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On the Oscillatory Behavior of a Class of Mixed Fractional Order Nonlinear Differential Equations

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Abstract: This article deals with the investigation of the oscillatory behavior of a class of mixed fractional order nonlinear differential equations based on conformable fractional derivative and the Liouville right - sided fractional derivative by using a generalized Riccati technique and an integral averaging method. Examples illustrating the significance and effectiveness of the results are also given.

Keywords: fractional differential equation; Riccati technique; oscillation

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1. Introduction

The theory and applications of fractional differential equations are contained in many monographs and articles [1–8,10–24]. Over the last years, the fractional order differential equations have proved to be the most valuable and effective tools in the modeling of several phenomena in various fields of science and engineering. In fact, we can find numerous applications on the design of fractional control systems. The electrical properties of nerve cell membranes and the propagation of electrical signals are both characterized by fractional order derivatives.The Fractional Adevective - Dispersive equation has been the model basis for simulating transport in porous media. This model has been applied to laboratory and field experiments. The fundamental explorations of the mechanical, electrical and thermal constitutive relations of various engineering materials such as viscoelastic polymers, are modeled successfully. In the area of financial markets, fractional order models have been recently used to describe the probability distribution of log prices in the long time - limit which is useful to characterize the natural variability in prices in the long term. See for example [1,7,8,12,13,16,18,20,21,25].

Fractional differentials and integrals provide more accurate models of the above aforementioned system. There are several kinds of definitions for fractional derivatives and integrals such as the Riemann - Liouville definition, the Caputo definition, the Liouville right-sided definition on the half axis $\mathbb{R}_+$, which are all based on integrals with singular kernels and exhibiting non - local behaviors which fail to satisfy the product, quotient and chain rules. In contrast, 2014, Khalil et al, introduced a limit based definition analogous to that for standard derivatives. See [2–4,11,19].

In recent years, there has been an increasing interest in obtaining sufficient conditions for oscillatory and nonoscillatory behavior of different classes of fractional differential equations. The oscillation theory of fractional differential equations with Liouville right - sided definition has been studied by many authors Chen [6], Xu [22], Han [9], Pan [15] and with damping term investigated by some other author's Qi [17], Zheng [23,24].

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In 2013, Xu [22] investigated the oscillatory behavior of class of nonlinear fractional differential equations of the following form

$$\left(a(t)\left[\left(r(t)g(D_{-}^{\alpha}x(t))'\right]\right]^{\eta}\right)' - F\left(t, \int_t^{\infty}(\nu - t)^{-\alpha}x(\nu)d\nu\right) = 0 \; for \; t \geq t_0 > 0.$$

In the same year, Zheng and Feng [23] discussed the oscillatory behavior of the following equation

$$\left(a(t)\left[\left(r(t)(D_{-}^{\alpha}x(t))'\right]\right]^{\gamma}\right)' + p(t)\left[\left(r(t)(D_{-}^{\alpha}x(t))'\right]\right]^{\gamma} - q(t)f\left(\int_t^{\infty}(\xi - t)^{-\alpha}x(\xi)d\xi\right) = 0$$

for $t \in [t_0, \infty)$, $0 < \alpha < 1$. In 2017, Pavithra and Muthulakshmi [14] studied the oscillatory behavior for the class of nonlinear fractional differential equations with damping term of the following form

$$\left(a(t)\left[\left(r(t)g(D_{-}^{\alpha}x(t))'\right]\right]^{\eta}\right)' + p(t)\left[\left(r(t)g(D_{-}^{\alpha}x(t))'\right]\right]^{\eta} - F\left(t, \int_t^{\infty}(\nu - t)^{-\alpha}x(\nu)d\nu\right) = 0 \; for \; t \geq t_0 > 0.$$

In 2020, G. E. Chatzarakis et al, investigated the Oscillatory Properties of a Certain Class of Mixed Fractional Differential Equations with the conformable fractional derivative and the Riemann-Liouville left-sided fractional derivative. From the above quoted literature, we have observed that the Liouville right-sided fractional derivative together with classical integer order derivative are used for $(2 + \alpha)$ order nonlinear differential equations. To the best of the author's knowledge, it seems that there has been no work done with conformable and Liouville right-sided derivatives in the fractional order differential equations.

Motivated by this gap, the authors have initiated the following oscillation problem of a class of mixed fractional order nonlinear differential equation of the form

$$T_{\alpha_3}\left[r_2(t)f_2\left(T_{\alpha_2}\left(r_1(t)f_1\left(D_{-}^{\alpha_1}x(t)\right)\right)\right)\right] + t^{1-\alpha_3}P(t)f_2\left(T_{\alpha_2}\left(r_1(t)f_1\left(D_{-}^{\alpha_1}x(t)\right)\right)\right) - F\left(t, \int_t^{\infty}(\nu - t)^{-\alpha_1}x(\nu)d\nu\right) = 0, t \geq t_0 > 0, 0 < \alpha_i < 1, i = 1,2,3 \quad (1.1)$$

where $D_{-}^{\alpha_1}$ denotes the Liouville right-sided fractional derivative and $T_{\alpha_2}, T_{\alpha_3}$ denotes the conformable fractional derivatives.

Throughout this paper, we will assume that the following conditions hold:

(A_1) $R(t) = \int_{t_0}^{t}\frac{P(s)}{r_2(s)}ds, P(t) \in C([t_0, \infty), \mathbb{R}_+))$;

(A_2) $r_2(t) \in C^{\alpha_3}([t_0, \infty), \mathbb{R}_+)), r_1(t) \in C^{\alpha_2+\alpha_3}([t_0, \infty), \mathbb{R}_+)), \int_{t_0}^{\infty}\frac{1}{f_2^{-1}[e^{R(s)}r_2(s)]}d_{\alpha_2}s = \infty$;

(A_3) $(a) f_i \in C^{\alpha_2+\alpha_3}(\mathbb{R}, \mathbb{R})$ is an increasing and odd function and there exist positive constants δ_i such that $\frac{x}{f_i(x)} \leq \delta_i > 0$ for $xf_i(x) \neq 0$, where i=1,2 and let $\delta = \delta_1\delta_2$;

(b) $f_i^{-1} \in C(\mathbb{R}, \mathbb{R})$ with $uf_i^{-1}(u) > 0$ for $u \neq 0$, and there exist some positive constants λ_i such that $f_i^{-1}(uv) \geq \lambda_i f_i^{-1}(u)f_i^{-1}(v)$ for $uv \neq 0$, i=1,2, where $\lambda = \lambda_1\lambda_2$;

(A_4) $F(t, K) \in C^1([t_0, \infty) \times \mathbb{R}, \mathbb{R}_+))$ there exists a function $Q(t) \in C^1([t_0, \infty), \mathbb{R}_+))$ such that $\frac{F(t,K)}{f_2(K)} \geq Q(t)$ for $K \neq 0$ $t \geq t_0$.

By a solution of (1.1), we mean a function $x(t) \in C(\mathbb{R}_+, \mathbb{R})$ such that $\int_t^{\infty}(\nu - t)^{-\alpha_1}x(\nu)d\nu \in C^1(\mathbb{R}_+, \mathbb{R}), r_1(t)f_1(D_{-}^{\alpha_1}x(t)) \in C^{\alpha_2+\alpha_3}(\mathbb{R}_+, \mathbb{R})$ and satisfies (1.1) on $[t_0, \infty)$.

A nontrivial solution of (1.1) is called *oscillatory* if it has arbitrary large zeros, otherwise it is called *nonoscillatory*. (1.1) is called oscillatory if all of its solutions are oscillatory.

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The main purpose of this paper is to extend and generalize all the results established in [6,9,14,15,17,22–24] to the mixed fractional differential equations (1.1) and to provide a detailed discussion of the main results by making use of the generalized Riccati technique and integral averaging method.

This paper is organized as follows: In Section 2, we recall the basic definitions of Liouville right-sided derivative and conformable fractional derivatives along with basic lemmas concerning the above set of derivatives. In Section 3, we present some new results of oscillation of solutions of (1.1). In Section 4, examples are provided to illustrate the main results.

2. Preliminaries

Before starting our analysis of (1.1), we have to explain the meaning of the operators $D^{\alpha}_{-}x(t)$ and $T_{\alpha}(f)(t)$. For the sake of completeness let us provide the essentials of fractional calculus according to Liouville right-sided approach and Khalil's conformable fractional derivative. Let us first define the Liouville right-sided operator.

Definition 2.1 ([12]). *The Liouville right-sided fractional derivative of order α of $x(t)$ is defined by*

$$D^{\alpha}_{-}x(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^{\infty}(\nu-t)^{-\alpha}x(\nu)d\nu, t \in \mathbb{R}_+ = (0,\infty),$$

where $\Gamma(.)$ is the gamma function defined by $\Gamma(t) = \int_0^{\infty} e^{-s}s^{t-1}ds, t \in \mathbb{R}_+$.

Lemma 2.1 ([6]). *Let x(t) be a solution of (1.1) and*

$$K(t) = \int_t^{\infty}(\nu-t)^{-\alpha}x(\nu)d\nu. \tag{2.1}$$

Then

$$K'(t) = -\Gamma(1-\alpha)D^{\alpha}_{-}x(t). \tag{2.2}$$

Next, we give the definition of the conformable fractional derivative proposed by Khalil et al [11].

Definition 2.2. *Given a function $f : [0,\infty) \to \mathbb{R}$. Then the conformable fractional derivative of f of order α, is defined by*

$$T_{\alpha}(f)(t) = \lim_{\epsilon\to 0}\frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t > 0, \alpha \in (0,1)$. If f is α- differentiable in some $(0,a), a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

We will sometimes write $f^{(\alpha)}(t)$ for $T_{\alpha}(f)(t)$, to denote the conformable fractional derivatives of f of order α.

Some properties of conformable fractional derivative:
Let $\alpha \in (0,1]$ and f and g be α - differentiable at a point $t > 0$. Then
(P_1) $T_{\alpha}(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.
(P_2) $T_{\alpha}(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
(P_3) $T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f)$.
(P_4) $T_{\alpha}(\frac{f}{g}) = \frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^2}$.
(P_5) If, in addition, f is differentiable, then $T_{\alpha}(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

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Definition 2.3 ([19]). *Let $\alpha \in (0,1]$ and $0 \le a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on [a,b] if the integral*

$$\int_a^b f(x)d_\alpha x := \int_a^b f(x)x^{\alpha-1}dx$$

exists and is finite.

Lemma 2.2 ([2]). *Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \le 1$. Then, for all $t > a$ we have*

$$I_\alpha^a T_\alpha^a(f)(t) = f(t) - f(a). \tag{2.3}$$

The following inequality is taken from Hardy et al [10], used in the sequel.

Lemma 2.3. *If X and Y are nonnegative, then*

$$mXY^{m-1} - X^m \le (m-1)Y^m, m > 1. \tag{2.4}$$

3. Main Results

In this section, we will present some new oscillation criteria for (1.1).

Lemma 3.1. *Assume that $x(t)$ is an eventually positive solution of (1.1). If*

$$\int_{t_0}^{\infty} f_1^{-1}\left(\frac{1}{r_1(s)}\right)ds = \infty \tag{3.1}$$

and

$$\int_{t_0}^{\infty} f_1^{-1}\left(\frac{1}{r_1(\tau)}\int_\tau^\infty f_2^{-1}\left(\frac{1}{e^{R(\xi)}r_2(\xi)}\int_\xi^\infty e^{R(s)}Q(s)d_{\alpha_3}s\right)d_{\alpha_2}\xi\right)d\tau = \infty, \tag{3.2}$$

then there exists a sufficiently large T, such that $T_{\alpha_2}(r_1(s)f_1(D_-^{\alpha_1}x(s))) < 0$ on $[T,\infty)$, and one of the following two conditions holds:

(i) $D_-^{\alpha_1}x(t) < 0$ on $[T,\infty)$
(ii) $D_-^{\alpha_1}x(t) > 0$ on $[T,\infty)$, and $\lim_{t\to\infty} K(t) = 0$.

Proof. Let $t_1 \ge t_0$ be such that $x(t) > 0$ on $[t_1,\infty)$ and so $K(t) > 0$ on $[t_1,\infty)$. Thus by (1.1) and (A_4), we have

$$T_{\alpha_3}\left[e^{R(t)}r_2(t)f_2(T_{\alpha_2}(r_1(t)f_1(D_-^{\alpha_1}x(t))))\right] \ge e^{R(t)}Q(t)f_2(K(t)) > 0, t \in [t_1,\infty), \tag{3.3}$$

which means that $e^{R(t)}r_2(t)f_2(T_{\alpha_2}(r_1(t)f_1(D_-^{\alpha_1}x(t))))$ is strictly increasing on $[t_1,\infty)$. Consequently, we can conclude that $T_{\alpha_2}(r_1(t)f_1(D_-^{\alpha_1}x(t)))$ is eventually of one sign. We claim that $T_{\alpha_2}(r_1(t)f_1(D_-^{\alpha_1}x(t))) < 0$ on $[t_2,\infty)$, where $t_2 > t_1$ is sufficiently large. Otherwise, there exists a sufficiently large $t_3 > t_2$ such that $T_{\alpha_2}(r_1(t)f_1(D_-^{\alpha_1}x(t))) > 0$ on $[t_3,\infty)$. Then for $t \in [t_3,\infty)$ and using Lemma 2.2, we get

$$r_1(t)f_1(D_-^{\alpha_1}x(t)) - r_1(t_3)f_1(D_-^{\alpha_1}x(t_3)) = \int_{t_3}^t \frac{f_2^{-1}\left[e^{R(s)}r_2(s)\right]T_{\alpha_2}(r_1(s)f_1(D_-^{\alpha_1}x(s)))}{f_2^{-1}\left[e^{R(s)}r_2(s)\right]}d_{\alpha_2}s$$

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$$
\begin{aligned}
r_1(t)f_1\big(D_-^{\alpha_1}x(t)\big)-r_1(t_3)f_1\big(D_-^{\alpha_1}x(t_3)\big)& \\
&\geq f_2^{-1}\Big[e^{R(t_3)}r_2(t_3)\Big]T_{\alpha_2}\big(r_1(t_3)f_1\big(D_-^{\alpha_1}x(t_3)\big)\big)\\
&\times\int_{t_3}^{t}\frac{1}{f_2^{-1}\big[e^{R(s)}r_2(s)\big]}d_{\alpha_2}s. \qquad (3.4)
\end{aligned}
$$

From (A_2), we have $\lim_{t\to\infty} r_1(t)f_1\big(D_-^{\alpha_1}x(t)\big)=\infty$, which implies that for some sufficiently large $t_4>t_3$, $r_1(t)f_1\big(D_-^{\alpha_1}x(t)\big)>0$. Thus, it is obvious that

$$
r_1(t)f_1\big(D_-^{\alpha_1}x(t)\big)\geq r_1(t_4)f_1\big(D_-^{\alpha_1}x(t_4)\big):=c>0, t\in[t_4,\infty). \qquad (3.5)
$$

From(A_3), we have

$$
-\frac{K'(t)}{\Gamma(1-\alpha_1)}=D_-^{\alpha_1}x(t)\geq f_1^{-1}\left[\frac{c}{r_1(t)}\right]\geq\lambda_1 f_1^{-1}(c)f_1^{-1}\left(\frac{1}{r_1(t)}\right), t\in[t_4,\infty)
$$

and therefore

$$
f_1^{-1}\left(\frac{1}{r_1(t)}\right)\leq-\frac{K'(t)}{\lambda_1\Gamma(1-\alpha_1)f_1^{-1}(c)},\ t\in[t_4,\infty). \qquad (3.6)
$$

Integrating (3.6) from t_4 to t, we obtain

$$
\int_{t_4}^{t} f_1^{-1}\left(\frac{1}{r_1(s)}\right)ds\leq\frac{K(t_4)}{\lambda_1\Gamma(1-\alpha_1)f_1^{-1}(c)},\ t\in[t_4,\infty).
$$

Letting $t\to\infty$, it follows that

$$
\int_{t_4}^{\infty} f_1^{-1}\left(\frac{1}{r_1(s)}\right)ds<\infty,\ t\in[t_4,\infty),
$$

which contradicts (3.1). Therefore $T_{\alpha_2}\big(r_1(t)f_1\big(D_-^{\alpha_1}x(t)\big)\big)<0$ on $[t_2,\infty)$.

From, (A_3), we get that $D_-^{\alpha_1}x(t)$ is eventually one sign. Consequently, there are two possibilities:(i) $D_-^{\alpha_1}x(t)<0$ on $[T,\infty)$ (ii) $D_-^{\alpha_1}x(t)>0$ for on $[T,\infty)$ for sufficiently large T. Suppose $D_-^{\alpha_1}x(t)>0$ for $t\in[T,\infty)$, for sufficiently large $T>t_2$. Thus, $K'(t)<0, t\in[T,\infty)$, and we have $\lim_{t\to\infty}K(t)=l$. Now we claim that $l=0$. Otherwise, assuming $l>0$ then $K(t)\geq l$ on $[T,\infty)$. By (3.3) we have

$$
T_{\alpha_3}\Big[e^{R(t)}r_2(t)f_2\big(T_{\alpha_2}\big(r_1(t)f_1\big(D_-^{\alpha_1}x(t)\big)\big)\big)\Big]\geq e^{R(t)}Q(t)f_2(l)
$$

for $t\in[T,\infty)$. α_3 - integrating the above inequality from t to ∞ and using Lemma 2.2, we can derive

$$
-e^{R(t)}r_2(t)f_2\big(T_{\alpha_2}\big(r_1(t)f_1\big(D_-^{\alpha_1}x(t)\big)\big)\big)\geq f_2(l)\int_t^{\infty}e^{R(s)}Q(s)d_{\alpha_3}s.
$$

From (A_3), we have that

$$
T_{\alpha_2}\big(r_1(t)f_1\big(D_-^{\alpha_1}x(t)\big)\big)\leq-\lambda_2 l f_2^{-1}\left[\frac{1}{e^{R(t)}r_2(t)}\int_t^{\infty}e^{R(s)}Q(s)d_{\alpha_3}s\right] \qquad (3.7)
$$

for $t\in[T,\infty)$.

α_2 -integrating both sides of (3.7) from t to ∞, we obtain

$$
-r_1(t)f_1\big(D_-^{\alpha_1}x(t)\big)\leq-\lambda_2 l\int_t^{\infty}f_2^{-1}\left[\frac{1}{e^{R(\xi)}r_2(\xi)}\int_{\xi}^{\infty}e^{R(s)}Q(s)d_{\alpha_3}s\right]d_{\alpha_2}\xi
$$

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for $t \in [T, \infty)$.

$$D_{-}^{\alpha_1}x(t) \geq f_1^{-1}\left(\frac{\lambda_2 l}{r_1(t)}\int_t^{\infty} f_2^{-1}\left[\frac{1}{e^{R(\xi)}r_2(\xi)}\int_{\xi}^{\infty} e^{R(s)}Q(s)d_{\alpha_3}s\right]d_{\alpha_2}\xi\right)$$

Applying Lemma 2.1 and $(A_3)(b)$, we have

$$-\frac{K'(t)}{\Gamma(1-\alpha_1)} \geq \lambda f_1^{-1}(l) f_1^{-1}\left(\frac{1}{r_1(t)}\int_t^{\infty} f_2^{-1}\left[\frac{1}{e^{R(\xi)}r_2(\xi)}\int_{\xi}^{\infty} e^{R(s)}Q(s)d_{\alpha_3}s\right]d_{\alpha_2}\xi\right). \tag{3.8}$$

Integrating both sides of (3.8) from T to t, we obtain

$$\begin{aligned} K(t) &\leq K(T) \\ &- \lambda\Gamma(1-\alpha_1)f_1^{-1}(l)\int_T^t f_1^{-1}\left(\frac{1}{r_1(\tau)}\int_{\tau}^{\infty} f_2^{-1}\left[\frac{1}{e^{R(\xi)}r_2(\xi)}\int_{\xi}^{\infty} e^{R(s)}Q(s)d_{\alpha_3}s\right]d_{\alpha_2}\xi\right)d\tau. \end{aligned} \tag{3.9}$$

Letting $t \to \infty$ and using (3.2), we get $\lim_{t\to\infty} K(t) = -\infty$. This contradicts $K(t) > 0$. The proof of the lemma is complete. □

Lemma 3.2. *Suppose that $x(t)$ is an eventually positive solution of (1.1) such that $T_{\alpha_2}(r_1(t)f_1(D_{-}^{\alpha_1}x(t))) < 0$, $D_{-}^{\alpha_1}x(t) < 0$ on $[t_1, \infty)$, where $t_1 > t_0$ is sufficiently large. Then*

$$K'(t) \geq \frac{-\delta\Gamma(1-\alpha_1)R_1(t_1,t)e^{R(t)}r_2(t)f_2(T_{\alpha_2}(r_1(t)f_1(D_{-}^{\alpha_1}x(t))))}{r_1(t)} \tag{3.10}$$

and

$$K(t) \geq -\delta\Gamma(1-\alpha_1)R_1(t_1,t)e^{R(t)}r_2(t)f_2(T_{\alpha_2}(r_1(t)f_1(D_{-}^{\alpha_1}x(t)))), \tag{3.11}$$

where $R_1(t_1,t) = \int_{t_1}^t \frac{1}{e^{R(s)}r_2(s)}d_{\alpha_2}s$, $R_2(t_1,t) = \int_{t_1}^t \frac{R_1(t,s)}{r_1(s)}ds$.

Proof. As in Lemma 3.1, we deduce that $e^{R(t)}r_2(t)f_2(T_{\alpha_2}(r_1(t)f_1(D_{-}^{\alpha_1}x(t))))$ is strictly increasing on $[t_1, \infty)$. So, we have

$$\begin{aligned} r_1(t)f_1(D_{-}^{\alpha_1}x(t)) &\leq \int_{t_1}^t \frac{e^{R(s)}r_2(s)T_{\alpha_2}(r_1(s)f_1(D_{-}^{\alpha_1}x(s)))}{e^{R(s)}r_2(s)}d_{\alpha_2}s \\ &\leq e^{R(t)}r_2(t)T_{\alpha_2}(r_1(t)f_1(D_{-}^{\alpha_1}x(t)))\int_{t_1}^t \frac{1}{e^{R(s)}r_2(s)}d_{\alpha_2}s \\ &= R_1(t_1,t)e^{R(t)}r_2(t)T_{\alpha_2}(r_1(t)f_1(D_{-}^{\alpha_1}x(t))). \end{aligned}$$

From (A_3), we obtain

$$\begin{aligned} r_1(t)\left(\frac{D_{-}^{\alpha_1}x(t)}{\delta_1}\right) &\leq r_1(t)f_1(D_{-}^{\alpha_1}x(t)) \\ &\leq R_1(t_1,t)e^{R(t)}r_2(t)T_{\alpha_2}(r_1(t)f_1(D_{-}^{\alpha_1}x(t))) \end{aligned}$$

which implies that,

$$K'(t) \geq -\frac{\delta\Gamma(1-\alpha_1)R_1(t_1,t)e^{R(t)}r_2(t)f_2(T_{\alpha_2}(r_1(t)f_1(D_{-}^{\alpha_1}x(t))))}{r_1(t)}.$$

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Integrating the above inequality from t_1 to t, we obtain

$$K(t) - K(t_1) \geq \int_{t_1}^{t} -\frac{\delta\Gamma(1-\alpha_1)R_1(t_1,s)e^{R(s)}r_2(s)f_2\big(T_{\alpha_2}\big(r_1(s)f_1(D_-^{\alpha_1}x(s))\big)\big)}{r_1(s)}ds$$
$$\geq -\delta\Gamma(1-\alpha_1)e^{R(t)}r_2(t)f_2\big(T_{\alpha_2}\big(r_1(t)f_1(D_-^{\alpha_1}x(t))\big)\big)\int_{t_1}^{t}\frac{R_1(t_1,s)}{r_1(s)}ds.$$

Consequently,

$$K(t) \geq -\delta\Gamma(1-\alpha_1)R_2(t_1,t)e^{R(t)}r_2(t)f_2\big(T_{\alpha_2}\big(r_1(t)f_1(D_-^{\alpha_1}x(t))\big)\big).$$

The proof of the lemma is complete. □

Theorem 3.1. *Assume that (3.1), (3.2) hold and suppose that $f_2'(v)$ exists such that $f_2'(v) \geq \mu$ for some $\mu > 0$ and for all $v \neq 0$. If there exist two functions $\phi(t) \in C^{\alpha_3}([t_0,\infty),\mathbb{R}_+), \eta(t) \in C^{\alpha_3}([t_0,\infty),[0,\infty))$ such that*

$$\int_T^{\infty}\Bigg(\phi(s)e^{R(s)}Q(s)s^{\alpha_3-1} - \phi(s)\eta'(s) + \frac{\phi(s)}{r_1(s)}s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s) - \frac{\big[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)+\phi'(s)r_1(s)\big]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)}\Bigg)ds = \infty, \quad (3.12)$$

for sufficiently large T, where $R_1(T,s)$ is defined in Lemma 3.2, then every solution of (1.1) is oscillatory or $\lim_{t\to\infty} K(t) = 0$.

Proof. Suppose that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$. Without loss of generality, we may assume that $x(t) > 0$ on $[t_1,\infty)$ for $t_1 > t_0$. By Lemma 3.1, we have $T_{\alpha_2}\big(r_1(t)f_1(D_-^{\alpha_1}x(t))\big) < 0$, $t \in [t_2,\infty)$ for some sufficiently large $t_2 > t_1$ and either $D_-^{\alpha_1}x(t) < 0$ on $[t_2,\infty)$ or $\lim_{t\to\infty} K(t) = 0$.

First, suppose that $D_-^{\alpha_1}x(t) < 0$ on $[t_2,\infty)$. We define the generalized Riccati function

$$w(t) = \phi(t)\left(-\frac{e^{R(t)}r_2(t)f_2\big(T_{\alpha_2}\big(r_1(t)f_1(D_-^{\alpha_1}x(t))\big)\big)}{f_2(K(t))} + \eta(t)\right), \quad (3.13)$$

when $w(t) > 0$ on $[t_2,\infty)$.

Now, differentiating (3.13) for α_3 times with respect to t for $t \in [t_2,\infty)$,

$$T_{\alpha_3}w(t) = T_{\alpha_3}\phi(t)\left(-\frac{e^{R(t)}r_2(t)f_2\big(T_{\alpha_2}\big(r_1(t)f_1(D_-^{\alpha_1}x(t))\big)\big)}{f_2(K(t))} + \eta(t)\right)$$
$$+ \phi(t)T_{\alpha_3}\left(-\frac{e^{R(t)}r_2(t)f_2\big(T_{\alpha_2}\big(r_1(t)f_1(D_-^{\alpha_1}x(t))\big)\big)}{f_2(K(t))}\right) + \phi(t)T_{\alpha_3}\eta(t).$$

Then, making use of (1.1), (3.3), (A_3) and (3.10) it follows that

$$w'(t) \leq \frac{\phi'(t)}{\phi(t)}w(t) - \phi(t)e^{R(t)}Q(t)t^{\alpha_3-1} + \phi(t)\eta'(t)$$
$$+ \phi(t)e^{R(t)}r_2(t)f_2\big(T_{\alpha_2}\big(r_1(t)f_1(D_-^{\alpha_1}x(t))\big)\big)t^{1-\alpha_3}\mu$$
$$\times \frac{-\delta\Gamma(1-\alpha_1)R_1(t_1,t)e^{R(t)}r_2(t)f_2\big(T_{\alpha_2}\big(r_1(t)f_1(D_-^{\alpha_1}x(t))\big)\big)}{r_1(t)(f_2(K(t)))^2}$$
$$\leq \frac{\phi'(t)}{\phi(t)}w(t) - \phi(t)e^{R(t)}Q(t)t^{\alpha_3-1} + \phi(t)\eta'(t)$$
$$- \frac{\phi(t)\mu t^{1-\alpha_3}\delta\Gamma(1-\alpha_1)R_1(t_1,t)}{r_1(t)}\left(\frac{w(t)}{\phi(t)} - \eta(t)\right)^2,$$

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or

$$w'(t) \leq -\phi(t)e^{R(t)}Q(t)t^{\alpha_3-1} + \phi(t)\eta'(t) - \frac{\phi(t)}{r_1(t)}t^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_1,t)\eta^2(t) + \frac{\left[2\eta(t)\phi(t)t^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_1,t) + \phi'(t)r_1(t)\right]^2}{4r_1(t)\phi(t)t^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_1,t)}. \tag{3.14}$$

Integrating the above inequality from t_2 to t, we obtain,

$$\int_{t_2}^{t}\left(\phi(s)e^{R(s)}Q(s)s^{\alpha_3-1} - \phi(s)\eta'(s) + \frac{\phi(s)}{r_1(s)}s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s) - \frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s) + \phi'(s)r_1(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)}\right)ds \leq w(t_2) - w(t) \leq w(t_2)$$

and letting $t \to \infty$, we get a contradiction to (3.12). The proof of the theorem is complete. □

Theorem 3.2. *Assume that (3.1) and (3.2) hold. If there exist two functions* $\phi(t) \in C^{\alpha_3}([t_0,\infty),\mathbb{R}_+), \eta(t) \in C^{\alpha_3}([t_0,\infty),[0,\infty))$ *such that*

$$\int_{T}^{\infty}\left(\frac{\phi(s)e^{R(s)}Q(s)s^{\alpha_3-1}}{\delta_2} - \phi(s)\eta'(s) + \frac{\phi(s)}{r_1(s)}\delta_1\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s) - \frac{\left[2\eta(s)\phi(s)\delta\Gamma(1-\alpha_1)R_1(T,s) + \phi'(s)r_1(s)\right]^2}{4r_1(s)\phi(s)\delta\Gamma(1-\alpha_1)R_1(T,s)}\right)ds = \infty \tag{3.15}$$

for sufficiently large T, where $R_1(T,s)$ is defined in Lemma 3.2, then every solution of (1.1) is oscillatory or $\lim_{t\to\infty} K(t) = 0$.

Proof. Suppose that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$. Without loss of generality, we may assume that $x(t) > 0$ on $[t_1,\infty)$ for $t_1 > t_0$. By Lemma 3.1, we have $T_{\alpha_2}(r_1(t)f_1(D_-^{\alpha_1}x(t))) < 0$, $t \in [t_2,\infty)$ for some sufficiently large $t_2 > t_1$ and either $D_-^{\alpha_1}x(t) < 0$ on $[t_2,\infty)$ or $\lim_{t\to\infty} K(t) = 0$.

Assume that $D_-^{\alpha_1}x(t) < 0$ on $[t_2,\infty)$. Let us define the generalized Riccati function as follows,

$$w(t) = \phi(t)\left(-\frac{e^{R(t)}r_2(t)f_2(T_{\alpha_2}(r_1(t)f_1(D_-^{\alpha_1}x(t))))}{K(t)} + \eta(t)\right), \tag{3.16}$$

when $w(t) > 0$ on $[t_2,\infty)$. The rest of the proof is similar to that of the Theorem 3.1. □

Next, we discuss some new oscillation criteria for (1.1) by using integral average method.

Theorem 3.3. *Let* $\mathbb{D}_0 = \{(t,s) : t > s \geq t_0\}$ *and* $\mathbb{D} = \{(t,s) : t \geq s \geq t_0\}$.

Assume that (3.1), (3.2) hold and there exists a function $H \in C'(\mathbb{D};\mathbb{R})$ is said to belong to the class $\mathbb{P}$ if

(T_1) $H(t,t) = 0$ *for* $t \geq t_0, H(t,s) > 0$ *on* $\mathbb{D}_0$,

(T_2) *H has a continuous and non positive partial derivative on $\mathbb{D}_0$ with respect to the second variable and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\left[\phi(s)e^{R(s)}Q(s)s^{\alpha_3-1} - \phi(s)\eta'(s) + \frac{\phi(s)}{r_1(s)}s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s) - \frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s) + \phi'(s)r_1(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)}\right]ds = \infty \tag{3.17}$$

for all sufficiently large T, where ϕ, η are defined as in Theorem 3.1. Then every solution of (1.1) is oscillatory or $\lim_{t\to\infty} K(t) = 0$.

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Proof. Suppose that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$. Without loss of generality, we may suppose that $x(t) > 0$ on $[t_1,\infty)$ for some large $t_1 > t_0$. By Lemma 3.1, we have $T_{\alpha_2}(r_1(t)f_1(D_-^{\alpha_1}x(t))) < 0$, $t \in [t_2,\infty)$ for some sufficiently large $t_2 > t_1$ and either $D_-^{\alpha_1}x(t) < 0$ on $[t_2,\infty)$ or $\lim_{t\to\infty} K(t) = 0$.

Now, we assume $D_-^{\alpha_1}x(t) < 0$ on $[t_2,\infty)$ for some sufficiently large $t_2 > t_1$. Let $w(t)$ be defined as in Theorem 3.1. By (3.14), we have

$$\phi(t)e^{R(t)}Q(t)t^{\alpha_3-1} - \phi(t)\eta'(t) + \frac{\phi(t)}{r_1(t)}t^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,t)\eta^2(t) - \frac{\left[2\eta(t)\phi(t)t^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,t)+\phi'(t)r_1(t)\right]^2}{4r_1(t)\phi(t)t^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,t)} \le -w'(t). \tag{3.18}$$

Multiplying both sides by $H(t,s)$ and then integrating it with respect to s from t_2 to t yields

$$\begin{aligned}&\int_{t_2}^{t} H(t,s)\Big[\phi(s)e^{R(s)}Q(s)s^{\alpha_3-1} - \phi(s)\eta'(s)\\ &+ \frac{\phi(s)}{r_1(s)}s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)\eta^2(s)\\ &- \frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)+\phi'(s)r_1(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)}\Big]ds \le -\int_{t_2}^{t} H(t,s)w'(s)ds\\ &\le H(t,t_2)w(t_2) \le H(t,t_0)w(t_2).\end{aligned}$$

Then,

$$\begin{aligned}&\int_{t_0}^{t} H(t,s)\Big[\phi(s)e^{R(s)}Q(s)s^{\alpha_3-1} - \phi(s)\eta'(s)\\ &+ \frac{\phi(s)}{r_1(s)}s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)\eta^2(s)\\ &- \frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)+\phi'(s)r_1(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)}\Big]ds\end{aligned}$$

$$\begin{aligned}\le &H(t,t_0)\int_{t_0}^{t_2}\Big|\phi(s)e^{R(s)}Q(s)s^{\alpha_3-1} - \phi(s)\eta'(s)\\ &+ \frac{\phi(s)}{r_1(s)}s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)\eta^2(s)\\ &- \frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)+\phi'(s)r_1(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)}\Big|ds + H(t,t_0)w(t_2).\end{aligned}$$

Therefore

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{H(t,t_0)}&\int_{t_0}^{t} H(t,s)\Big[\phi(s)e^{R(s)}Q(s)s^{\alpha_3-1} - \phi(s)\eta'(s)+\\ &\frac{\phi(s)}{r_1(s)}s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)\eta^2(s)-\\ &\frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)+\phi'(s)r_1(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)}\Big]ds < \infty,\end{aligned}$$

which contradicts (3.17). The proof of the theorem is complete. □

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In this theorem, if we take $H(t,s)$ for some special functions such as $(t-s)^m$ or $log\left(\frac{t}{s}\right)$, then we can obtain some corollaries as follows.

Corollary 3.1. *Assume that (3.1), (3.2) hold and*

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)^m}\int_{t_0}^{t}(t-s)^m\Big[\phi(s)e^{R(s)}Q(s)s^{\alpha_3-1}-\phi(s)\eta'(s) + \frac{\phi(s)}{r_1(s)}s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s) - \frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)+\phi'(s)r_1(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)}\Big]ds=\infty$$

for sufficiently large T. Then every solution of (1.1) is oscillatory or $\lim_{t\to\infty}K(t)=0$.

Corollary 3.2. *Assume that (3.1), (3.2) hold and*

$$\limsup_{t\to\infty}\frac{1}{logt-logt_0}\int_{t_0}^{t}(logt-logs)\Big[\phi(s)e^{R(s)}Q(s)s^{\alpha_3-1}-\phi(s)\eta'(s) + \frac{\phi(s)}{r_1(s)}s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s) - \frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)+\phi'(s)r_1(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)}\Big]ds=\infty$$

for sufficiently large T. Then every solution of (1.1) is oscillatory or $\lim_{t\to\infty}K(t)=0$.

4. Examples

In this section, we give some examples to illustrate our main results.

Example 4.1. *Consider the fractional differential equation*

$$T_{\frac{5}{7}}\left[T_{\frac{1}{3}}\left(D^{\frac{1}{2}}_{-}x(t)\right)\right]+t^{1-\frac{5}{7}}t^{-\frac{16}{7}}\left(T_{\frac{1}{3}}\left(D^{\frac{1}{2}}_{-}x(t)\right)\right) - \frac{M}{t^2}\left(\int_t^\infty(\nu-t)^{-\frac{1}{2}}x(\nu)d\nu\right)=0\ for\ t\geq 1. \tag{4.1}$$

This corresponds to (1.1) with $r_2(t)=r_1(t)=1, \alpha_1=\frac{1}{2}, \alpha_2=\frac{1}{3}, \alpha_3=\frac{5}{7}$, $f_1(x)=x, \frac{x}{f_1(x)}\leq\delta_1=1, f_2(x)=x, P(t)=t^{-\frac{16}{7}}, F(t,K)=Q(t)\left(\int_t^\infty(\nu-t)^{-\alpha_1}x(\nu)d\nu\right)$, *where* $\frac{M}{t^2}=Q(t)$. *Then, we have*

$$R(t)=\int_{t_0}^{t}\frac{P(s)}{r_2(s)}ds=-\frac{7}{9}[t^{-\frac{9}{7}}-1]\leq 1,$$

$$\int_{t_0}^{\infty}\frac{1}{f_2^{-1}[e^{R(s)}r_2(s)]}d_{\alpha_2}s=\int_1^\infty\frac{s^{\frac{1}{3}-1}}{e}ds=\infty$$

and

$$\int_{t_0}^{\infty}f_1^{-1}\left(\frac{1}{r_1(s)}\right)ds=\int_1^\infty ds=\infty.$$

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Furthermore,

$$\begin{aligned}
&\int_{t_0}^{\infty} f_1^{-1}\left(\frac{1}{r_1(\tau)}\int_{\tau}^{\infty} f_2^{-1}\left(\frac{1}{e^{R(\xi)}r_2(\xi)}\int_{\xi}^{\infty} e^{R(s)}Q(s)d_{\alpha_3}s\right)d_{\alpha_2}\xi\right)d\tau \\
&= \int_{t_0}^{\infty} f_1^{-1}\left(\frac{1}{r_1(\tau)}\int_{\tau}^{\infty} f_2^{-1}\left(\frac{1}{e^{R(\xi)}r_2(\xi)}\int_{\xi}^{\infty} e^{R(s)}Q(s)s^{\alpha_3-1}ds\right)\xi^{\alpha_2-1}d\xi\right)d\tau \\
&= \int_{1}^{\infty}\int_{\tau}^{\infty}\left[\frac{1}{e}\int_{\xi}^{\infty}\frac{eM}{s^2}s^{\frac{5}{7}-1}ds\right]\xi^{\frac{1}{3}-1}d\xi d\tau = \frac{7M}{9}\int_{1}^{\infty}\int_{\tau}^{\infty}\frac{1}{\xi^{\frac{9}{7}}}\xi^{\frac{1}{3}-1}d\xi d\tau \\
&= \frac{7M}{9}\int_{1}^{\infty}\int_{\tau}^{\infty}\xi^{-\frac{41}{21}}d\xi d\tau = \frac{49M}{60}\int_{1}^{\infty}\frac{1}{\tau^{\frac{20}{21}}}d\tau = \infty.
\end{aligned}$$

On the other hand, for sufficiently large T, we get

$$R_1(T,t) = \int_T^t \frac{1}{e^{R(s)}r_2(s)}d_{\alpha_2}s = \int_T^t \frac{1}{e}s^{\alpha_2-1}ds = \frac{1}{e}\int_T^t s^{\frac{1}{3}-1}ds \to \infty (t\to\infty).$$

Thus we can take $T^* > T$ *such that* $R_1(T,t) > 1$ *for* $t \in [T^*, \infty)$.

Letting $\phi(s) = s^{\frac{9}{7}}, \eta(s) = 0, \mu = 1, \delta = 1.$

$$\begin{aligned}
&\int_T^{\infty} \phi(s)e^{R(s)}Q(s)s^{\alpha_3-1} - \phi(s)\eta'(s) + \frac{\phi(s)}{r_1(s)}s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s) - \\
&\quad \frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)+\phi'(s)r_1(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)}ds \\
&\quad \geq \int_T^{\infty}\left[s^{\frac{9}{7}}eMs^{-2}s^{-\frac{2}{7}} - \frac{1}{4\sqrt{\pi}}\left(\frac{9}{7}\right)^2 s^{\frac{-9}{7}}s^{\frac{4}{7}}s^{\frac{-2}{7}}\right]ds \\
&\quad = \int_T^{\infty}\frac{1}{s}\left[eM - \frac{81}{196\sqrt{\pi}}\right]ds \\
&\quad = \int_T^{T*}\frac{1}{s}\left[eM - \frac{81}{196\sqrt{\pi}}\right]ds + \int_{T*}^{\infty}\frac{1}{s}\left[eM - \frac{81}{196\sqrt{\pi}}\right]ds \to \infty,
\end{aligned}$$

provided $M > \frac{81}{196e\sqrt{\pi}}$. *Hence, all the conditions of Theorem 3.1 are satisfied. Therefore, every solution of (4.1) is oscillatory or* $\lim_{t\to\infty} K(t) = 0$.

Example 4.2. *Consider the fractional differential equation*

$$\begin{aligned}
&T_{\frac{5}{7}}\left[(2+\cos t)\left(T_{\frac{3}{7}}\left(D_{-}^{\frac{1}{2}}x(t)\right)\right)\right] + t^{1-\frac{5}{7}}t^{-3}\left(T_{\frac{3}{7}}\left(D_{-}^{\frac{1}{2}}x(t)\right)\right) \\
&-\frac{M}{t^{\frac{5}{7}}}\left(\int_t^{\infty}(v-t)^{-\frac{1}{2}}x(v)dv\right) = 0 \ for\ t \geq 1.
\end{aligned} \tag{4.2}$$

This corresponds to (1.1) with $r_2(t) = 2+\cos t, r_1(t) = 1, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{3}{7}$,
$\alpha_3 = \frac{5}{7}, f_1(x) = x, \frac{x}{f_1(x)} \leq \delta_1 = 1, f_2(x) = x, P(t) = t^{-3}$,
$F(t,K) = Q(t)(\int_t^{\infty}(v-t)^{-\alpha_1}x(v)dv)$, *where* $\frac{M}{t^{\frac{5}{7}}} = Q(t)$. *Then, we have*

$$R(t) = \int_{t_0}^t \frac{P(s)}{r_2(s)}ds = \int_1^t \frac{s^{-3}}{2+\cos s}ds \leq \frac{1}{2}$$

and

$$\int_{t_0}^{\infty}\frac{1}{f_2^{-1}[e^{R(s)}r_2(s)]}d_{\alpha_2}s = \int_1^{\infty}\frac{s^{\frac{3}{7}-1}}{\sqrt{e}(2+\cos s)}ds \geq \frac{1}{3\sqrt{e}}\int_1^{\infty}s^{\frac{3}{7}-1}ds = \infty.$$

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In addition, we can get

$$\int_{t_0}^{\infty} f_1^{-1}\left(\frac{1}{r_1(s)}\right)ds = \int_1^{\infty} ds = \infty.$$

Furthermore,

$$\begin{aligned}
&\int_{t_0}^{\infty} f_1^{-1}\left(\frac{1}{r_1(\tau)}\int_{\tau}^{\infty} f_2^{-1}\left(\frac{1}{e^{R(\xi)}r_2(\xi)}\int_{\xi}^{\infty} e^{R(s)}Q(s)d_{\alpha_3}s\right)d_{\alpha_2}\xi\right)d\tau \\
&= \int_{t_0}^{\infty} f_1^{-1}\left(\frac{1}{r_1(\tau)}\int_{\tau}^{\infty} f_2^{-1}\left(\frac{1}{e^{R(\xi)}r_2(\xi)}\int_{\xi}^{\infty} e^{R(s)}Q(s)s^{\alpha_3-1}ds\right)\xi^{\alpha_2-1}d\xi\right)d\tau \\
&= \int_1^{\infty}\int_{\tau}^{\infty}\left[\frac{1}{\sqrt{e}(2+\cos\xi)}\int_{\xi}^{\infty}\frac{\sqrt{e}M}{s^{\frac{5}{7}}}s^{\frac{5}{7}-1}ds\right]\xi^{\frac{3}{7}-1}d\xi d\tau \\
&\geq \frac{M}{3}\int_1^{\infty}\int_{\tau}^{\infty}\left(\int_{\xi}^{\infty}\frac{1}{s}ds\right)\xi^{\frac{3}{7}-1}d\xi d\tau = \infty.
\end{aligned}$$

On the other hand, for sufficiently large T, we get

$$R_1(T,t) = \int_T^t \frac{1}{e^{R(s)}r_2(s)}d_{\alpha_2}s = \int_T^t \frac{1}{\sqrt{e}(2+\cos s)}s^{\frac{3}{7}-1}ds \to \infty (t\to\infty).$$

Thus we can take $T^* > T$ such that $R_1(T,t) > 1$ for $t \in [T^*, \infty)$.

Letting $\phi(s) = 1, \eta(s) = 0, \mu = 1, \delta_2 = \delta = 1$.

$$\begin{aligned}
&\int_T^{\infty}\left(\frac{\phi(s)e^{R(s)}Q(s)s^{\alpha_3-1}}{\delta_2} - \phi(s)\eta'(s) + \frac{\phi(s)}{r_1(s)}\delta_1\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s)\right. \\
&\quad\left. - \frac{[2\eta(s)\phi(s)\delta\Gamma(1-\alpha_1)R_1(T,s)+\phi'(s)r_1(s)]^2}{4r_1(s)\phi(s)\delta\Gamma(1-\alpha_1)R_1(T,s)}\right)ds \\
&\quad\geq \int_T^{\infty}\sqrt{e}Ms^{-\frac{5}{7}}s^{-\frac{2}{7}}ds = \int_T^{\infty}\frac{1}{s}ds \to \infty,
\end{aligned}$$

Hence, all the conditions of Theorem 3.2 are satisfied. Therefore, every solution of (4.2) is oscillatory or $\lim_{t\to\infty} K(t) = 0$.

Example 4.3. *Consider the fractional differential equation*

$$\begin{aligned}
&T_{\frac{5}{7}}\left[(2+\cos t)\left(T_{\frac{1}{4}}\left(D_-^{\frac{1}{2}}x(t)\right)\right)\right] + t^{1-\frac{5}{7}}t^{-3}\left(T_{\frac{1}{4}}\left(D_-^{\frac{1}{2}}x(t)\right)\right) \\
&- \frac{M}{t^2}\left(\int_t^{\infty}(v-t)^{-\frac{1}{2}}x(v)dv\right) = 0 \text{ for } t \geq 1.
\end{aligned} \tag{4.3}$$

This corresponds to (1.1) with $r_2(t) = 2+\cos t, r_1(t) = 1, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{4}$,
$\alpha_3 = \frac{5}{7}, f_1(x) = x, \frac{x}{f_1(x)} \leq \delta_1 = 1, f_2(x) = x, P(t) = t^{-3}$,
$F(t,K) = Q(t)\left(\int_t^{\infty}(v-t)^{-\alpha_1}x(v)dv\right)$, where $\frac{M}{t^2} = Q(t)$.
Then, we have

$$R(t) = \int_{t_0}^t \frac{P(s)}{r_2(s)}ds = \int_1^t \frac{s^{-3}}{2+\cos s}ds \leq \frac{1}{2}$$

and

$$\int_{t_0}^{\infty}\frac{1}{f_2^{-1}[e^{R(s)}r_2(s)]}d_{\alpha_2}s = \int_1^{\infty}\frac{s^{\frac{1}{4}-1}}{\sqrt{e}(2+\cos s)}ds \geq \frac{1}{3\sqrt{e}}\int_1^{\infty}s^{\frac{1}{4}-1}ds = \infty.$$

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In addition, we can get

$$\int_{t_0}^{\infty} f_1^{-1}\left(\frac{1}{r_1(s)}\right)ds = \int_1^{\infty} ds = \infty.$$

Furthermore,

$$\begin{aligned}
&\int_{t_0}^{\infty} f_1^{-1}\left(\frac{1}{r_1(\tau)}\int_{\tau}^{\infty} f_2^{-1}\left(\frac{1}{e^{R(\xi)}r_2(\xi)}\int_{\xi}^{\infty} e^{R(s)}Q(s)d_{\alpha_3}s\right)d_{\alpha_2}\xi\right)d\tau \\
&= \int_{t_0}^{\infty} f_1^{-1}\left(\frac{1}{r_1(\tau)}\int_{\tau}^{\infty} f_2^{-1}\left(\frac{1}{e^{R(\xi)}r_2(\xi)}\int_{\xi}^{\infty} e^{R(s)}Q(s)s^{\alpha_3-1}ds\right)\xi^{\alpha_2-1}d\xi\right)d\tau \\
&= \int_1^{\infty}\int_{\tau}^{\infty}\left[\frac{1}{\sqrt{e}(2+\cos\xi)}\int_{\xi}^{\infty}\frac{\sqrt{e}M}{s^{\frac{5}{7}}}s^{\frac{5}{7}-1}ds\right]\xi^{\frac{1}{4}-1}d\xi d\tau \\
&\geq \frac{M}{3}\int_1^{\infty}\int_{\tau}^{\infty}\left(\int_{\xi}^{\infty}\frac{1}{s}ds\right)\xi^{\frac{1}{4}-1}d\xi d\tau = \infty.
\end{aligned}$$

On the other hand, for sufficiently large T, we get

$$R_1(T,t) = \int_T^t \frac{1}{e^{R(s)}r_2(s)}d_{\alpha_2}s = \int_T^t \frac{1}{\sqrt{e}(2+\cos s)}s^{\frac{1}{4}-1}ds \to \infty(t\to\infty).$$

Thus, we can take $T^* > T$ such that $R_1(T,t) > 1$ for $t \in [T^*,\infty)$.

Letting $\phi(s) = s^{\frac{9}{7}}, \eta(s) = 0, \mu = 1, \delta = 1$ and $H(t,s) = log(\frac{t}{s})$.

$$\begin{aligned}
\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\Big[&\phi(s)e^{R(s)}Q(s)s^{\alpha_3-1} - \phi(s)\eta'(s) \\
&+ \frac{\phi(s)}{r_1(s)}s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s) \\
&- \frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)+\phi'(s)r_1(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)}\Big]ds
\end{aligned}$$

$$\begin{aligned}
&\geq \limsup_{t\to\infty}\frac{1}{logt}\left(\int_T^{\infty} log\frac{t}{s}\left[s^{\frac{9}{7}}\sqrt{e}Ms^{-2}s^{-\frac{2}{7}} - \frac{1}{4\sqrt{\pi}}\left(\frac{9}{7}\right)^2 s^{\frac{-9}{7}}s^{\frac{4}{7}}s^{\frac{-2}{7}}\right]ds\right) \\
&= \limsup_{t\to\infty}\frac{1}{logt}\left(\int_T^{\infty} log\frac{t}{s}\left[\sqrt{e}M - \frac{81}{196\sqrt{\pi}}\right]\frac{1}{s}ds\right) \\
&= \limsup_{t\to\infty}\frac{1}{logt}\left(\int_T^{T*} log\frac{t}{s}\left[\sqrt{e}M - \frac{81}{196\sqrt{\pi}}\right]\frac{1}{s}ds\right. \\
&\quad \left.+ \int_{T*}^{\infty} log\frac{t}{s}\left[\sqrt{e}M - \frac{81}{196\sqrt{\pi}}\right]\frac{1}{s}ds\right) \to \infty,
\end{aligned}$$

provided $M > \frac{81}{196e\sqrt{e\pi}}$. Hence all the conditions of Theorem 3.3 are satisfied. Therefore, every solution of (4.3) is oscillatory or $\lim_{t\to\infty} K(t) = 0$.

Conclusion

In this article, the authors have derived some new oscillation results for a class of mixed fractional order nonlinear differential equations with conformable fractional derivative and Liouville right-sided fractional derivative, by using the generalized Riccati technique and integral averaging method. This work extends and generalizes some of the results in the known literature [5,8,13,14,16,21–23] to the mixed fractional differential equations. Some illustrative examples are given to test the effectiveness of our newly established results.

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