Article

Not peer-reviewed version

# A Short Proof of Knuth's Old Sum

Kunle Adegoke *

Posted Date: 18 November 2024

doi: 10.20944/preprints202411.1249.v1

Keywords: Knuth's old sum; Reed Dawson identity; Catalan number; Beta function; Combinatorial identity; polynomial identity

*Article*

# A Short Proof of Knuth's Old Sum

**Kunle Adegoke**

Department of Physics and Engineering Physics, Obafemi Awolowo University, Ile-Ife 220005, Nigeria; adegoke00@gmail.com

**Abstract:** We give a short proof of the well-known Knuth's old sum and provide some generalizations. Our approach utilizes the binomial theorem and integration formulas derived using the Beta function. Several new polynomial identities and combinatorial identities are derived.

**Keywords:** Knuth's old sum; Reed Dawson identity, Catalan number; Beta function; Combinatorial identity; polynomial identity

**MSC:** Primary 05A10; Secondary 05A19

## 1. Introduction

There appears to be a renewed interest [1,2,4,6,8] in the famous Knuth's old sum (also known as the Reed Dawson identity),

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{-k}\binom{2k}{k}=\begin{cases}2^{-n}\binom{n}{n/2}, & \text{if } n \text{ is even;}\\ 0, & \text{if } n \text{ is odd.}\end{cases}\tag{1.1}$$

Many different proofs of this identity and various generalizations exist in the literature (see [5] for a survey).

In this paper we give a very short proof of (1.1) and offer the following generalization:

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{-k-m}\binom{2(k+m)}{k+m}\\ &\quad=\begin{cases}\sum\limits_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}2^{-n-2k}\binom{2k+n}{(2k+n)/2}, & \text{if } n \text{ is even;}\\ -\sum\limits_{k=1}^{\lceil m/2\rceil}\binom{m}{2k-1}2^{-n-2k+1}\binom{2k+n-1}{(2k+n-1)/2}, & \text{if } n \text{ is odd;}\end{cases}\end{aligned}\tag{1.2}$$

where $m$ and $n$ are non-negative integers and, as usual, $\lfloor z\rfloor$ is the greatest integer less than or equal to $z$ while $\lceil z\rceil$ is the smallest integer greater than or equal to $z$.

The following special cases of (1.2) were also reported in Riordan [7, p.72, Problem 4(b)]:

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{n-2k}\binom{2k}{k}=\binom{2n}{n},\tag{1.3}$$

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}2^{n-2k}\binom{2k}{k}=\frac{1}{2}\binom{2n+2}{n+1}-\binom{2n}{n}=\frac{n}{n+1}\binom{2n}{n}.\tag{1.4}$$

Identity (1.3) corresponds to setting $n=0$ in (1.2) and re-labeling $m$ as $n$; while (1.4) follows from setting $n=1$ in (1.2).

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 November 2024 doi:10.20944/preprints202411.1249.v1



Identity (1.2) is itself a particular case of a more general identity, stated in Theorem 2, which has many interesting consequences, including another generalization of Knuth's old sum, namely,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{-k}\binom{2k+v}{(2k+v)/2}\binom{k+v}{v/2}^{-1}=\begin{cases}2^{-n}\binom{n}{n/2}\binom{(n+v)/2}{v/2}^{-1}, & \text{if } n \text{ is even;}\\ 0, & \text{if } n \text{ is odd;}\end{cases}$$

for a real number $v$, as well as simple, apparently new combinatorial identities such as

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}2^{n-2k}C_k=\frac{1}{2}C_{n+2}-C_{n+1};$$

where, here and throughout this paper,

$$C_j=\frac{\binom{2j}{j}}{j+1},$$

defined for every non-negative integer $j$, is a Catalan number.

Based on the binomial theorem, we will derive, in Section 5, some presumably new polynomial identities, including the following:

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}2^{-k}\binom{2k}{k}(1-x)^{n-k}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{2k}{k}x^{n-2k}. \tag{1.5}$$

Identity (1.5) subsumes Knuth's old sum (1.1) (at $x=0$), as well as (1.3) (at $x=1$).

Finally, in Section 6, the polynomial identities will facilitate the derivation of apparently new combinatorial identities such as

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{1}{2k+1}=\frac{2^{n-1}}{2^n-1}\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\frac{1}{k}, \quad n\neq 0,$$
$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}C_k=\frac{2^{-n+1}}{n+2}(2n+1)C_n,$$

and

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{2(2k+1)}{k+2}C_k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_k.$$

## 2. Required Identities

In order to give the short proof of Knuth's old sum, we need a couple of definite integrals which we establish in Lemma 1.

The binomial coefficients are defined, for non-negative integers $m$ and $n$, by

$$\binom{m}{n}=\begin{cases}\dfrac{m!}{n!(m-n)!}, & m\geq n;\\ 0, & m<n;\end{cases}$$

the number of distinct sets of $n$ objects that can be chosen from $m$ distinct objects.

Generalized binomial coefficients are defined for complex numbers $u$ and $v$, excluding the set of negative integers, by

$$\binom{u}{v}=\frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}, \tag{2.1}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 November 2024 doi:10.20944/preprints202411.1249.v1



where $\Gamma(z)$ is the Gamma function defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = \int_0^\infty (\log(1/t))^{z-1} dt$$

and extended to the rest of the complex plain, excluding the non-positive integers, by analytic continuation.

**Lemma 1.** *Let $u$ and $v$ be complex numbers such that $\Re u > -1$ and $\Re v > -1$. Let $m$ be a non-negative integer. Then*

$$\int_0^\pi \cos^u(x/2)\,dx = 2^{-u}\,\pi \binom{u}{u/2} = \int_0^\pi \sin^u(x/2)\,dx, \tag{2.2}$$

$$\int_0^\pi \cos^m x\,dx = \begin{cases} 2^{-m}\,\pi\binom{m}{m/2}, & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd;} \end{cases} \tag{2.3}$$

*and, more generally,*

$$I(u,v) := \int_0^\pi \cos^u\left(\frac{x}{2}\right)\sin^v\left(\frac{x}{2}\right) dx = 2^{-u-v}\,\pi \binom{u}{u/2}\binom{v}{v/2}\binom{(u+v)/2}{u/2}^{-1}, \tag{2.4}$$

*and*

$$J(m,v) := \int_0^\pi \cos^m x \sin^v x\,dx = \begin{cases} 2^{-m-v}\,\pi\binom{m}{m/2}\binom{v}{v/2}\binom{(m+v)/2}{m/2}^{-1}, & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd.} \end{cases} \tag{2.5}$$

*Obviously $I(v,u) = I(u,v)$, a symmetry property that is not possessed by $J(m,v)$.*

**Proof.** Identities (2.4) and (2.5) are immediate consequences of the well-known Beta function integral ([3] Entry 3.621.5, Page 397):

$$K(u,v) := \int_0^{\pi/2} \cos^u x \sin^v x\,dx = 2^{-u-v-1}\,\pi \binom{u}{u/2}\binom{v}{v/2}\binom{(u+v)/2}{u/2}^{-1}, \tag{2.6}$$

valid for $\Re u > -1$, $\Re v > -1$, with the symmetry property $K(u,v) = K(v,u)$.

Identity (2.4) is obtained via a simple change of the integration variable from $x$ to $y$ in (2.6), with $x = y/2$.

To prove (2.5), write

$$J(m,v) = \int_0^\pi \cos^m x \sin^v x\,dx = \int_0^{\pi/2} \cos^m x \sin^v x\,dx + \int_{\pi/2}^\pi \cos^m x \sin^v x\,dx.$$

Change the integration variable in the second integral on the right hand side from $x$ to $y$ via $x = y + \pi/2$; this gives

$$\begin{aligned} J(m,v) &= \int_0^{\pi/2} \cos^m x \sin^v x\,dx + (-1)^m \int_0^{\pi/2} \sin^m y \cos^v y\,dy \\ &= K(m,v) + (-1)^m K(v,m) \\ &= (1 + (-1)^m) K(m,v); \end{aligned}$$

and hence (2.5).

□

 doi:10.20944/preprints202411.1249.v1



**Remark 1.** *Since, for a real number $u$,*

$$1+(-1)^u = 2\cos^2\Big(\frac{\pi u}{2}\Big) + i\sin(\pi u),$$

*the $J(m,v)$ stated in (2.5) is a special case of the following more general result:*

$$\begin{aligned} J(u,v) &= \int_0^\pi \cos^u x \sin^v x\,dx \\ &= \frac{\pi}{2^{u+v+1}}\binom{u}{u/2}\binom{v}{v/2}\binom{(u+v)/2}{u/2}^{-1}\Big(2\cos^2\Big(\frac{\pi u}{2}\Big)+i\sin(\pi u)\Big), \end{aligned} \tag{2.7}$$

*which is valid for $u>-1$ and $\Re v > -1$.*

## 3. A Short Proof of Knuth's Old Sum

**Theorem 1.** *If $n$ is a non-negative integer, then*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} 2^{-k}\binom{2k}{k} = \begin{cases} 2^{-n}\binom{n}{n/2}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

**Proof.** Substitute $-\cos x - 1$ for $y$ in the binomial theorem

$$\sum_{k=0}^n \binom{n}{k} y^k = (1+y)^n,$$

to obtain

$$\sum_{k=0}^n (-1)^k \binom{n}{k} 2^k \cos^{2k}(x/2) = (-1)^n \cos^n x. \tag{3.1}$$

Thus

$$\sum_{k=0}^n (-1)^k \binom{n}{k} 2^k \int_0^\pi \cos^{2k}(x/2)\,dx = (-1)^n \int_0^\pi \cos^n x\,dx,$$

and hence (1.1) on account of (2.2) and (2.3). $\square$

## 4. A Generalization of Knuth's Old Sum

In this section we extend (1.1) by introducing an arbitrary non-negative integer $m$ and a real number $v$.

**Theorem 2.** *If $m$ and $n$ are non-negative integers and $v$ is a real number, then*

$$\begin{aligned} &\sum_{k=0}^n (-1)^k \binom{n}{k} 2^{-k-m}\binom{2k+2m+v}{(2k+2m+v)/2}\binom{k+m+v}{v/2}^{-1} \\ &= \begin{cases} \displaystyle\sum_{k=0}^{\lfloor m/2\rfloor} \tbinom{m}{2k} 2^{-n-2k}\tbinom{2k+n}{(2k+n)/2}\tbinom{(2k+n+v)/2}{(2k+n)/2}^{-1}, & \text{if } n \text{ is even;} \\ \displaystyle -\sum_{k=1}^{\lceil m/2\rceil} \tbinom{m}{2k-1} 2^{-n-2k+1}\tbinom{2k+n-1}{(2k+n-1)/2}\tbinom{(2k+n-1+v)/2}{(2k+n-1)/2}^{-1}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{4.1}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 November 2024 doi:10.20944/preprints202411.1249.v1



In particular,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{-k-m}\binom{2(k+m)}{k+m}$$
$$=\begin{cases}\sum\limits_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}2^{-n-2k}\binom{2k+n}{(2k+n)/2}, & \text{if } n \text{ is even;}\\ -\sum\limits_{k=1}^{\lceil m/2\rceil}\binom{m}{2k-1}2^{-n-2k+1}\binom{2k+n-1}{(2k+n-1)/2}, & \text{if } n \text{ is odd.}\end{cases}$$

**Proof.** Since

$$(1+\cos x)^m = 2^m\cos^{2m}\left(\frac{x}{2}\right) = \sum_{k=0}^{m}\binom{m}{k}\cos^k x$$

and

$$\sin^v x = 2^v\sin^v\left(\frac{x}{2}\right)\cos^v\left(\frac{x}{2}\right),$$

multiplication of the left hand side of (3.1) by

$$2^{m+v}\cos^{2m+v}\left(\frac{x}{2}\right)\sin^v\left(\frac{x}{2}\right)$$

and the right hand side by

$$\sin^v x\sum_{k=0}^{m}\binom{m}{k}\cos^k x$$

gives

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{k+m+v}\cos^{2k+2m+v}(x/2)\sin^v(x/2)$$
$$=(-1)^n\sum_{k=0}^{m}\binom{m}{k}\cos^{k+n}x\sin^v x;$$

so that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{k+m+v}\cos^{2k+2m+v}(x/2)\sin^v(x/2)$$
$$=(-1)^n\sum_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}\cos^{2k+n}x\sin^v x+(-1)^n\sum_{k=1}^{\lceil m/2\rceil}\binom{m}{2k-1}\cos^{2k-1+n}x\sin^v x,$$

from which (4.1) now follows by termwise integration from 0 to $\pi$, according to the parity of $n$, using Lemma 1.
□

**Corollary 3.** *If $n$ is a non-negative integer and $v$ is a real number, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{-k}\binom{2k+v}{(2k+v)/2}\binom{k+v}{v/2}^{-1}=\begin{cases}2^{-n}\binom{n}{n/2}\binom{(n+v)/2}{v/2}^{-1}, & \text{if } n \text{ is even;}\\ 0, & \text{if } n \text{ is odd;}\end{cases} \tag{4.2}$$

**Corollary 4.** *If $n$ is a non-negative integer and $v$ is a real number, then*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{2k}{k}\binom{(2k+v)/2}{k}^{-1}=2^{-n}\binom{2n+v}{(2n+v)/2}\binom{n+v}{v/2}^{-1}, \tag{4.3}$$

 doi:10.20944/preprints202411.1249.v1



*and*

$$\begin{aligned}&\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}2^{n-2k}\binom{2k}{k}\binom{(2k+v)/2}{k}^{-1}\\&\quad=\frac{1}{2}\binom{2n+v+2}{(2n+v+2)/2}\binom{n+v+1}{v/2}^{-1}-\binom{2n+v}{(2n+v)/2}\binom{n+v}{v/2}^{-1}.\end{aligned}\tag{4.4}$$

**Proof.** Identity (4.3) is obtained by setting $n=0$ in (4.1) and re-labeling $m$ as $n$ while (4.4) is the evaluation of (4.1) at $n=1$ with a re-labeling of $m$ as $n$. □

**Proposition 5.** *If $n$ is a non-negative integer, then*

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\frac{1}{2k+1}=\frac{2^{n+1}}{n+2}-\frac{2^n}{n+1},\tag{4.5}$$

$$\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}2^{n-2k}C_k=\frac{1}{2}C_{n+2}-C_{n+1}.\tag{4.6}$$

**Proof.** Evaluation of (4.4) at $v=1$ gives (4.5) while evaluation at $v=2$ yields (4.6). In deriving (4.5), we used the following relationships between binomial coefficients:

$$\binom{r}{1/2}=\frac{2^{2r+1}}{\pi}\binom{2r}{r}^{-1},\tag{4.7}$$

$$\binom{r}{r/2}=\frac{2^{2r}}{\pi}\binom{r}{(r-1)/2}^{-1},\tag{4.8}$$

$$\binom{r+1/2}{r}=(2r+1)\,2^{-2r}\binom{2r}{r},\tag{4.9}$$

and

$$r\binom{s}{r}=s\binom{s-1}{r-1};\tag{4.10}$$

all of which can be derived by using the Gamma function identities:

$$\Gamma\left(u+\frac{1}{2}\right)=\sqrt{\pi}\,2^{-2u}\binom{2u}{u}\Gamma(u+1),$$

and

$$\Gamma\left(-u+\frac{1}{2}\right)=(-1)^u\,2^{2u}\binom{2u}{u}^{-1}\frac{\sqrt{\pi}}{\Gamma(u+1)},$$

together with the definition of the generalized binomial coefficients as given in (2.1). □

**Proposition 6.** *If $m$ and $n$ are non-negative integers, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2^{k+m}}{k+m+1}=\begin{cases}\sum\limits_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}\frac{1}{2k+n+1}, & \text{if } n \text{ is even;}\\ -\sum\limits_{k=1}^{\lceil m/2\rceil}\binom{m}{2k-1}\frac{1}{2k+n}, & \text{if } n \text{ is odd.}\end{cases}\tag{4.11}$$

In particular,

$$\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}2^k}{k+1}=\begin{cases}\frac{1}{n+1}, & \text{if } n \text{ is even;}\\ 0, & \text{if } n \text{ is odd;}\end{cases}\tag{4.12}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 November 2024 doi:10.20944/preprints202411.1249.v1



and

$$\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}2^{k+1}}{k+2}=\begin{cases}\frac{1}{n+1}, & \text{if } n \text{ is even;}\\ -\frac{1}{n+2}, & \text{if } n \text{ is odd.}\end{cases} \tag{4.13}$$

**Proof.** Evaluate (4.1) at $v=1$.

□

## 5. Polynomial Identities

In this section, by employing Lemma 1 we derive new polynomial identities from the binomial theorem.

**Theorem 7.** *If $n$ is a non-negative integer, $v$ is a real number and $x$ is a complex variable, then*

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}2^{-k}\binom{2k+v}{(2k+v)/2}\binom{k+v}{v/2}^{-1}(1-x)^{n-k}\\&\quad=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{2k}{k}\binom{(2k+v)/2}{k}^{-1}x^{n-2k}.\end{aligned} \tag{5.1}$$

**Proof.** Consider the following variation on the binomial theorem:

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(1+y)^k(1-x)^{n-k}=\sum_{k=0}^{n}\binom{n}{k}y^kx^{n-k}.$$

Write $\cos y$ for $y$, multiply through by $\sin^v y$ and integrate from 0 to $\pi$ with respect to $y$, using Lemma 1. □

**Corollary 8.** *If $n$ is a non-negative integer and $x$ is a complex variable, then*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{2^k}{k+1}(1-x)^{n-k}=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\binom{n}{2k}}{2k+1}x^{n-2k}, \tag{5.2}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{\binom{n}{k}}{k+2}2^{-k}\binom{2(k+1)}{k+1}(1-x)^{n-k}=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\binom{n}{2k}}{k+1}2^{-2k}\binom{2k}{k}x^{n-2k}. \tag{5.3}$$

**Proof.** Identity (1.5) on page 2 and identities (5.2) and (5.3) correspond to the evaluation of (5.1) at $v=0$, $v=1$ and $v=2$, respectively.

In deriving (5.2), we used (4.7)–(4.10) to obtain

$$\binom{2k+1}{k+1/2}=\frac{2^{4k+4}}{\pi(k+1)}\binom{2(k+1)}{k+1}^{-2}\binom{2k+1}{k},$$

$$\binom{k+1}{1/2}=\frac{2^{2k+3}}{\pi}\binom{2(k+1)}{k+1}^{-1},$$

and

$$\binom{2k+1}{k}=\frac{1}{2}\binom{2(k+1)}{k+1}.$$

□

**Theorem 9.** *if $n$ is a non-negative integer, $v$ is a real number and $x$ is a complex variable, then*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}2^{-k}\binom{2k+v}{(2k+v)/2}\binom{k+v}{v/2}^{-1}(1-x)^k x^{n-k}\\&=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{2k}{k}\binom{(2k+v)/2}{k}^{-1}(1-x)^{2k}.\end{aligned}\tag{5.4}$$

**Proof.** Write $\cos y$ for $y$ in the following variation on the binomial theorem:

$$\sum_{k=0}^{n}\binom{n}{k}(1-x)^k(1+y)^k x^{n-k}=\sum_{k=0}^{n}\binom{n}{k}y^k(1-x)^k,$$

multiply through by $\sin^v y$ and integrate from 0 to $\pi$ with respect to $y$, using Lemma 1. □

**Corollary 10.** *If $n$ is a non-negative integer and $x$ is a complex variable, then*

$$\sum_{k=0}^{n}\binom{n}{k}2^{-k}\binom{2k}{k}(1-x)^k x^{n-k}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{2k}{k}(1-x)^{2k},\tag{5.5}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{2^k}{k+1}(1-x)^k x^{n-k}=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\binom{n}{2k}}{2k+1}(1-x)^{2k},\tag{5.6}$$

$$\sum_{k=0}^{n}\frac{\binom{n}{k}}{k+2}2^{-k}\binom{2(k+1)}{k+1}(1-x)^k x^{n-k}=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\binom{n}{2k}}{k+1}2^{-2k}\binom{2k}{k}(1-x)^{2k}.\tag{5.7}$$

**Proof.** Identities (5.5), (5.6) and (5.7) correspond to the evaluation of (5.4) at $v=0$, $v=1$ and $v=2$, respectively.
□

## 6. More Combinatorial Identities

**Theorem 11.** *If $n$ is an integer and $v$ is a real number, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-2k}\binom{2k+v}{(2k+v)/2}\binom{k+v}{v/2}^{-1}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{2k}{k}\binom{(2k+v)/2}{k}^{-1},\tag{6.1}$$

*and*

$$\sum_{k=0}^{n}\binom{n}{k}2^{-k}\binom{2k+v}{(2k+v)/2}\binom{k+v}{v/2}^{-1}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{n-4k}\binom{2k}{k}\binom{(2k+v)/2}{k}^{-1}.\tag{6.2}$$

**Proof.** Evaluate (5.1) at $x=-1$ and $x=2$, respectively. □

**Remark 2.** *Setting $x=0$ in (5.1) reproduces identity (4.2) while setting $x=1$ reproduces (4.3).*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 November 2024 doi:10.20944/preprints202411.1249.v1



**Proposition 12.** *If n is a non-negative integer, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-2k}\binom{2k}{k}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{2k}{k}, \tag{6.3}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{1}{2k+1}=\frac{2^{n-1}}{2^n-1}\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\frac{1}{k},\quad n\neq 0, \tag{6.4}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2k+1}{k+2}2^{n-2k+1}C_k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}C_k. \tag{6.5}$$

**Proof.** Set $x=-1$ in each of identities (1.5)–(5.3). □

**Proposition 13.** *If n is a non-negative integer, then*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{-2k}\binom{2k}{k}=2^{-n}\binom{2n}{n},$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\binom{n}{2k}}{2k+1}=\frac{2^n}{n+1}, \tag{6.6}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\binom{n}{2k}}{k+1}2^{-2k}\binom{2k}{k}=\frac{2^{-n+1}}{n+2}(2n+1)C_n. \tag{6.7}$$

**Proof.** Set $x=1$ in each of identities (1.5)–(5.3). □

**Proposition 14.** *If n is a non-negative integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}2^{-k}\binom{2k}{k}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{n-4k}\binom{2k}{k}, \tag{6.8}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{2^{n-2k+1}-2^{2k+1}}{2k+1}=\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\frac{2^{2k-1}}{k}, \tag{6.9}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{2k+1}{k+2}2^{-k+1}C_k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}2^{n-4k}C_k. \tag{6.10}$$

**Proof.** Set $x=2$ in each of identities (1.5)–(5.3).
□

**Theorem 15.** *if n is a non-negative integer and v is a real number, then*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{2k+v}{(2k+v)/2}\binom{k+v}{v/2}^{-1}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}\binom{(2k+v)/2}{k}^{-1}. \tag{6.11}$$

**Proof.** Set $x=-1$ in (5.4). □

 doi:10.20944/preprints202411.1249.v1



**Proposition 16.** *If $n$ is a non-negative integer, then*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{2k}{k}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}, \tag{6.12}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{2^{2k}}{k+1}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{2^{2k}}{2k+1}, \tag{6.13}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{2(2k+1)}{k+2}C_k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_k. \tag{6.14}$$

**Proof.** Set $x=-1$ in each of identities (5.5)–(5.7) or what is the same thing, $v=0$, $v=1$ and $v=2$ in (6.11).
□

## References


1. K. Adegoke, Editorial Correction to "A note on a generalization of Riordan's combinatorial identity via a hypergeometric series approach", *Notes on Number Theory and Discrete Mathematics* **30**:1 (2024), 211–212.
2. H. Alzer, Combinatorial identities and hypergeometric functions, II, *Discrete Mathematics Letters* **13** (2024), 1–5.
3. I. Gradshteyn and I. Ryzhik, Table of Integrals, Series, and Products, 7th edition, Elsevier Academic Press, Amsterdam, 2007.
4. D. Lim, A note on a generalization of Riordan's combinatorial identity via a hypergeometric series approach, *Notes on Number Theory and Discrete Mathematics* **29**:3 (2023), 421–425.
5. H. Prodinger, Knuth's old sum - a survey, *EACTS Bulletin* **52** (1994), 232–245.
6. A. K. Rathie, I. Kim and R. B. Paris, A note on a generalization of two well-known combinatorial identities via a hypergeometric series approach, *Integers* **22**, #A28 (2022).
7. J. Riordan, *Combinatorial Identities*, John Wiley & Sons, Inc., New York, (1971).
8. A. Tefera and A. Zeleke, On proofs of generalized Knuth's old sum, *Integers* **23**, #A99 (2023).