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Exotic Particle Dynamics using Novel Hermitian Spin Matrices

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Abstract: In this work, a generalized analogue to the Pauli spin matrices is presented and investigated. The proposed Hermitian spin matrices exhibit the following general symmetry: $\frac{n^2}{2}(\sigma'_1)^2 = n(\sigma'_2)^2 = n(\sigma'_3)^2 = -\frac{n^3}{2}(\sigma'_1)^2\sigma'_2\sigma'_3 = I_2$ for $n \in \mathbb{R}$. The generalized spin projection operators are derived and the electrodynamics for hypothetical fermions ($n = 2$) are explored using the proposed spin matrices. The fermionic quantum Heisenberg model is constructed using the proposed spin matrices and comparative studies against simulation results using the Pauli spin matrices are conducted. Further analysis on key findings as well as discussions on extending the proposed spin matrix framework to describe hypothetical bosonic systems (spin-1) are provided.

Keywords: exotic particle dynamics; generalized hermitian spin matrices; Pauli matrices; electrodynamics; fermionic quantum Heisenberg model; bosonic spin systems

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Introduction

In this work, four novel Hermitian spin matrices as analogues to the Pauli spin matrices are introduced. The proposed spin matrices exhibit the following symmetry: $\frac{n^2}{2}(\sigma'_1)^2 = n(\sigma'_2)^2 = n(\sigma'_3)^2 = -\frac{n^3}{2}(\sigma'_1)^2\sigma'_2\sigma'_3 = I_2$ for $n > 0$:

$$\sigma'_1 = \begin{bmatrix} 0 & \frac{1}{n}(1+i) \\ \frac{1}{n}(1-i) & 0 \end{bmatrix}, \quad \sigma'_2 = \begin{bmatrix} \frac{1}{\sqrt{n}} & 0 \\ 0 & -\frac{1}{\sqrt{n}} \end{bmatrix}, \sigma'_3 = \begin{bmatrix} -\frac{1}{\sqrt{n}} & 0 \\ 0 & \frac{1}{\sqrt{n}} \end{bmatrix} \tag{1}$$

This work aims to explore hypothetical fermion dynamics using the proposed Hermitian spin matrices in equation (1) at $n = 2$. Research works in the past decade have focused on the non-Hermitian formulation of quantum mechanics (Moiseyev, 2011; Bender, 2007; Ashida et al., 2020). Non-Hermitian physics has found various applications in phenomena related to optics, photonics, and condensed matter systems. In the recent work of Ju et al., (2022), the authors proposed a formalism to transform a non-Hermitian Hamiltonians to Hermitian ones without altering the underlying physics. Another interesting work is presented in Fring and Tenney, (2021). In that work the authors explored exactly solvable time-dependent non-Hermitian quantum systems. They employed complex point transformations for constructing non-Hermitian first integrals, metric operators and time-dependent Dyson maps for non-Hermitian quantum systems. A similar line of investigation is pursued by the authors of Koussa et al., (2018) where the time evolution of quantum systems was analyzed with respect to the time-dependent non-Hermitian Hamiltonian. This non-Hermitian Hamiltonian exhibits SU(1,1) and SU(2) dynamical symmetry. In Koussa et al., (2018), the exact solutions for the Schrödinger equations for both symmetries with respect to the eigenstates of the pseudo-Hermitian operators were obtained. In Luiz et al., (2020), the unitarity of time-evolution and the observability of non-Hermitian Hamiltonians were explored in the context of time-dependent Dyson maps. The authors in that work derived the time-dependent Dyson map for two instances. The first one via a constructed Schrödinger-like equation while the second instance was carried out using the non-Hermitian Hamiltonian.

An interesting review on the investigation of non-Hermitian dynamics in magnetic systems is presented in Hurst and Flebus, (2022). In that work, the authors describe non-Hermitian frameworks in magnonic and hybrid magnonic systems – e.g., magnon-qubit coupling schemes and cavity magnonic systems. The mentioned review also discusses recent advances in the dynamics of inherently lossy magnetic systems as well as systems with gain induced by external application of spin currents. In Zhang et al., (2021), the authors theoretically investigate the critical phases in steady states of non-unitary free fermion dynamics. The authors of that work explored the physics of such critical phases by developing a solvable static/Brownian quadratic Sachdev-Ye-Kitaev chains with non-Hermitian dynamics. Another interesting research review on non-Hermitian dynamics of open Markovian quantum systems is seen in Roccati et al., (2022). In that review, the authors outline some critical developments in the last two decades in research related to non-Hermitian Hamiltonians and their connections to the Gorini-Kossakowski-Sudarshan-Lindblad master equation. Besides non-Hermitian quantum dynamics, pseudo-Hermitian systems have also been a subject of recent investigations. For instance, in the work of Cius et al., (2022), the authors analyzed the pseudo-Hermitian dynamical Casimir effect. The authors present a novel non-Hermitian version of the effective Law's Hamiltonian to describe the mentioned effect.

Besides non-Hermitian matrices, researchers have also begun exploring pseudo-Hermitian frameworks. For instance, in He et al., (2023) the authors explored the topology of pseudo-Hermitian Chern insulator defined using the basis of q-deformed Pauli matrices (related to deformed algebras). The key finding s of that work was obtained for a completely nonequilibrium case where the quantum evolution after quenching was dictated by the Floquet pseudo-Hermitian Hamiltonian. Similarly, in Fring and Taira, (2020) the authors employed a pseudo-Hermitian approach to Goldstone's theorem in non-Abelian non-Hermitian quantum field theories. In that work, a detailed analysis for a non-Hermitian field theory with two complex scalar field (two-component) exhibiting SU(2) symmetry was presented. In Zhu et al., (2021), the authors demonstrated that several two and three-dimensional pseudo-Hermitian phases could be constructed using q-deformed matrices. In addition to investigations related to topological bulk states , quantum metrics and non-Abelian tensor Berry connections, an experimental protocol was proposed for empirical validation (of the proposed models). The following works provide current detailed developments on research efforts in the direction of non-Hermitian and pseudo-Hermitian frameworks: Okuma and Sato, (2023), Ashida et al., (2020), Kunst and Dwivedi (2019) and Feinberg and Riser, (2021).

This paper is organized as follows: The second and third sections describe the properties of the proposed spin matrices and their respective projection operators. In the fourth section, the electrodynamics of fermions is explored using the proposed spin matrices as operators in the Schrödinger–Pauli and Dirac equations. The fifth section compares spin chain simulation results using the proposed spin matrices and the Pauli spin matrices. This paper ends with further analysis on bosonic systems and some ideas on directions for future research.

Analogue Spin Matrices

Pauli spin matrices serve as quantum operators corresponding to observables for the spin of fermions at each spatial direction: $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. In this work, three generalized spin matrices are proposed as analogues to the Pauli spin matrices, σ_i (see equation (1)). The proposed spin matrices are Hermitian where the following complex conjugate transpose relations hold:

$$\sigma'_i = \overline{(\sigma'_i)^T} \quad \text{for } i = 1,2,3 \qquad (2)$$

The proposed matrices also exhibit the following symmetry:

$$\frac{n^2}{2}(\sigma'_1)^2 = n(\sigma'_2)^2 = n(\sigma'_3)^2 = -\frac{n^3}{2}(\sigma'_1)^2\sigma'_2\sigma'_3 = I_2 \text{ for any } n > 0$$

where the identity matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (3)$$

The parameter, $n > 0$ is real-valued. If the parameter, n is complex-valued, then the proposed matrices will lose their Hermitian property. Similar to the Pauli matrices, the following matrix properties apply:

$$\det\left(\frac{n}{\sqrt{2}}\sigma'_1\right) = -1 \text{ and } \mathrm{Tr}\left(\frac{n}{\sqrt{2}}\sigma'_1\right) = 0$$

$$\det(\sqrt{n}\sigma'_i) = -1 \text{ and } \mathrm{Tr}(\sqrt{n}\sigma'_i) = 0 \quad \text{for } i = 2, \tag{4}$$

However, unlike the Pauli matrices, the proposed matrices are not involutory. In addition, the equivalence relation in equation (3) yields the following relationship: $\sigma'_2 = -\frac{I_2}{n}(\sigma'_3)^{-1}$. The proposed matrices have the following commutative properties:

$$(\sigma'_k)^2\,\sigma'_i = \sigma'_i(\sigma'_k)^2 \text{ and } \sigma'_i\sigma'_j = \sigma'_i\sigma'_i$$
$$(\sigma'_k)^2\,\sigma'_i = \frac{2}{n^5}\sigma'_i \text{ and } \sigma'_i\sigma'_j = -\frac{1}{n}I_2 \text{ for } i,j = \{2,3\} \text{ and } k = 1 \tag{5}$$

This introduces a possibility for simplifying analysis via dimensional reduction (from three to two dimensions). The commutation relations for the proposed matrices where $[a,b] = ab - ba$ are as follows:

$$[\sigma'_1, \sigma'_2] = [\sigma'_3, \sigma'_1]$$
$$[\sigma'_2, \sigma'_3] = [(\sigma_1)^2, \sigma'_2] = [(\sigma'_1)^2, \sigma'_3] = O_2 \quad \text{where } O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{6}$$

On the other hand, the anti-commutation relations for the proposed matrices where $\{a,b\} = ab + ba$ are as follows:

$$\{\sigma'_1, \sigma'_2\} = \{\sigma'_1, \sigma'_3\} = O_2$$
$$-\{\sigma'_2, \sigma'_3\} = \{\sigma'_2, \sigma'_2\} = \{\sigma'_3, \sigma'_3\} = \frac{2}{n}I_2$$
$$\{\sigma'_1, \sigma'_1\} = \frac{4}{n^2}I_2, \tag{7}$$

The Pauli matrix, σ_3 is seen to be recovered using the anti-commutator operation:

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$\{(\sigma'_1)^2, \sigma'_2\} = -\{(\sigma'_1)^2, \sigma'_3\} = \frac{4}{\sqrt{n^5}}\sigma_3 \tag{8}$$

Using the proposed matrices, an analogue to the gamma matrices in the Dirac basis is then constructed as follows:

$$\gamma'_j = \begin{bmatrix} 0 & \sigma'_j \\ \sigma'_j & 0 \end{bmatrix} \text{ where } \gamma'_2 = -\frac{I_2}{n}(\gamma'_3)^{-1} \tag{9}$$

The proposed time-like gamma matrix is then as follows:

$$\gamma'_0 = -\frac{1}{\sqrt{n}}\begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix} \qquad \text{where,}$$
$$\frac{n^2}{2}(\gamma'_1)^2 = n(\gamma'_2)^2 = n(\gamma'_3)^2 = n(\gamma'_0)^2 = \frac{\sqrt{n^7}}{2}(\gamma'_1)^2\gamma'_2\gamma'_3\gamma'_0 = \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix} = I_4 \tag{10}$$

It is important to note that as with the Pauli matrices, all the proposed gamma matrices are Hermitian. Hence, these matrices yield real eigenvalues (real quantum energy states). The anti-commutation relations for the proposed gamma matrices are:

$$\{(\gamma'_1)^2, \gamma'_2\} = \{(\gamma'_1)^2, \gamma'_3\} = \begin{bmatrix} O_2 & 0 \\ 0 & O_2 \end{bmatrix}$$
$$\{(\gamma'_1)^2, \gamma'_0\} = -\frac{4}{\sqrt{n^5}}I_4\,;\, \{\gamma'_2, \gamma'_0\} = \{\gamma'_3, \gamma'_0\} = \frac{2}{n}\begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix};$$
$$\{\gamma'_1, \gamma'_1\} = \frac{4}{n^2}I_2\,;\, \{(\gamma'_1)^2, (\gamma'_1)^2\} = \frac{8}{n^4}I_4$$
$$\{\gamma'_j, \gamma'_j\} = \{\gamma'_2, \gamma'_3\} = \frac{2}{n}I_4 \qquad \text{for } j \in \{0,2,3\} \tag{11}$$

As with the conventional gamma matrices, the following matrix properties hold:

$$\det\left(\frac{n}{\sqrt{2}}\gamma'_1\right) = 1$$
$$\det(\sqrt{n}\gamma'_j) = 1 \text{ and } \mathrm{Tr}(\sqrt{n}\gamma'_j) = 0 \;\; j \in \{0,2,3\} \tag{12}$$

Spin Projection Operators

In this section, the spin projection operators are obtained for the exotic fermions using the proposed spin matrices. A generalized formulation is given for spin projection operators for fermions with spin-$^1/_n$ at $n = 2$. The two-component spinor is then employed to represent the quantum state of a fermion using the proposed spin matrices. The spin projection operator, S_1 is expressed a s follows:

$$S_1 = \frac{\hbar}{n}\left(\frac{n}{\sqrt{2}}\sigma'_1\right) = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & \frac{1}{n}(1+i) \\ \frac{1}{n}(1-i) & 0 \end{bmatrix} \tag{13}$$

where $\hbar$ is the reduced Planck's constant. The eigenvalues for S_1 are $\lambda_1 = \frac{\hbar}{n}$ and $\lambda_2 \approx \frac{\hbar}{n}(-1 + 7.85046 \times 10^{-17} i)$. The eigenvectors for S_1 are $v_1 = [(\sqrt{2} + \sqrt{2}i), 1]$ and $v_2 = [(-\sqrt{2} - \sqrt{2}i), 1]$. The eigenspinor representations are then as follows:

$$\chi_1^1 = \begin{bmatrix} (\sqrt{2} + \sqrt{2}i) \\ 1 \end{bmatrix} = \left| s_1 = +\frac{1}{n} \right\rangle = |\uparrow\rangle$$

$$\chi_2^1 = \begin{bmatrix} -(\sqrt{2} - \sqrt{2}i) \\ 1 \end{bmatrix} = \left| s_1 = \left(-\frac{1}{n} + 7.85046 \times 10^{-17} i \right) \right\rangle = |\downarrow\rangle \quad (14)$$

The spin projection operator, S_2 is expressed as follows:

$$S_2 = \frac{\hbar}{n}\left(\sqrt{n}\sigma'_2\right) = \frac{\hbar}{\sqrt{n}}\sigma'_2 = \frac{\hbar}{\sqrt{n}} \begin{bmatrix} \frac{1}{\sqrt{n}} & 0 \\ 0 & -\frac{1}{\sqrt{n}} \end{bmatrix} \quad (15)$$

The eigenvectors for S_2 are $v_1 = [0, 1]$ and $v_2 = [1,0]$, and the eigenspinor representations are then as follows:

$$\chi_1^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left| s_2 = +\frac{1}{n} \right\rangle = |\uparrow\rangle$$

$$\chi_2^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left| s_2 = -\frac{1}{n} \right\rangle = |\downarrow\rangle \quad (16)$$

The spin projection operator, S_3 is expressed as follows:

$$S_3 = \frac{\hbar}{n}\left(\sqrt{n}\sigma'_3\right) = \frac{\hbar}{\sqrt{n}}\sigma'_3 = \frac{\hbar}{\sqrt{n}} \begin{bmatrix} -\frac{1}{\sqrt{n}} & 0 \\ 0 & \frac{1}{\sqrt{n}} \end{bmatrix} \quad (17)$$

The eigenvectors for S_3 are $v_1 = [1, 0]$ and $v_2 = [0,1]$, and the eigenspinor representations are then:

$$\chi_1^3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left| s_3 = +\frac{1}{n} \right\rangle = |\uparrow\rangle$$

$$\chi_2^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left| s_3 = -\frac{1}{n} \right\rangle = |\downarrow\rangle \quad (18)$$

An interesting difference with the Pauli spin projection operators is that for the operator S_1, the $|\downarrow\rangle$ spin state contains a small imaginary contribution: $7.85046 \times 10^{-17} i$.

Electrodynamics

In this section, the non-relativistic electrodynamics of exotic fermions are explored using the Schrödinger–Pauli equation (Niederle and Nikitin, 1999; Mourad and Sazdjian, 1994). The Schrödinger–Pauli equation is presented using the proposed spin matrices at $n = 2$:

$$\frac{1}{2m}\left[\left(\sqrt{2}\boldsymbol{\sigma}' \cdot (\hat{\mathbf{p}} - q\vec{\boldsymbol{A}})\right)^2 + q\phi\right]|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle \quad (19)$$

$$\hat{H}_0 = \left[\left(\sqrt{2}\boldsymbol{\sigma}' \cdot (\hat{\mathbf{p}} - q\vec{\boldsymbol{A}})\right)^2 + q\phi\right] \quad (20)$$

where m is the fermion mass, $\hat{\mathbf{p}}$ is the vector form of the momentum operator, $\vec{\boldsymbol{A}}$ is the magnetic vector potential, q is the fermion's electric change, ϕ is the electric scalar potential, $\hat{H}_0$ is the Hamiltonian (using the proposed spin matrices) and $|\psi\rangle$ is the quantum state. The spin matrices are represented as the following vector: $\boldsymbol{\sigma}' = [\sigma'_i, \sigma'_j, \sigma'_k]$. Implementing the proposed spin matrices, the Schrödinger–Pauli equation yields solutions for the Hamiltonian, $\hat{H}$. Considering the fermion (at $n = 2$) subjected to a constant magnetic field within the Landau gauge (Blasi et al., 1991):

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} \text{ with the possible solution: } \vec{\mathbf{A}} = \begin{pmatrix} 0 \\ Bx \\ 0 \end{pmatrix} \quad (21)$$

where B is the uniform magnetic field. Solving the Schrödinger–Pauli equation for $\hat{H}_0$ gives the following Hamiltonian operators:

$$\hat{H}_0 = \frac{1}{2nm}\left[2\hat{p}_1{}^2 + n(\hat{p}_2 - \hat{p}_3 - qBx)^2\right]I_2 + q\phi \quad (22)$$

This analysis is extended using the Landau symmetric gauge where the magnetic vector potential is given as follows:

$$\vec{\mathbf{A}} = \frac{1}{2}\mathbf{B} \times \vec{\mathbf{r}} = \frac{1}{2}\begin{pmatrix} -By \\ Bx \\ 0 \end{pmatrix} \quad (23)$$

Solving the Schrödinger–Pauli equation within the Landau symmetric gauge for each spin configuration, $\boldsymbol{\sigma}'$ gives the following Hamiltonian operators:

$$\hat{H}_0 = \frac{1}{2nm}[2(\hat{p}_1 + qBy)^2 + n(\hat{p}_2 - \hat{p}_3 - qBx)^2]I_2 + q\phi \quad (24)$$

In the context of relativistic quantum electrodynamics, the following Dirac equation is considered to determine the rest energy for the fermion placed in an electric potential, cqA^0:

$$\hat{H}_0 = \sqrt{2}\gamma'_0 \left[mc^2 + c\sum_j \sqrt{2}\gamma'_j \hat{p}_j\right] + cqA^0 \quad \text{where } j = 1,2,3 \quad (25)$$

where m is the mass of the fermion, c is the speed of light, γ'_j is the proposed gamma matrices and. As with the conventional gamma matrices in the Dirac basis, it can be shown using equation (25) that the rest energy of the exotic fermion at $\vec{A} = 0$ is equivalent to the energy of a particle placed in an electric potential, cqA^0.

Simulation with Spin Chains

The proposed spin matrices are compared with the Pauli spin matrices by analyzing their respective behaviors in the context of a spin chain using the spin$-{}^1/_2$ quantum Heisenberg model (Dahbi et al., 2023; Mohamed et al., 2023). The one-dimensional Heisenberg model is utilized where magnetic interactions take place specifically between adjacent dipoles:

$$\hat{H} = -h\sum_{j=1}^{N} \sigma_j - J\sum_{j=1}^{N} \sigma_j\sigma_{j+1} \quad (26)$$

where J is the coupling constant, h is the external magnetic field and the dipoles are described as the quantum operator acting on the Kronecker product of dimensions, 2^N. Considering the coupling constant, $J = (J_x, J_y, J_z)$ to be real-valued, the Hamiltonian operator is represented as follows:

$$\hat{H} = -\frac{1}{2}\sum_{j=1}^{N}\left(h\sigma_j^3 + J_x\sigma_j^1\sigma_{j+1}^1 + J_y\sigma_j^2\sigma_{j+1}^2 + J_z\sigma_j^3\sigma_{j+1}^3\right) \quad (27)$$

where σ_j^k is the k^{th} Pauli matrix on the j^{th} lattice point with periodic boundary conditions. The Pauli matrices, σ_j^k is defined as: $\sigma_j^k = I_2^{\otimes j-1} \otimes \sigma_j^k \otimes I_2^{\otimes N-j}$. In this work the Heisenberg XXX model with $J > 0$ is employed where the coupling constants conform to the following equivalence: $J = J_x = J_y = J_z$. The simulation is carried out using the Python programming language. To simplify the simulation, a qubit system is considered: where the number of spins in the chain is limited to two. The following parameters are fixed in the simulation: coupling constant, $J = 1 \times 10^{-19}$ and $B = 1 \times 10^{-19}$ T. The simulation is performed on three temperature values: $T_1 = 10$ K (low), $T_2 = 1000$ K (medium) and $T_3 = 10{,}000$ K (high). The spectrum of the Hamiltonian (i.e., energy states, $\lambda(\boldsymbol{\sigma}) = H(\boldsymbol{\sigma})$) is then obtained for each spin state: $\boldsymbol{\sigma} = \{(1,1), (1,-1), (-1,1), (1,1)\}$. The configuration probability for each spin state, $P(\boldsymbol{\sigma})$ is determined as follows:

$$P(\boldsymbol{\sigma}) = Z^{-1}\exp[-\beta\lambda(\boldsymbol{\sigma})] \quad (28)$$

where Z is the partition function and k_B is the Boltzmann constant:

$$Z = \sum_{\boldsymbol{\sigma}} \exp[-\beta\lambda(\boldsymbol{\sigma})] \quad \text{and} \quad \beta = (k_B T)^{-1} \quad (29)$$

Computational experiments were performed by executing the simulation using Pauli matrices and the proposed spin matrices. Table 1 provides the state probabilities generated from the simulations performed for each configuration of the proposed spin matrices:

Table 1. State probabilities from simulations for each configuration of the proposed spin matrices.

Spin matrix configuration (*i,j,k*)	Spin States at $T = 10{,}000$K	(1,1)	(1,-1)	(-1,1)	(-1,-1)
State Probabilities using proposed spin matrices		3.13E-02	5.66E-01	7.65E-02	3.26E-01
State Probabilities using Pauli ma-		2.68E-01	2.32E-01	2.68E-01	2.32E-01

trices					
Spin matrix configuration (*i,j,k*)	Spin States at *T*=1000K	(1,1)	(1,-1)	(-1,1)	(-1,-1)
State Probabilities using proposed spin matrices		2.61E-13	9.96E-01	2.01E-09	3.94E-03
State Probabilities using Pauli matrices		5.00E-01	0.00E+00	5.00E-01	0.00E+00
Spin matrix configuration (*i,j,k*)	Spin States at *T*=10K	(1,1)	(1,-1)	(-1,1)	(-1,-1)
State Probabilities using proposed spin matrices		0.00E+00	1.00E+00	0.00E+00	5.00E-241
State Probabilities using Pauli matrices		5.00E-01	0.00E+00	5.00E-01	0.00E+00

In Table 1, the comparison of the state probabilities obtained using the proposed spin matrices and the state probabilities generated using the proposed spin matrices for temperatures, T = 10,000 K, 1000 K and 10 K is shown. It can be observed for that the overall dynamics of the system is similar to the dynamics of spin chains using Pauli matrices – where the particles experience losses in magnetic orientation at higher temperatures while the restoration of orientation arises at lower temperatures. However, the key distinguishing feature of the simulations using the proposed spin matrices is that the results show a completely different selection of state orientation probabilities; as compared to the simulation results of the system using the Pauli spin matrices.

Analysis & Future Work

One of the key distinctive features of the proposed spin matrices as compared to the Pauli matrices is that these matrices introduce a different type of algebra in relation to their commuting and anti-commuting properties (see Section 2). As seen in Section 3, the proposed spin matrices also produce spin states which contain a constant small imaginary contribution of : $7.85046 \times 10^{-17} i$. The electrodynamics exploration using the Schrödinger–Pauli equation conducted in Section 4 using the proposed spin matrices yield different Hamiltonian expressions as compared to Pauli matrices. However, the computation of the rest energy of a theoretical fermion using the proposed gamma matrices is consistent with the analysis on the Dirac equation performed using conventional gamma matrices. In this line of reasoning, it is also possible to construct higher spin systems (e.g., for bosons) using the proposed spin matrices. For instance, a set of Hermitian spin-1 matrices for triplet states using the proposed spin matrices at $n = 1$ is constructed. The spin projection operator, S_1 is expressed a s follows:

$$S_1 = \frac{\hbar}{n}\left(\frac{n}{\sqrt{2}}\sigma'_1\right) = \hbar \begin{bmatrix} 0 & \frac{1}{n}(1+i) & 0 \\ \frac{1}{n}(1-i) & 0 & \frac{1}{n}(1+i) \\ 0 & \frac{1}{n}(1-i) & 0 \end{bmatrix} \quad (30)$$

The eigenvalues for S_1 are $\lambda_1 = -\frac{\hbar\sqrt{2}}{n}$, $\lambda_2 = \frac{\hbar\sqrt{2}}{n}$ and $\lambda_3 = 0$ The eigenvectors for S_1 are $v_1 = [i, -i-1{,}1]$, $v_2 = [i, i+1{,}1]$ and $v_3 = [-i, 0{,}1]$. The eigenspinor representations are then as follows:

$$\chi_1^1 = \begin{bmatrix} i \\ -i-1 \\ 1 \end{bmatrix} = \left|s_1 = -\frac{\sqrt{2}}{n}\right\rangle$$

$$\chi_2^1 = \begin{bmatrix} i \\ i+1 \\ 1 \end{bmatrix} = \left|s_1 = +\frac{\sqrt{2}}{n}\right\rangle$$

$$\chi_3^1 = \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix} = |s_1 = 0\,\rangle \quad (31)$$

The spin projection operator, S_2 is expressed a s follows:

$$S_2 = \frac{\hbar}{n}\left(\sqrt{n}\sigma'_1\right) = \frac{\hbar}{n}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix} \quad (32)$$

The eigenvalues for S_2 are $\lambda_1 = -\frac{\hbar}{n}$, $\lambda_2 = \frac{\hbar}{n}$ and $\lambda_3 = 0$ The eigenvectors for S_1 are $v_1 = [0,0,1]$, $v_2 = [1,0,0]$ and $v_3 = [0,1,0]$. The eigenspinor representations are then as follows:

$$\chi_1^2 = \begin{bmatrix}0\\0\\1\end{bmatrix} = \left|s_2 = -\frac{1}{n}\right\rangle$$

$$\chi_2^2 = \begin{bmatrix}1\\0\\0\end{bmatrix} = \left|s_2 = +\frac{1}{n}\right\rangle$$

$$\chi_3^2 = \begin{bmatrix}0\\1\\0\end{bmatrix} = |s_2 = 0\,\rangle \quad (33)$$

The spin projection operator, S_2 is expressed a s follows:

$$S_3 = \frac{\hbar}{n}\left(\sqrt{n}\sigma'_1\right) = \frac{\hbar}{n}\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} \quad (34)$$

The eigenvalues for S_3 are $\lambda_1 = -\frac{\hbar}{n}$, $\lambda_2 = \frac{\hbar}{n}$ and $\lambda_3 = 0$ The eigenvectors for S_1 are $v_1 = [1,0,0]$, $v_2 = [0,0,1]$ and $v_3 = [0,1,0]$. The eigenspinor representations are then as follows:

$$\chi_1^3 = \begin{bmatrix}0\\0\\1\end{bmatrix} = \left|s_2 = -\frac{1}{n}\right\rangle$$

$$\chi_2^3 = \begin{bmatrix}1\\0\\0\end{bmatrix} = \left|s_2 = +\frac{1}{n}\right\rangle$$

$$\chi_3^3 = \begin{bmatrix}0\\1\\0\end{bmatrix} = |s_2 = 0\,\rangle \quad (35)$$

The proposed spin matrices also allow theoretical explorations for particles with arbitrary spins by defining the appropriate n. In Table 1, it can be observed that the spin states obtained by the simulations of quantum Heisenberg model using the proposed spin matrices differ from the ones generated using the Pauli spin matrices. In addition, since the system of proposed spin matrices are completely Hermitian, real eigenstates (energy states) are consistently obtained. As observed in Section 2, the relationship: $\sigma'_2 = -\frac{I_2}{n}(\sigma'_3)^{-1}$ could be established. This introduces the possibility of reducing a three-dimensional analysis to a two-dimensional one. Considering this dimensional reduction result, the proposed spin matrices may have applications in describing quasiparticles in two-dimensional systems – e.g., anyons and the fractional quantum Hall effect (Manna et al., (2020); Stern, 2008). It is also conjectured that the proposed spin matrices and their underlying symmetry may have applications in the particle physics of more exotic forms of matter – e.g., dark matter.

Future work could be directed towards exploring other spin state systems (where $n \neq \frac{1}{2}$) as provided in this section. Further generalizations of the proposed matrices to produce analogues to the Gell-Mann matrices for theoretically investigating the particle physics involving strong interactions could be conducted. In addition, the proposed gamma matrices and their formulations in the Weyl and Majorana basis could also be carried out. Finally, the implications of the proposed gamma matrices on quantum interactions via field theory would be an interesting avenue for potential research.

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