and $\lambda\in\rho(B)$.

Lemma 8. *[39] A linear closed densely defined sectorial operator B generates a strongly analytic semigroup $\{K(\delta):\delta\geq 0\}$. Moreover,*

$$K(\delta)=\frac{1}{2\pi i}\int_\gamma e^{\lambda\delta}R(\lambda,B)d\lambda, \tag{36}$$

where γ is a suitable path inside $\rho(B)$

Lemma 9. *[30]. Let B be a linear closed densely defined sectorial operator on the Banach space $\mho$ of type (M,ω,σ),where $M^*>0,\tau\in\mathbb{R},\sigma\in(0,\frac{\pi}{2})$ and $f:[0,b]\to\mho$ be continuous. The continuous function*

$$\zeta(\delta)=K(\frac{\delta^\alpha}{\alpha})(\zeta_0+g(\zeta))+\int_0^\delta s^{\alpha-1}f(s)ds. \tag{37}$$

is the mild solution for the semilinear Cauchy problem:

$$\begin{cases} D^\alpha\zeta(\delta)=B\zeta(\delta)+f(\delta),\ \delta\in\Upsilon,\\ \zeta(0)=\zeta_0+g(\zeta), \end{cases} \tag{38}$$

where $K(\delta),\delta\geq 0$ is given by (36).
Based on this Lemma, we present the concept of mild solutions to Problem (4).

Definition 6. *A function $\zeta\in PC(\Upsilon,\mho)$ is called a mild solution to the problem(4),if*

$$\zeta(\delta)=\begin{cases} K(\frac{\delta^\alpha}{\alpha})(\zeta_0+g(\zeta))+\int_0^\delta s^{1-\alpha}K(\frac{\delta^\alpha-s^\alpha}{\alpha})f(s)ds,\delta\in[0,b_1],\\ g_i(\delta,\zeta(b_i^-)),\delta\in(b_i,s_i],i\in\Delta_{1,r},\\ g_i(s_i,\zeta(b_i^-))-\int_0^{s_i}s^{1-\alpha}K(\frac{s_i^\alpha-s^\alpha}{\alpha})f(s)ds\\ +\int_0^\delta s^{1-\alpha}K(\frac{\delta^\alpha-s^\alpha}{\alpha})f(s)ds,\delta\in(s_i,b_{i+1}],i\in\Delta_{1,r}, \end{cases} \tag{39}$$

where $f\in S^1_{\Theta(.,\zeta(.))}=\{z\in L^1([0,T],\mho):f(\delta)\in\Theta(\delta,\zeta(\delta)),a.e.\}$.
Now, since every analytic semigroup is a C_0 - semigroup, then by following the same arguments used in the proof of Theorems (1)-(3), we can demonstrate the following existence results of mild solutions to Problem (4).

Theorem 4. *In addition of assumptions $(H\Theta)_1$, $(Hg)_1$ and $(H)_1$, we suppose the following condition:*

(HB) Let B be a linear, closed and densely defined sectorial operator on the Banach space $\mho$ of type (M^*,ω,θ),where $M^*>0,\tau\in\mathbb{R},\sigma\in(0,\frac{\pi}{2})$.
Then, $S_4(B,\Theta)$ is not empty and compact in $PC(\Upsilon,\mho)$ provided that

$$M^*a+2M^*b^{1-\alpha}||\varphi||_{L^p(J,\mathbb{R}^+)}+h<1, \tag{40}$$

and

$$4M^*b^{1-\alpha}||\beta||_{L^1(\Upsilon,\mathbb{R}^+)}<1, \tag{41}$$

where $M^*=\sup\{||K(\delta)||,\delta\geq 0\}$, and $h=\sum_{1=1}^{i=r}h_i$.

Theorem 5. *If the assumptions (HB), $(H\Theta)_2$, $(Hg)_2$ and $(H)_2$, then $S_4(B,\Theta)$ is not empty in the Banach space $PC(\Upsilon,\mho)$ provided that (40) is satisfied.*

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Theorem 6. *If the assumptions (HB), $(H\Theta)_1$, $(Hg)_1$ and $(H)_1$, then $S_4(B,\Theta)$ is not empty in the Banach space $PC(\Upsilon,\mho)$ provided that*

$$M^* a + h < 1. \tag{42}$$

5. Examples

Example 1. *Let $\alpha = \frac{1}{2}, \mho = L^2[0,\infty)$, $\Upsilon = [0,1]$, and $s_0 = 0, s_i = \frac{2i}{9}, t_i = \frac{2i-1}{9}, i = 1,2,3,4, t_5 = 1$, $r = 4$, $\Delta : \mho \to \mho$ be a linear bounded compact operator and $\Lambda \in P_{ck}(\mho)$ with $0 \in \Lambda$. On E, define the translation C_0- semigroup:$T(t)f(s) = f(t+s); f \in \mho$. If A is the infinitesimal generator of this semigroup, then $Af = f\prime$,where*

$$D(A) = \{f \in L^2[0,\infty) : \text{ the weak derivative } f\prime \text{ exists and } f\prime \in L^2[0,\infty)\}.$$

Define $\Theta : \Upsilon \times \mho \to P_{ck}(\mho)$, and $g : PC(\Upsilon,\mho) \to \mho$ and $g_i : [t_i, s_i] \times \mho \to \mho$as follows:

$$\Theta(\delta, z) = \frac{v\ ||z|| \sin\delta}{\omega\ (1 + ||z||)} \Lambda\,; (\delta, z) \in \Upsilon \times \mho, \tag{43}$$

$$g(x) = \sum_{i=1}^{i=4} \kappa_i \Delta(x(b_i)), \tag{44}$$

$$g_i(\delta, z) := i\kappa_5 \delta \Delta(z), \forall (\delta, z) \in [b_i, s_i] \times \mho, i = 1,2,..,4, \tag{45}$$

where $v, \kappa_i (i = 1,2,..,5)$ are positive real numbers and $\omega = \sup\{||u|| : u \in \Lambda\}$.Notice that, for any $z \in \mho$, the function $f(\delta) = \frac{v\ ||z|| \sin\delta}{\omega\ (1+||z||)} u_0$; $u_0 \in Z$ is a strongly measurable selection for the multi-valued function $\delta \to \Theta(\delta, z)$. Also,, for any $\delta \in \Upsilon$ and any $z, y \in \mho$, we have

$$\sup_{z \in \Theta(\delta,z)} ||y|| \leq \frac{v\ ||z||\ |\sin\delta|}{(1 + ||z||)} \leq v < v(1 + ||z||), \tag{46}$$

and

$$\begin{aligned} H(\Theta(\delta,z), \Theta(\delta,y)) &\leq v\ || \frac{||z||}{(1 + ||z||)} - \frac{||y||}{(1 + ||y||)} || \\ &\leq v||z - y||, \end{aligned} \tag{47}$$

It results from (46 and (47) that, $\Theta(\delta, .)$ is upper semicontinuous, and for any bounded subset $D \subseteq \mho$,

$$\chi(\Theta(\delta, D)) \leq v\, \chi(D), \text{ for } \delta \in [0,1].$$

Then, the assumption $(H\Theta)_1$ is verified with $\varphi(\delta) = \beta(\delta) = v$, for $\delta \in [0,1]$. Furthermore, the compactness of the operator Δ implies the compactness of g and for any $x \in PC(\Upsilon,\mho)$,

$$||g(x)|| \leq || \sum_{i=1}^{i=4} \kappa_i \Delta(x(b_i))|| \leq \sum_{i=1}^{i=4} \kappa_i ||\Delta||\ ||x(b_i)|| \leq ||\Delta|| ||x||_{PC(\Upsilon,\mho)} \sum_{i=1}^{i=4} \kappa_i.$$

and hence, $(Hg)_1$ is satisfied with $a = ||\Delta|| \sum_{i=1}^{i=4} \kappa_i$ and $d = 0$. Next, the compactness of the operator Δ implies the compactness of $g_i; i = 1,2,3,4$. Moreover, in view of (47),

$$||g_i(\delta, z)|| = i\kappa_5\ ||(\Delta(z))|| \leq i\kappa_5 ||\Delta||\ ||z||_{\mho}, \forall (\delta, z) \in [b_i, s_i] \times \mho.$$

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Consequently, the assumption $(H)_1$is verified, where $h_i = i\kappa_5||\Delta||, i = 1,2,3,4$. By Applying Theorem 1,the set of mild solutions to the following nonlocal impulsive conformable fractional semilinear differential inclusion :

$$\begin{cases} \frac{d^{\frac{1}{2}}}{d\zeta^{\frac{1}{2}}}\zeta(\delta) \in \zeta\prime(\delta) + \frac{\upsilon}{\omega}\frac{||\zeta|| \sin\delta}{(1+||\zeta||)}\Lambda, a.e.\ \delta \in \cup_{i=0}^{i=4}(s_i, b_{i+1}], \\ \zeta(\delta) = i\kappa_5\delta\Delta(\zeta), \delta \in (b_i, s_i];\ i \in \{1,2,3,4\}, \\ \zeta(0) = \zeta_0 + \sum_{i=1}^{i=4}\kappa_i\Delta(\zeta(b_i)), \end{cases} \tag{48}$$

is not empty and compact in $PC(\Upsilon, \mho)$, provided that

$$M||\Delta||\sum_{i=1}^{i=4}\kappa_i + 2M\,\upsilon + 4\kappa_5||\Delta|| < 1, \tag{49}$$

and

$$4M\upsilon < 1, \tag{50}$$

where $M = \sup\{||T(\delta)||, \delta \geq 0\}$.By choosing $\kappa_i, i = 1,..,5$,and υ small enough, both (49) and (50) will be satisfied.

Example 2. *Let $\alpha = \frac{2}{3}$,$\mho = L^2([0,\pi])$ and Υ, s_i, t_i ,$i = 1,2,3,4, t_5 = 1$, ,Δ, and Λ be as in in Example 1. On E, define $A : D(A) \subseteq \mho \to \mho$, $Az = z''$ with*

$$D(A) = \{z \in E : z, z' \text{ are absolutely continuous}, z(0) = z(\pi) = 0\}.$$

The operator A is sectorial and it is the infinitesimal generator of an Analytic semigroup$\{K(t) : t \geq 0\}$,

$$K(t)(x) = \sum_{k=1}^{\infty}\cos kt < x\,,\ x_k > x_k, x \in E,$$

where $x_k(y) = \sqrt{2}\,\sin ky, k = 1,2,...$, is the orthonormal set of eigen functions of A. Moreover, $M = 1$. Let $\Theta, g, g_i, 1 = 1,2,3,4$ be as in the Example 1. Then, by following the same arguments in Example1 and applying Theorem 4, the set of mild solutions to the following nonlocal impulsive conformable fractional semilinear differential inclusion :

$$\begin{cases} \frac{d^{\frac{2}{3}}}{d\zeta^{\frac{3}{3}}}\zeta(\delta) \in \zeta\prime\prime(\delta) + \frac{\upsilon}{\omega}\frac{||\zeta|| \sin\delta}{(1+||\zeta||)}\Lambda, a.e.\ \delta \in \cup_{i=0}^{i=4}(s_i, b_{i+1}], \\ \zeta(\delta) = i\kappa_5\delta\Delta(\zeta), \delta \in (b_i, s_i];\ i \in \{1,2,3,4\}, \\ \zeta(0) = \zeta_0 + \sum_{i=1}^{i=4}\kappa_i\Delta(\zeta(b_i)), \end{cases} \tag{51}$$

is not empty and compact in $PC(\Upsilon, \mho)$, provided that

$$||\Delta||\sum_{i=1}^{i=4}\kappa_i + 2\,\upsilon + 4\kappa_5||\Delta|| < 1, \tag{52}$$

and

$$4\upsilon < 1. \tag{53}$$

By choosing $\kappa_i, i = 1,..,5$,and υ small enough, both (52) and (53) will be satisfied.

6. Discussion and Conclusion

Unlike all known fractional derivatives, the conformable fractional derivative, was introduced by Khalil et al. [15], satisfies many basic properties of the usual derivative, such as the product rule, quotient rule, mean value theorem, chain rule and Taylor power series expansion. Therefore, the conformable fractional derivative is the most natural fractional derivative. For this reason, many researchers have shown interest in exploring more properties of the conformable fractional derivative and studying differential equations and involving. In this work, six existential results are presented for mild solutions of two types of fractional differential inclusions with non-local conditions involving the conformable fractional derivative in infinite-dimensional Banach spaces and in the attending of non-instantaneous impulses. In contrast to [30], we did not assume that the semigroup generated by the linear part is compact. We considered the case when the linear part is the infinitesimal generated of semigroup of linear bounded operators, as well as a sectorial operator. Also, we considered the case when the values of Θ are convex, as well as, nonconvex.

We propose the following future directions:

1- Study the controllability of problems (3) and (4).

2- Prove that the set of mild solutions to problems (3) and (4) are R_{δ} - sets.

3- Generalize the results obtained in [40] to the case of replacing the single-valued function f with a multi-valued function.

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