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Article

Universe as a Graph (Ramsey Approach to Analysis of Physical Systems)

Nir Shvalb[1], Mark Frenkel[2], Shraga Shoval[3], Edward Bormashenko[2]

[1] Department of Mechanical Engineering & Mechatronics, Faculty of Engineering, Ariel University, P.O.B. 3, Ariel 407000, Israel
[2]Chemical Engineering Department, Engineering Faculty, Ariel University, P.O.B. 3, 407000 Ariel, Israel
[3]Department of Industrial Engineering and Management, Faculty of Engineering, Ariel University, P.O.B. 3, Ariel 407000, Israel
* Correspondence: edward@ariel.ac.il

Abstract: Application of the Ramsey graph theory to the analysis of physical systems is reported. Physical interactions may be very generally classified as attractive and repulsive. This classification creates the premises for the application of the Ramsey theory to the analysis of physical systems built of electrical charges, electric and magnetic dipoles. The notions of mathematical logic, such as transitivity and intransitivity relations, become crucial for understanding of behavior of physical systems. The Ramsey approach may be applied to the analysis of mechanical systems, when actual and virtual paths between the states in configurational space are considered. Irreversible mechanical and thermodynamic processes are seen within the reported approach as directed graphs. Chains of irreversible processes appear as transitive tournaments. These tournaments are acyclic; the transitive tournaments necessarily contain the Hamiltonian path. The set of states in the phase space of the physical system, between which irreversible processes are possible, is considered. Hamiltonian path of the tournament emerging from the graph uniting these states is a relativistic invariant. Applications of the Ramsey theory to the general relativity become possible when the discrete changes in the metric tensor are assumed. Reconsideration of the concept of "simultaneity" within the Ramsey approach is reported.

Keywords: physical system; attraction; repulsion; Ramsey theory; transitivity; complete graph; relativity; Hamiltonian path.

1. Introduction

In discrete mathematics, graph theory is framework unifying the study of graphs, which are mathematical structures used to model pairwise relations between objects [1, 2]. Pairwise interactions constitute the core of physics; thus, it is well expected, that the graphs theory will play an important role in the modern physical picture of the world. Graphs are one of the principal objects of study in discrete mathematics [1-2]. We adopt the reasonable hypothesis that both physics and the corresponding mathematics have to be described by means of discrete concepts on the Planck-scale; thus, theory of graphs is well-expected to supply powerful instruments for understanding the discrete physical Universe [3,4]. A graph in this context is made up of vertices representing physical bodies which are connected by edges (also called links) representing physical interactions, which may differ in their nature. A distinction should be made between undirected graphs, where edges link two vertices symmetrically, and directed graphs, where edges link two vertices asymmetrically.

In particular, we apply the Ramsey theory for the analysis of physical behavior of a physical system built of interacting particles [5]. These interactions very generally may be classified as attractions or repulsions; thus, providing the base for application of the Ramsey theory. Ramsey theory is a field of graph theory that investigates the emergence of interconnected/interrelated sub-structures within a structure/graph of a known size [6-

10]. Ramsey's theorem, in one of its graph-theoretic forms, states that one will find monochromatic cliques in any edge labelling (with colors) of a sufficiently large complete graph [6, 8, 9]. An accessible introduction to the Ramsey theory is found in refs. [6, 9]. More rigorous approach is supplied in ref. [10]. We demonstrate that the Ramsey theory is useful for the analysis of a broad variety of physical systems, including thermodynamics and relativity problems. Thus, notions and concepts of mathematical logic become crucial for understanding of behavior of physical systems.

2. Results

2.1. Ramsey theory for a set of interacting bodies

Consider a set of interacting bodies between which attractive and repulsive interactions are possible. Let us pose following question: what is the minimal number of bodies, giving rise to appearance of sub-systems, in which only attractive or repulsive interactions are acting? These interactions may be homogeneous and heterogeneous in their physical nature. In the case of heterogeneous interactions, for example, the attractive force may be gravity and repulsion may be electrostatic in their origin. Until now, we do not specify the kind of these interactions (we will demonstrate below that the physical nature of these interactions is important for answering the posed question). The solution of the aforementioned problem is supplied by the Ramsey theory. Consider the system of n physical bodies, interacting one with another *via* attraction or repulsion. What is the minimal number of bodies giving rise to m attractions and l repulsions in the system? From the pure mathematical point of view, we have to answer the question: what is the Ramsey number $R(m,l)$?

We start from the analysis of a set of $n = 6$ interacting physical bodies, illustrated with **Figure 1**. We adopt that two kind of interactions are possible within the system: namely: attraction and repulsion. Attraction is depicted in **Figure 1** with the red solid line/edge, whereas repulsion is shown with the green solid line/edge.

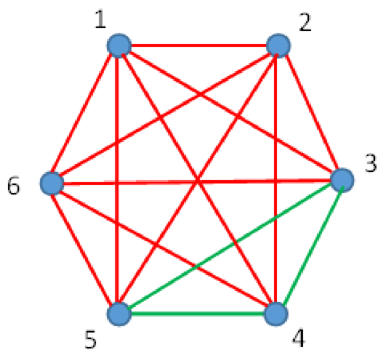

Figure 1. System built of six interacting bodies is represented. Red lines depict attractive interaction between the bodies; green lines depict repulsive interactions between the bodies.

The physical interactions acting within the system form the complete graph, i.e. a graph in which each pair of graph vertices (masses) is connected by an edge/link (interaction, i.e. repulsion or attraction); the situation when zero interaction between the bodies is possible gives rise to the three-color Ramsey problem and it will be considered below. Let us pose the following question: what is the minimal number of bodies for which three mutual attractions or three mutual repulsions will necessarily appear in the graph? The Ramsey theory supplies an exact answer to this question, namely $R(3,3) = 6$. Indeed, we recognize that within the subsystem labeled “345” only repulsions are present; whereas, in the subsystems “123”, “124”,”125”, “126”, ”136”, “146” ,”156”, “236”, “246” and “256” only attractive interactions are present.

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Now let us specify the exact kind of these interactions. It seems natural to consider the Coulomb forces acting between the point electrical charges as an attractive/repulsive interactions. It also seems from the first glance, that the Coulomb interactions represent the particular case of the aforementioned case, shown in **Figure 1**, and $R(3,3) = 6$ is kept. However, this answer will be wrong, due to the fact, that the Coulomb forces represent physical interactions, which may be transitive or intransitive in their physical nature. Consider the system of five point electrical charges, depicted in **Figure 2A**. It is immediately recognized from **Figure 2A**, that in the sub-system labeled *"135"* only repulsive Coulomb interactions are present. Moreover, in any system of five point charges (whatever are their signs) we will find ate least one sub-system built of three point charges interacting *via* repulsive forces. No triangle built of attractions (red links) is recognized. Let us understand this observation.

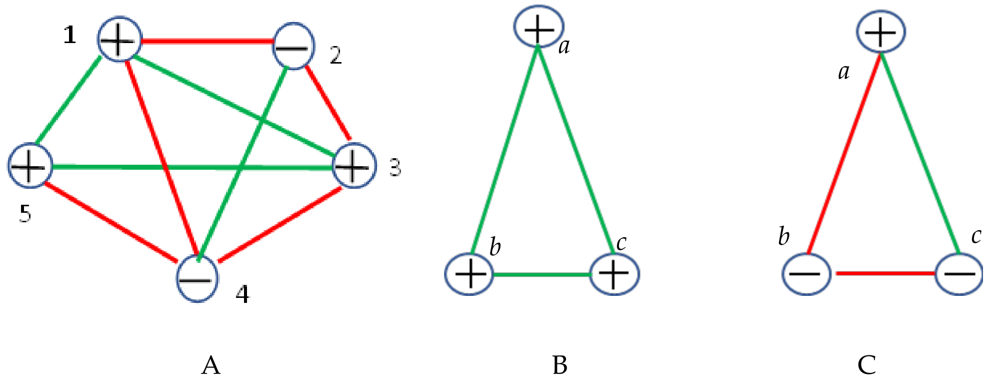

Figure 2. A. System built of five point electrical charges is represented. Red lines depict attractive Coulomb interaction between the bodies; green lines depict repulsive Coulomb interactions between the point charges. **B.** Coulomb interactions in the system of the three point charges of the same sign are shown. **C**. Coulomb interactions in the system of the three point charges of the different signs are shown .

When charges *"a"*, *"b"* and *"c"* are of the same sign, interactions $a \leftrightarrow b, b \leftrightarrow c$ and $a \leftrightarrow c$ are necessarily repulsive, in other words: when a positive charge *"a"* repulses a positive point charge *"b"*, and a positive charge *"b"* repulses a positive charge *"c"*, necessarily, charge *"a"* repulses charge *"c"* (see **Figure 2B**). Thus, when the signs of the point charges are the same the Coulomb interactions are transitive.

Now, consider the situation, when the signs of the point charges are different, as illustrated with **Figure 2C**. In this case, when a positive charge *"a"* attracts a negative point charge *"b"*, and negative charge *"b"* attracts positive charge *"c"*, necessarily charge *"a"* repulses charge *"c"*. This kind of relations is called in mathematical logic "intransitivity". Let us illustrate this property with the following logical example, involving three groups of experts, labeled *"A"*, *"B"* and *"C"* correspondingly. Consider the situation when group of experts *"A"* recognizes group *"B"*, and group *"B"* recognizes group *"C"*, but group *A* does not recognize group *C*. In this case, the recognition relation among the expert groups is defined as "intransitive". This is exactly the situation inherent for Coulomb interaction of three point charges of different signs, shown in **Figure 2C**. It should be emphasized, that no monochromatic triangle will appear when point charges of various signs are located in its vertices; however, the clique built of two monochromic edges will be necessarily present, as shown in **Figure 2C**. On the other hand, the monochromatic triangle will necessarily appear when the three point charges of the same sign are placed in the vertices, as depicted in **Figure 2B**. Using the notions of the Ramsey theory we conclude $R_{trans,intrans}(2,3) = 3$ is true for the Coulomb interactions acting between point charges.

In other words, when interactions between charges *"a"* and *"b"* are *"b"* and *"c"* are known, the kind of interaction between charges *"a"* and *"c"* is pre-scribed unequivocally, irrespectively of the spatial location of the charges; however, these interactions may be transitive, giving rise to monochromatic triangles, and intransitive, which do not enable monochromatic triangles. It is easily seen, that in any system built of five point charges at least one monochromatic triangle will necessarily appear. This is true for any odd number of point electrical charges.

The transitivity/intransitivity of the Coulomb force immediately follows from the expression for the potential energy of interaction $U(r)$ between two point charges q_1 and q_2 separated by the distance r:

$$U(r) = k\frac{q_1 q_2}{r} \tag{1}$$

where k is the constant depending of the adopted system of units. It is seen from Eq. 1 that the energy is increased with the decrease of the separation between the charges of the same sign (which corresponds to repulsion); whereas, the energy is decreased with the decrease of the separation between the charges of the different signs (which corresponds to attraction). Thus, the Coulomb interaction within a triad of electrical charges is transitive/intransitive, depending on the signs of charges, as shown in **Figure 2B-C**. This will be true for general Coulomb-like interaction, described by Eq. 2:

$$U(r) = \zeta\frac{\Phi_1\Phi_2}{r^n} \tag{2}$$

where ζ is the coefficient depending on the system of units, Φ_1 and Φ_2 are "effective charges", which may be of the different signs. Thus, the Ramsey theory may be also applied for the analysis of the physical situation, in which the "Coulomb-like" interactions between the effective charges, described by Eq. 2, are involved.

Finally, when the notions of the Ramsey theory are used, we formulate the obtained result as follows: the transitive/intransitive Ramsey number, describing Coulomb (or Coulomb-like) interactions between points charges/effective charges equals three, i.e. $R_{trans,intrans}(2,3) = 3$, whereas $R_{trans,intrans}(3,3)$ does not exist.

It should be emphasized that electrostatic interactions may be non-transitive. For example, interactions between electrical dipoles are non-transitive. Energy of interaction between two dipoles, illustrated with **Figure 3**, is described by Eq. 3 [12-13]:

$$U_{el}(r) = -k\frac{p_1 p_2}{r^3}(2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\varphi) \tag{3}$$

where p_1 and p_2 are permanent dipole moments, r is the separation between dipoles and the angles θ_1, θ_2 and φ are shown in **Figure 3**.

Electric dipoles may attract or repel each other; when $\theta_1 = \frac{\pi}{2}; \theta_2 = \frac{\pi}{2}$ and $\varphi = 0$ corresponding to the $\vec{p}_1 \upuparrows \vec{p}_2$ parallel configuration; the energy of the repulsive interaction stems from Eq. 2 and it is given by Eq. 4:

$$U_{el}(r) = k\frac{p_1 p_2}{r^3} \tag{4}$$

In turn, when $\theta_1 = \frac{\pi}{2}; \theta_2 = -\frac{\pi}{2}$ and $\varphi = 0$ corresponding to the $\vec{p}_1 \uparrow\downarrow \vec{p}_2$ anti-parallel configuration we derive for the energy of the attractive interaction (see Eq. 5):

$$U_{el}(r) = -k\frac{p_1 p_2}{r^3} \tag{5}$$

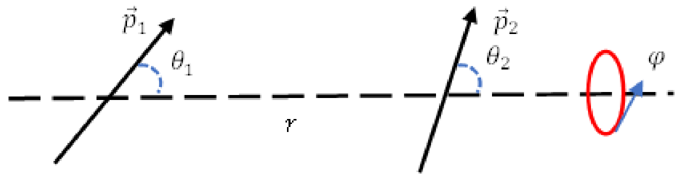

Figure 3. Geometric relations in a schematic dipole-dipole interaction are shown; $\vec{p}_1$ and $\vec{p}_2$ are permanent dipole moments, r is the separation between the dipoles, θ_1 and θ_2 are the angles of each dipole in polar coordinates and φ is the rotation angle around the axis; all three angles describe orientation of the two dipoles to each other.

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And, it is noteworthy, that the dipole-dipole interactions may by not-transitive as illustrated in **Figure 4A**.

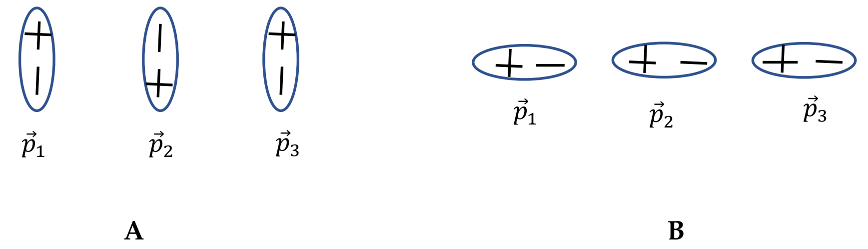

Figure 4. Sketch illustrating the non-transitive nature of interaction between electric dipoles is shown. **A**. Dipole $\vec{p}_1$ attracts dipole $\vec{p}_2$, dipole $\vec{p}_2$ attracts dipole $\vec{p}_3$ whereas dipole $\vec{p}_1$ repels dipole $\vec{p}_3$. **B**. All of the dipoles attract one another.

Indeed, in the scheme depicted in **Figure 4A** dipole $\vec{p}_1$ attracts dipole $\vec{p}_2$ and dipole $\vec{p}_2$ attracts dipole $\vec{p}_3$, whereas dipole $\vec{p}_1$ repels dipole $\vec{p}_3$. Thus, the dipole-dipole interaction is non-transitive. However, in the scheme depicted in **Figure 4B** all of the dipoles attract one another, consequently the interaction is transitive. Thus, the kind of dipole-dipole interaction depends on their mutual spatial orientation, in contrast to the situation with the interaction of point charges. Consider the system of electrical dipoles, depicted in **Figure 5**, in which red lines depict attractive dipole-dipole interaction and green lines, in turn, depict repulsive dipole-dipole interactions; now the nature of interaction depends on the spatial orientation between the dipoles, as follows from Eq. 3. For dipoles monochromatic triangles built of red links, corresponding to attraction become possible, and this is again in contrast to graphs, emerging from the interaction of point charges. Thus, we return to the situation described in **Figure 1,** when the transitivity of interaction is not unambiguously prescribed within the triad of interacting bodies. It brings us back to the complete non-transitive graphs, such as that depicted in **Figure 1**.

Recall that $R(3,3) = 6$ for complete non-transitive graphs. Indeed, triple sub-systems of dipoles labeled "125", "135", "124", "235", "265" and "146", in which only attractive interactions act are recognized in **Figure 5**. No subsystem in which only repulsive interactions appear is present in the complete graph of interactions, shown in **Figure 5**.

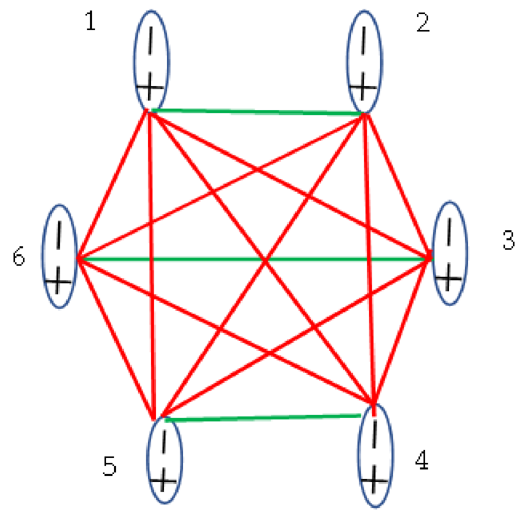

Figure 5. System of dipoles is shown. Red lines depict attractive interactions; green lines depict repulsive interactions between the dipoles (see Eq. 3).

Interaction between two magnetic dipoles is also of the non-transitive nature. This becomes clear when the energies of electrical $\boldsymbol{U_{el}}(\boldsymbol{r})$ and magnetic interaction $\boldsymbol{U_{magn}}(\boldsymbol{r})$ between dipoles are written in the symmetrized form [14]:

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$$U_{el}(r) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})}{r^3}, \tag{6}$$

$$U_{magn}(r) = \frac{\mu_0}{4\pi}\frac{\vec{\mu}_1 \cdot \vec{\mu}_2 - 3(\vec{\mu}_1 \cdot \hat{r})(\vec{\mu}_2 \cdot \hat{r})}{r^3} \tag{7}$$

where $\vec{\mu}_1$ and $\vec{\mu}_2$ are the magnetic moments and $\hat{r}$ is a unit vector parallel to the line joining the centers of the two dipoles.

The energy of interaction of two magnetic dipoles depends on the mutual orientation, and it may give rise to attractive and repulsive forces, which are non-transitive, as shown in Figure 6. Again, sub-systems of magnetic dipoles labeled "125", "135", "124", "235", "265" and "146", in which only attractive interactions act are recognized in Figure 5. No subsystem in which only repulsive interactions appear is not present in the complete graph of interactions, shown on Figure 6. Thus, the Ramsey theory may supply additional insights to the Ising problem[15].

We conclude that for the complete graphs depicting repulsive and attractive interactions between electric and magnetic dipoles $R(3,3) = 6$ is true and monochromatic triangles corresponding to attractive and repulsive interactions are possible. And this in contrast to Coulomb interaction between point electrical charges, where monochromatic triangles representing attraction are forbidden, by the logical structure of interaction between point electrical charges.

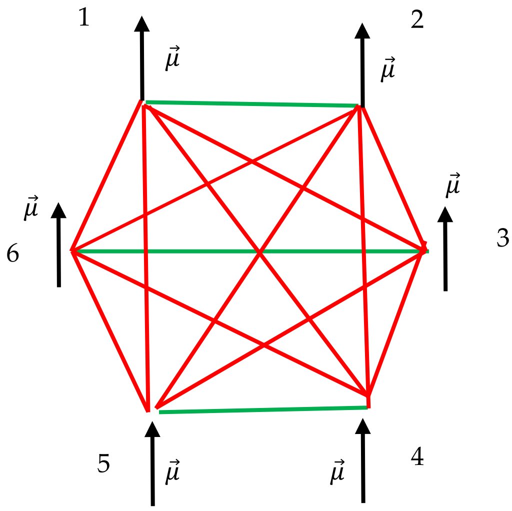

Figure 6. System of magnetic dipoles $\vec{\mu}$ resembling the system of electric dipoles shown in **Figure 5** is shown (see Eq. 6 and Eq. 7). Red lines depict attractive interactions; green lines depict repulsive interactions between the dipoles. The nature of the interaction depends on the mutual orientation of magnetic dipoles (see Eq. 7).

2.2. Triple interacting systems: the Ramsey approach

The Ramsey Theory supplies the general framework for more complicated systems in which arbitrary number of interactions are possible. For example, consider the system in which three kinds of interactions are possible, namely: attraction, repulsion and zero interaction, which are supposed to be non-transitive. Consider the system depicted in Figure 7. Red lines in Figure 7 correspond to the attractive interaction between, green lines depict repulsive interactions and yellow lines connect the bodies, which do not interact. It is easily seen from Figure 7 that no mono-chromatic triangle appear in Figure 7.

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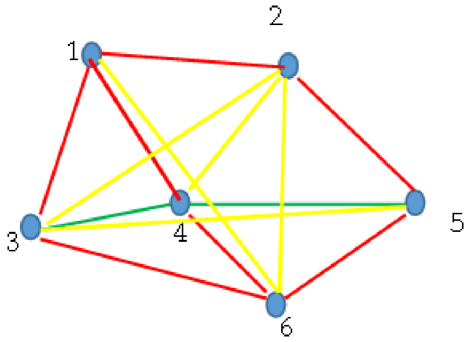

Figure 7. System built of six bodies is represented. Red lines depict attractive interaction between the bodies; green lines depict repulsive interactions between the bodies; yellow lines correspond to the zero interaction between the bodies (bodies do not interact). .

Let us address the following question: what is the minimal number of bodies, giving rise to appearance of sub-systems, in which monochromatic triangles will necessarily appear, in other words: sub-systems including only attracting or repulsing or non-interacting bodies are present. From the point of view of the Ramsey theory, the question is formulated as follows what the value of $\boldsymbol{R}(\mathbf{3},\mathbf{3},\mathbf{3})$ number? The problem was solved by mathematicians: $\boldsymbol{R}(\mathbf{3},\mathbf{3},\mathbf{3}) = \mathbf{17}$ [**16**]. It is noteworthy that a restricted set of Ramsey numbers is known until now [11, 16].

2.3. Ramsey theory and dynamics of mechanical system

Now consider the Ramsey re-interpretation of the classical mechanics. The dynamics of mechanical systems is determined by the Hamiltonian principle, stating that the true evolution $\boldsymbol{q}(\boldsymbol{t})$ of a system described by *N* generalized coordinates $\boldsymbol{q} = (\boldsymbol{q}_{\mathbf{1}},\boldsymbol{q}_{\mathbf{2}} \dots \boldsymbol{q}_{\boldsymbol{n}})$ between two specified states $\boldsymbol{q}(\boldsymbol{t}_{\mathbf{1}}) = (\boldsymbol{q}_{\mathbf{1}}(\boldsymbol{t}_{\mathbf{1}}), \boldsymbol{q}_{\mathbf{2}}(\boldsymbol{t}_{\mathbf{1}}) \dots \boldsymbol{q}_{\boldsymbol{n}}(\boldsymbol{t}_{\mathbf{1}}))$ and $\boldsymbol{q}(\boldsymbol{t}_{\mathbf{2}}) = (\boldsymbol{q}_{\mathbf{1}}(\boldsymbol{t}_{\mathbf{2}}), \boldsymbol{q}_{\mathbf{2}}(\boldsymbol{t}_{\mathbf{2}}) \dots \boldsymbol{q}_{\boldsymbol{n}}(\boldsymbol{t}_{\mathbf{2}}))$ at two specified times t_1 and t_2 is a stationary point (a point where the variation $\boldsymbol{\delta S}$ is zero) of the action functional *S*, defined by Eq. 8:

$$\boldsymbol{S} = \int_{\boldsymbol{t}_{\mathbf{1}}}^{\boldsymbol{t}_{\mathbf{2}}} \boldsymbol{L}(\boldsymbol{q}(\boldsymbol{t}), \dot{\boldsymbol{q}}(\boldsymbol{t}))\boldsymbol{dt} \tag{8}$$

Consider that the motion of the system may be pictured as that of the single point (labeled usually *C*-point) in the extended configurational space comprising the generalized coordinated and time as independent variables [17, 18]. In this space the successive phases of the motion show up as successive points of a curve. This curve, the "world-line" of the *C*–point, contains in geometrical form the entire physical history of the mechanical system [17, 18]. The Hamilton principle states that the motion of the system between the initial time t_1 and final time t_2 follows a path that minimizes the scalar action integral defined as the time integral of the Lagrangian, provided the initial and final configurations of the system are prescribed. Thus, from the point of view of a pure logic two kinds of pathways are possible in the configurational space, namely: i) pathways which minimize the action integral (at these pathways $\delta S = 0$ takes place); we call these paths the "actual paths" and ii) paths which do not minimize the action functional, given by Eq. 8. We call below these paths the "virtual paths" and $\delta S \neq 0$ hold along these paths. Thus, premises for application of the Ramsey approach are created, as illustrated with Figure 8.

Consider the map of the states, available for the system in the configurational space.

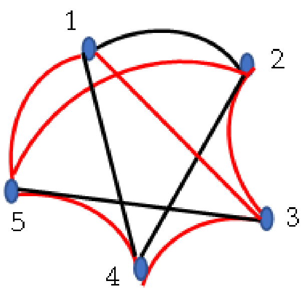

Figure 8. Map depicting five states in the configuration space is shown. Red links correspond to the true evolution of the system, .i.e. providing $\delta S = 0$ (**actual paths**); black links illustrate virtual pathways in the configurational space, $\delta S \neq 0$ along black pathways (virtual paths).

The map emerges from five points in the configurational space of the mechanical system. The points are interconnected by paths, corresponding to the actual paths, corresponding to the actual motions of the mechanical system ($\delta S = 0$) and virtual paths ($\delta S \neq 0$) which were not chosen by nature for actual motions of the system. Actual paths are shown with red links, whereas virtual motions are shown with black links. These paths form the complete graph. It is recognized from the map supplied in Figure 8, that is possible create a graph in which no monochrome triangle is present. However, it will be already impossible for the map comprising six points, due to the fact that $R(3,3) = 6$. Thus, in the graph built of the six vertices, representing *C*-points in the configurational space and interconnected by actual and virtual paths, cycles will necessarily appear. These cycles ("red cycles" or alternatively "black" ones) may correspond to actual or virtual (motions) of the *C*-point in the configurational space. Thus, any evolution of any mechanical system may be represented with the coresspomding Ramsey graph.

2.4. Irreversible processes and graph theory

Until now, we did not address reversibility of the addressed mechanical processes. Now consider the physical system in which only irreversible processes are possible (as a matter of fact in any macro scale mechanical system friction is inevitable, and the processes are irreversible to a greater or lesser extent). Again, we consider the map of the states in the configurational space available to the system, shown in Figure 9.

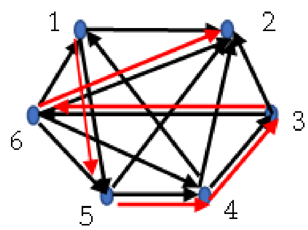

Figure 9. Map depicting six states in the configurational space is depicted. Only irreversible transitions between the states are possible. The transitive tournament is shown with black arrows. Red arrows indicate the Hamiltonian path.

Black arrows indicate directions of the irreversible processes. We assume that irreversible transitions between all of the states, corresponding to the points in the configurational space are possible, as shown in Figure 9. Thus, a tournament which is a directed graph (digraph) obtained by assigning a direction for each edge emerges [1]. We assume that the emerging tournament is transitive, namely $(a \rightarrow b)$ **and** $(b \rightarrow c) \Rightarrow (a \rightarrow c)$ takes place in such a tournament (for example: $(6 \rightarrow 1)$ **and** $(1 \rightarrow 2) \Rightarrow (6 \rightarrow 2)$ is true for the discussed tournament). If the tournament is transitive, the theory of graphs predicts three consequences: i) the tournament is acyclic, i.e. it is a directed graph with no directed cycles; in particular, the tournament does not contain a cycle of length 3. Indeed, we recognize from Figure 9, that no cyclic process is possible for the presented tournament; ii) the tournament contains the Hamiltonian path. Hamiltonian path is the directed path on all n vertices of the graph, which is shown with red arrows in Figure 9. And it should be emphasized that the transitive directed graph has the only one Hamiltonian path. Thus, an irreversible process which passes over all available states in the configurational space of the system is possible.

Now consider the graph theory interpretation of thermodynamic processes. Consider n bodies which are in a thermal contact, the temperatures of the bodies are labeled $\boldsymbol{T_i}$ $(\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2} \ldots \boldsymbol{n})$. We accept that no pair of bodies is in the thermal equilibrium, in other words $\boldsymbol{T_i} \neq \boldsymbol{T_k}$, when $\boldsymbol{i} \neq \boldsymbol{k}$. According to the Clausius statement "heat can never pass from a colder to a warmer body without some other change, connected therewith, occurring at the same time" [19]. Thus, directions of the heat transfer give rise to the tournament; we assume that all of the bodies are in a thermal contact one with another. For a sake of simplicity consider the system built of four bodies, $\boldsymbol{T_1} > \boldsymbol{T_2} > \boldsymbol{T_3} > \boldsymbol{T_4}$ is adopted is shown in Figure 10.

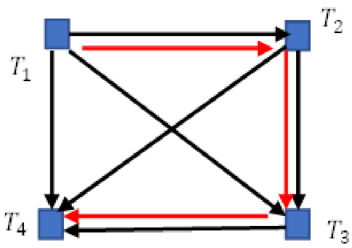

Figure 10. Graph illustrating thermal contact of four bodies is depicted, $\boldsymbol{T_1} > \boldsymbol{T_2} > \boldsymbol{T_3} > \boldsymbol{T_4}$ is assumed. Black arrows depict the direction of the heat transfer. The graph represents the transitive tournament. Red arrows depict the Hamiltonian path.

The graph shown in Figure 10 is a transitive tournament; no cycles of length 3 are recognized in the graph and the single Hamiltonian path (shown with the red arrows) is inherent for this graph. The generalization for n bodies in thermal contact is straightforward. Thus, re-shaping of the Second Law of Thermodynamics with the graph theory becomes possible, as follows: the heat transfer in the system of n bodies $\boldsymbol{T_i}$ $(\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2} \ldots \boldsymbol{n}), \boldsymbol{T_i} \neq \boldsymbol{T_k}$, when $\boldsymbol{i} \neq \boldsymbol{k}$ generates a transitive tournament. No cycles with a length of 3 are present in this graph. Thus, no cyclic processes are possible in the system. A single Hamiltonian path is possible in the graph.

2.5. Ramsey theory and general relativity

The Ramsey theory enables a new interpretation of the general relativity. An interval ds between two events in the general relativity is given by Eq. 9:

$$-ds^2 = g_{ik} dx_i dx_k \tag{9}$$

where g_{ik} is the metric tensor (we use the definition of interval adopted in the classical textbook by Landau and Lifshitz [20]). Generally speaking, g_{ik} is the continuous function of the space coordinates and time [20]. We consider the situation of the discrete change in the metric tensor, in other words, the situation, when the interval between two events is given by Eqs. 10-11:

$$-ds^2 = g_{ik}^{(1)} dx_i dx_k \tag{10}$$

$$-ds^2 = g_{ik}^{(2)} dx_i dx_k \tag{11}$$

where $g_{ik}^{(1)}$ and $g_{ik}^{(2)}$, are the metric tensors, which are not-equal each to another. For a sake of simplicity the components of the tensors may be taken as constant. This situation is depicted in Figure 11, in which six events separated by different metric tensors $g_{ik}^{(1)}$ and $g_{ik}^{(2)}$ are depicted.

The events form complete, non-transitive, non-directed graph, shown in Figure 11. Let us pose the following fundamental question: what is the minimal number of physical events providing appearance of triangles within the events' map, interconnected by the same metric tensor ($\boldsymbol{g_{ik}^{(1)}}$ or $\boldsymbol{g_{ik}^{(2)}}$?). The answer to this question is supplied by the Ramsey theory: $\boldsymbol{R(3,3) = 6}$. Indeed, in the events' map presented in Figure 11 red triangles "135" and "246" corresponding to the events connected by the metric tensor $\boldsymbol{g_{ik}^{(2)}}$ are recognized.

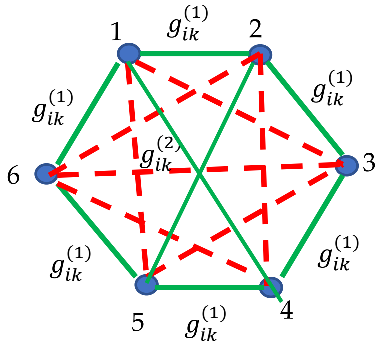

Figure 11. The map depicting six events is depicted. The events are separated by the intervals supplied by Eqs. 10-11. Green links correspond to the events separated by the interval, defined by the metric tensor $\boldsymbol{g_{ik}^{(1)}}$; red links, in turn, correspond to the events separated by the interval, defined by the metric tensor $\boldsymbol{g_{ik}^{(2)}}$. Red triangles "135" and "246" are recognized in the map.

2.6. Graph theory and simultaneous events in classical physics and relativity: the Ramsey theory and causality

Now we address the Ramsey interpretation of the notion of simultaneity. Consider five events which occurred in the given frame of references. Two kinds of the time relationship between the events are possible: the first relationship occurs, when the events occurred non-simultaneously, i.e. $\Delta\tau \neq 0$ takes place, where $\Delta\tau$ is the time span between the events (we consider now the classical meaning of simultaneity of events; the relativistic extension of the Ramsey approach to simultaneity of events will be treated immediately below). These events are connected in **Figure 12** with the red line. The second situation takes place when the events are simultaneous, i.e. $\Delta\tau = 0$. These events are connected with the green line (as shown in **Figure 12**). Let us address the following question: what is the minimal set of events in which three events took place simultaneously ($\Delta\tau = 0$) or three events occurred non-simultaneously ($\Delta\tau \neq 0$). Simultaneity of events is the transitive property in the classical physics (the relativistic extension of the problem is more complicated and it will be treated below). The answer to this question again is supplied by the Ramsey theory, and it is formulated as follows: what is the minimal transitive Ramsey $R_{tr}(3,3)$? The answer to this question was addressed in Section 2.1 and it is $R_{tr}(3,3) = 5$ (see ref. 11). Indeed, we recognize in the example illustrated with **Figure 12**, that in the set built of five events, in which the relationships "to be simultaneous" and "to be non-simultaneous" necessarily present we find a triad of simultaneous events, connected with green links. The triad of simultaneous events appears as a green triangle in **Figure 12**.

Indeed, we recognize in the example illustrated with **Figure 12**, that in the set built of five events, in which the relationships "to be simultaneous" and "to be non-simultane-

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ous" are necessarily present we necessarily find the triad of simultaneous or non-simultaneous ones. This fact imposes the restrictions on the causality of the aforementioned events. The events forming the green triangle in **Figure 12** cannot influence one another.

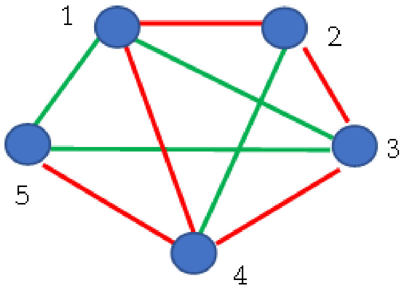

Figure 12. Graph representing five events which took place in the same frame of references is presented. Red lines link events which are non-simultaneous $\Delta\tau \neq 0$; green lines link simultaneous events $(\Delta\tau = 0)$.

Let us consider the relativistic extension of the aforementioned approach. The special relativity-based generalization is trivial: the synchronization of clocks with the light beam should be carried out [20]. This will reduce the situation to that presented in **Figure 12**. Synchronization of clocks in the general relativity is a more complicated problem [20, 21]. In the general theory of relativity, proper time elapses differently even at different points of space in the same reference system [20]. This means that the interval of proper time between two events occurring at some point in space, and the interval of time between two events simultaneous with these at another point in space, are in general different from one another. The time difference between two events, occurring at infinitely near points is given by:

$$\Delta\tau = -\frac{g_{0i}dx^i}{g_{00}}, (i = 1,2,3) \tag{12}$$

where $\boldsymbol{g_{ik}}$ is the metric tensor. Eq. 12 enables synchronization of clocks in any infinitesimal region of space. Carrying out a similar synchronization from the given point, we can synchronize clocks, i.e. we can define simultaneity of events, along any open curve. However, synchronization of clocks along a closed contour turns out to be impossible in general; indeed, starting out along the contour and returning to the initial point, we would obtain for $\boldsymbol{\Delta\tau}$ a value different from zero [20, 21]. Thus it is, impossible to synchronize clocks over all space. The exceptional cases are those reference systems in which all the components of the metric tensor $\boldsymbol{g_{oi}}$ are equal to zero (i.e. so called the time-orthogonal coordinate systems). However, in any gravitational field, it is possible to choose the reference system so that the three components of the metric tensor $\boldsymbol{g_{oi}}$ are equal to zero [20]; thus, making possible a complete synchronization of clocks. Thus, simultaneity is transitive if and only if a space-time is time-orthogonal. Thus, the graph analysis supplied in Figure 12 will apply only to the time-orthogonal coordinate systems. In these systems the aforementioned conclusions arising from the Ramsey-theory-based analysis remain true.

2.6. Irreversible processes in the relativity: the graph theory analysis

Now consider relativistic generalization of the thermodynamic processes already considered in Section 2.4. Consider the chain of the irreversible thermodynamic processes, depicted in Figure 13. We define now the processes as "irreversible", when they create new entropy, denoted S [22-23]. The hierarchy of entropies, supplied by Eq. 13 is assumed:

$$\boldsymbol{S_3 > S_4 > S_5 > S_1 > S_2} \tag{13}$$

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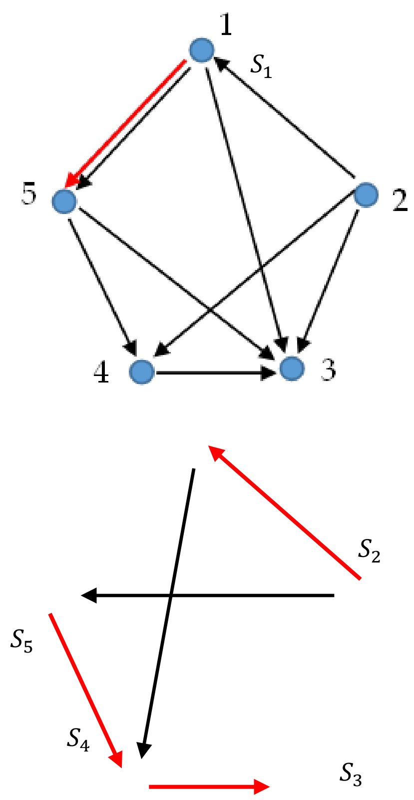

Figure 13. Graph representing the chain built of five irreversible processes is shown. Hierarchy of entropies is given by: $\boldsymbol{S_3} > \boldsymbol{S_4} > \boldsymbol{S_5} > \boldsymbol{S_1} > \boldsymbol{S_2}$. Black arrows show the direction of the processes; red arrows demonstrate the Hamiltonian path.

In this case, we obviously deal with the directed transitive tournament (see Section 2.4). Thus, no cycles are present in the graph. The Hamiltonian path inherent for this tournament is shown with red arrows. Now we are interested in the relativistic analysis of this graph. Entropy of the physical system is relativistic invariant [24]. Thus, the ordering of the graph with entropy is also relativistic invariant, hence the Hamiltonian path is relativistic invariant. Thus, an important theorem is proved for any set of states in the phase space of a physical system, between which irreversible processes are possible, namely: Hamiltonian path of the tournament emerging from the graph uniting these states is a relativistic invariant.

4. Discussion

The concepts of modern mathematics turn out to be extremely useful for understanding of the physical reality [25, 26]. Galileo Galilei stated that nature to be "written in the language of mathematics" and Eugene Wigner stressed the "unreasonable effectiveness of mathematics in physical sciences" [26]. Tegmark suggested that our physical reality is a pure mathematical structure [25]. That is, the physical universe is not merely described by mathematics, but is mathematics itself [25]. As an example the physical Universe may be seen as a hologram [27]. The classical example of this kind of physical thinking is identifying of gravity with the geometry of time-space continuum in the general relativity. We propose to view the physical reality as a graph and suggest application of the graph theory to physical problems. Thus, notions of mathematical logic, such as transitivity and intransitivity, start to play decisive role in the treatment of physical problems.

In a majority of physical problems various kinds of fundamental relationships between physical bodies are present; these relationship may be: attraction and repulsion between physical objects, simultaneous and non-simultaneous events in the special and general relativity, etc. This fact makes possible applications of the Ramsey theory to the analysis of physical systems. Ramsey theory, named after the British mathematician and philosopher Frank P. Ramsey, is a is a branch of combinatorics that focuses on the appearance of order in a substructure given a structure of a known size [8-10]. Thus, the problem may be formulated as follows: consider physical system in which repulsions and attractions between the bodies are present. How large must addressed system be to guarantee appearance of triads of bodies interconnected by attractive and repulsive force? The answer supplied by the Ramsey theory is $R(3,3) = 6$. However, when the kind of interactions is specified, the possible transitivity/intransitivity of these interactions should be considered. And this is the case when Coulomb interactions between point electrical charges are addressed. When electrical charges of various signs are involved in these interactions, they are intransitive, and no monochromatic triangle corresponding to attractions will appear in the complete graph representing Coulomb interactions between point charges. We addressed several examples in which the Ramsey analysis of the physical system is useful including the relativity and thermodynamics problems.

The notions of mathematical logic occupy the central place in the Ramsey theory, thus these notions also are of the primary importance for the Ramsey-based analysis of physical problems. In particular considering of the physical properties becomes extremely important for analysis of physical problems. The deep treatment of the analogy between transitivity of the heat transfer and simultaneity of events in the general relativity was carried out in ref. 21. And it becomes extremely important for Ramsey-analysis of transitive and non-transitive graphs, representing physical problems; the transitive and non-transitive Ramsey numbers are different [11]. Applications of the Ramsey theory to the analysis physical problems are rare [28, 29]; we demonstrate the possibility of these applications in the various sub-fields of fundamental physics.

5. Conclusions

The ideas of discrete mathematics become ubiquitous in the analysis of physical systems. We demonstrate that methods supplied by the graph theory are applicable for analysis of fundamental physical problems, in particular we focus on the application of the Ramsey theory to discrete physical systems and processes. Interactions between physical bodies may be very generally classified as repulsive and attractive. This makes possible formulation of the typical Ramsey-shaped question: how larger should be physical system in order to provide appearance of triads of bodies/particles interconnected by repulsion or attraction? An answer to this question has a fine structure: the interactions may be transitive and non-transitive (and it is also possible that bodies/particles do not interact). Coulomb interactions between point charges may be transitive or intransitive, depending on the signs of electric charges, irrespectively on the spatial location of the charges. On the other hand, static interactions between electrical and magnetic dipoles may be transitive and non-transitive, depending on their mutual spatial orientation. The transitive/intransitive Ramsey number, describing Coulomb (or Coulomb-like) interactions between points charges/effective charges equals three, i.e. $R_{trans,intrans}(2,3) = 3$, whereas $R_{trans,intrans}(3,3)$ does not exist. Thus, no monochromatic triangle corresponding to attractions will appear in the complete graph representing Coulomb interactions between point electrical charges.

The non-transitive Ramsey number $R(3,3) = 6$ is applicable for the graph describing static interactions between electric and magnetic dipoles. This result supplies an additional insight into the Ising problem, when interaction between magnetic dipoles is considered. Considering the triple interaction problem, i.e. allowing for the bodies/particles attraction, repulsion and zero interaction, gives rise to the three-color Ramsey problem, the Ramsey number for this problem is $R(3,3,3) = 17$.

The Ramsey approach may be applied to the analysis of mechanical systems, when actual (i. e. emerging from the Hamilton principle) and virtual paths between the states in configurational space are taken into account. The Ramsey number $R(3,3) = 6$. Thus, in the graph built of the six vertices, representing C-points in the configurational space and interconnected by actual and virtual paths, cycles of actual or virtual paths will necessarily appear. These cycles may correspond to actual or virtual motions of the *C*-point in the configurational space. Thus, evolution of any mechanical system may be represented with the Ramsey graph. Ramsey theory enables reconsideration of the concept of "simultaneity" in the classical mechanics and relativity. In the classical mechanics simultaneity is transitive; thus, in the set built of five events, in which the relationships "to be simultaneous" and "to be non-simultaneous" are present we necessarily find the triad of simultaneous or non-simultaneous ones. This fact stems from the fact that the transitive Ramsey number $R_{tr}(3,3) = 5$. Triads of "simultaneous graph vertices" in this case represent events which can not influence each other. In turn, simultaneity is transitive in the general relativity only if a space-time is time-orthogonal (i.e. we mean the reference systems in which all the components of the metric tensor $g_{0i}(i = 1,2,3)$ are equal to zero). In these systems, again in the set built of five events we necessarily find the triad of simultaneous or non-simultaneous ones. Graph theory is extremely useful for the analysis of the chains of irreversible processes (whatever, mechanical or thermodynamic). These chains form transitive tournaments; thus, no cycles of length 3 are possible in these directed graphs. The only one Hamiltonian path is possible in these graphs, and this path is a relativistic invariant for the directed graphs ordered according to the entropies of discrete thermodynamic states. Restrictions inherent to the Ramsey theory should be considered. Firstly, the results supplied by the Ramsey theory are non-constructive: they may show that some sub-structure exists, but they give no process for finding this structure (other than brute-force search). Secondly, the Ramsey theory states that sufficiently large objects must necessarily contain a given sub-structure, often the proof of these results requires these objects to be enormously large, giving rise to bounds that grow exponentially. Anyway, the Ramsey approach enables the fresh glance on the physical systems and processes seen as discrete entities and re-shaping of the fundamental physical problems with the notions of mathematical logic.

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