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A Related Fixed Point Theorem for Multivalued Mappings and Its Consequences

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Abstract: In the present paper, the concept related orbitally completeness of two metric spaces for multi-valued mappings is introduced. A new related fixed point theorem for multi-valued mappings is proved and some important results are obtained as the corollaries of the present main theorem. Single-valued version of the present main theorem is obtained like a simple corollary and also two illustrative examples are given.

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MSC: 47H09; 47H10; 54H25

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1. Introduction

In 1981, Fisher [1] established a theorem concerning the fixed points of compositions of two mappings on two complete metric spaces, providing a relationship between the fixed points of these mappings. Since then, numerous researchers have extended this result in various ways, exploring different types of the theorem for two or more mappings. Examples of such works include [2–9] and others.

In 2000, Fisher and Türkoğlu [10], by considering the multi-valued version of related fixed point theorem for single-valued mappings in [2], gave some related fixed point theorems for multi-valued mappings on two complete and compact metric spaces. Also Chourasia and Fisher [11], Jain and Fisher [12], Popa [13], Rohen and Murthy [14] and Biçer et al. [15] are proved some related fixed point theorems for multi-valued mappings, using the some contractive conditions.

In this paper, by considering the multi-valued version of the related fixed point theorem for single-valued mappings, we give a new different related fixed point theorem for multi-valued mappings in two related complete metric spaces.

The following are some characteristics of the present work.

- The concept related orbitally completeness of two metric spaces for multi-valued mappings is introduced.
- A new related fixed point theorem for multi-valued mappings is established.
- While in existing related fixed point theorems for multi-valued mappings where used at least two or more contraction conditions, in the present main theorem is used only one contraction condition.
- Unlike other existing theorems for multi-valued mappings, only the classical metric is used in the contraction conditions in the main results presented.
- Multi-valued version of the Bollenbacher and Hicks's result [16] is obtained as a corollary of the present main theorem.
- Single-valued version of the present main theorem is obtained like a simple corollary.
- Two illustrative examples are given.

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2. Preliminaries

Let (Z, ρ) be a metric space. Throughout the paper we denote by $P(Z)$ the family of all nonempty subsets of Z, by $CL(Z)$ the family of all nonempty closed subset of Z and by $CB(Z)$ the family of all nonempty closed bounded subset of Z.

Let $z_0 \in Z$ and let F be a mapping of Z into $P(Z)$. We shall use the following definitions.

Definition 1 ([17]). *An orbit of the multi-valued mapping F at the point of $z_0 \in Z$ is a sequence*

$$O(z_0) = \{z_n : z_n \in Fz_{n-1}\}.$$

Definition 2 ([17]). *A metric space (Z, ρ) is said to be F- orbitally complete iff every Cauchy sequence of the form $\{z_{n_i} : z_{n_i} \in Fz_{n_i - 1}\}$ converges in Z.*

Definition 3. *Let (Z, ρ) and (Y, ϱ) be two metric spaces, let F be a mapping of Z into $P(Y)$ and G be a mapping of Y into $P(Z)$. Let $z_0 \in Z$ and $y_0 \in Y$.*

Consider the following sets,

$$\begin{aligned} O_Z(z_0, y_0) &= \{z_n : z_n \in Gy_{n-1}, n = 1, 2, \dots\}, \\ O_Y(z_0, y_0) &= \{y_n : y_n \in Fz_{n-1}, n = 1, 2, \dots\}. \end{aligned} \tag{1}$$

Then the metric spaces (Z, ρ) and (Y, ϱ) are called related FG-orbitally complete for $(z_0, y_0) \in Z \times Y$ iff every Cauchy sequence in $O_Z(z_0, y_0)$ and $O_Y(z_0, y_0)$ converges to a point in Z and converges to a point in Y respectively.

Note that two metric spaces (Z, ρ) and (Y, ϱ) of course related FG-orbitally complete. But related FG-orbitally complete (Z, ρ) and (Y, ϱ) metric spaces do not necessarily complete as in shown by the following example.

Example 1. *Let $Z = Y = [0, 1)$ with Euclidean metric ρ. Let the mappings $F, G : Z \to P(Z)$ be defined by $F(z) = [0, z/2], G(z) = [0, z/4]$ for all $z \in Z$. Then for $z_0 = 0$ and $y_0 = 0$, we have*

$$O_Z(0, 0) = \{0, 0, 0, \dots\}, \quad O_Y(0, 0) = \{0, 0, 0, \dots\}.$$

Therefore, (Z, ρ) and (Y, d) are related FG-orbitally complete for all $(0, 0) \in Z \times Y$. But (Z, ρ) is not complete.

In a recent paper [18], Romaguera introduced the definition of 0-lower semicontinuity as a generalization of lower semicontinuity.

Similarly, we now the following definition.

Definition 4. *Let (Z, ρ) and (Y, ϱ) be two metric spaces. We shall say that a real-valued function $\mu : Z \times Y \to [0, \infty)$ is FG-orbitally 0-lower semicontinuous (briefly 0-lsc) at $(z, y) \in Z \times Y$ with respect to (z_0, y_0) if $\lim_{n\to\infty} z_n = z, \lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} \mu(z_n, y_n) = 0$, then $\mu(z, y) = 0$, where $\{z_n\}$ and $\{y_n\}$ are sequences in $O_Z(z_0, y_0)$ and $O_Y(z_0, y_0)$, respectively.*

Definition 5. *Let F be a mapping of Z into $P(Y)$ and G be a mapping of Y into $P(Z)$. Then the composition of the mappings F and G defined by*

$$(GF)(z) = \bigcup_{v \in F(z)} G(v) \text{ for } z \in Z, \quad (FG)(y) = \bigcup_{u \in G(y)} F(u) \text{ for } y \in Y.$$

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3. Main Results

In this section, firstly we give the following related fixed point theorem in two (Z,ρ) and (Y,ϱ) related orbitally complete metric spaces.

Theorem 1. *Let (Z,ρ) and (Y,ϱ) be two metric spaces and let F be a mapping of Z into $P(Y)$ and G be a mapping of Y into $P(Z)$. Suppose there exist $z' \in G(y)$ and $y' \in F(z)$ such that*

$$\max\{\rho(z,z'),\varrho(y,y')\} \leq \gamma(z,y) - \gamma(z',y') \tag{2}$$

for all $z \in Z$ and $y \in Y$, where $\gamma : Z \times Y \to [0,\infty)$. If (Z,ρ) and (Y,ϱ) are related FG-orbitally complete for some $(z_0,y_0) \in Z \times Y$, then we have;

(a) *There exist two sequence $\{z_n\}$ in $O_Z(z_0,y_0)$ and $\{y_n\}$ in $O_Y(z_0,y_0)$ such that*

$$\lim_{n\to\infty} z_n = u \in Z \text{ and } \lim_{n\to\infty} y_n = v \in Y,$$

(b) $\max\{\rho(z_n,u),\varrho(y_n,v)\} \leq \gamma(z_0,y_0)$,

(c) *If F is a mapping of Z into $CL(Y)$ and G is a mapping of Y into $CL(Z)$, then the following statements are equivalent;*

- (i) *$u \in G(v)$ and $v \in F(u)$.*
- (ii) *$\mu : Z \times Y \longrightarrow [0,\infty)$, $\mu(z,y) = \rho(z,G(y))$ and $\eta : Z \times Y \longrightarrow [0,\infty)$, $\eta(z,y) = \varrho(y,F(z))$ are FG-orbitally 0-lsc at (u,v) with respect to (z_0,y_0), where $\rho(z,G(y)) = \inf\{\rho(z,w) : w \in G(y)\}$ and $\varrho(y,F(z)) = \inf\{\varrho(y,x) : x \in F(z)\}$.*
- (iii) *$\rho(u,G(v)) = 0$ and $\varrho(v,F(u)) = 0$.*

Further, if $u \in G(v)$ and $v \in F(u)$, then $u \in (GF)(u)$ and $v \in (FG)(v)$.

Proof. Suppose that (Z,ρ) and (Y,ϱ) are related FG-orbitally complete for $z_0 \in Z$ and $y_0 \in Y$. Then from inequality (2), there exist $z_1 \in G(y_0)$ and $y_1 \in F(z_0)$ such that

$$\max\{\rho(z_0,z_1),\varrho(y_0,y_1)\} \leq \gamma(z_0,y_0) - \gamma(z_1,y_1).$$

Similarly, there exist $z_2 \in G(y_1)$ and $y_2 \in F(z_1)$ such that

$$\max\{\rho(z_1,z_2),\varrho(y_1,y_2)\} \leq \gamma(z_1,y_1) - \gamma(z_2,y_2)$$

and continuing in this way, we obtain two sequences $\{z_n\}$ in $O_Z(z_0,y_0)$ and $\{y_n\}$ in $O_Y(z_0,y_0)$ such that $z_n \in G(y_{n-1})$ and $y_n \in F(z_{n-1})$, and

$$\max\{\rho(z_{n-1},z_n),\varrho(y_{n-1},y_n)\} \leq \gamma(z_{n-1},y_{n-1}) - \gamma(z_n,y_n) \tag{3}$$

for all $n = 1,2,\ldots$.

Now we shall show that the sequences $\{z_n\}$ and $\{y_n\}$ are the Cauchy sequences.

Using inequality (3), we get

$$\begin{aligned} r_n &= \sum_{k=1}^{n} \max\{\rho(z_{k-1},z_k),\varrho(y_{k-1},y_k)\} \\ &\leq \sum_{k=1}^{n} [\gamma(z_{k-1},y_{k-1}) - \gamma(z_k,y_k)] \\ &= \gamma(z_0,y_0) - \gamma(z_n,y_n) \leq \gamma(z_0,y_0). \end{aligned}$$

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Therefore, $\{r_n\}$ is bounded and also non-decreasing. Thus $\{r_n\}$ is convergent. Let m, n be any two positive integers with $m > n$. From triangle inequality property of the metrics ρ and ϱ, we have

$$\begin{aligned}\max\{\rho(z_n, z_m), \varrho(y_n, y_m)\} &\leq \max\left\{\sum_{i=n}^{m-1} \rho(z_i, z_{i+1}), \sum_{i=n}^{m-1} \varrho(y_i, y_{i+1})\right\} \\ &\leq \sum_{i=n}^{m-1} \max\{\rho(z_i, z_{i+1}), \varrho(y_i, y_{i+1})\}. \end{aligned} \tag{4}$$

Since $\{r_n\}$ is convergent, for any $\varepsilon > 0$, we can choose a positive integer n_0 such that

$$\sum_{i=n}^{\infty} \max\{\rho(z_i, z_{i+1}), \varrho(y_i, y_{i+1})\} < \varepsilon$$

for all $n \geq n_0$. Thus inequality (4), we get

$$\max\{\rho(z_n, z_m), \varrho(y_n, y_m)\} < \varepsilon$$

for all $m, n \geq n_0$, and thus $\{z_n\}$ and $\{y_n\}$ are two Cauchy sequences in $O_Z(z_0, y_0)$ and $O_Y(z_0, y_0)$, respectively. Since (Z, ρ) and (Y, ϱ) are related FG-orbitally complete, the sequence $\{z_n\}$ has a limit u in Z and the sequence $\{y_n\}$ has a limit v in Y. Thus, the proof of (a) is complete.

To prove (b), let $m > n$. Then from inequalities (2) and (3), we get

$$\begin{aligned}\max\{\rho(z_n, z_m), \varrho(y_n, y_m)\} &\leq \sum_{i=n}^{m-1} \max\{\rho(z_i, z_{i+1}), \varrho(y_i, y_{i+1})\} \\ &\leq \sum_{i=0}^{m-1} [\gamma(z_i, y_i) - \gamma(z_{i+1}, z_{i+1})] \\ &= \gamma(z_0, y_0) - \gamma(z_m, y_m) \leq \gamma(z_0, y_0).\end{aligned}$$

Letting m tends to infinity it follows that

$$\max\{\rho(z_n, u), \varrho(y_n, v)\} \leq \gamma(z_0, y_0)$$

Thus the proof of (b) is complete.

Now suppose that F is a mapping of Z into $CL(Y)$ and G is a mapping of Y into $CL(Z)$.
(i) $\Rightarrow$ (ii): Assume that $u \in G(v)$ and $v \in F(u)$. Clearly

$$\rho(u, G(v)) = 0 \quad \text{and} \quad \varrho(v, F(u)) = 0.$$

Let $\{z_n\}$, $\{y_n\}$ be two sequences in $O_Z(z_0, y_0)$ and $O_Y(z_0, y_0)$ respectively with $z_n \to u$, $y_n \to v$. Then we get,

$$\lim_{n\to\infty} \mu(z_n, y_n) = \lim_{n\to\infty} \rho(z_n, G(y_n)) \leq \lim_{n\to\infty} \rho(z_n, z_{n+1}) = 0$$

and

$$\lim_{n\to\infty} \eta(z_n, y_n) = \lim_{n\to\infty} \varrho(y_n, F(z_n)) \leq \lim_{n\to\infty} \varrho(y_n, y_{n+1}) = 0$$

and so μ and η are FG-orbitally 0-lsc at (u, v) with respect to (z_0, y_0) since

$$\mu(u, v) = \rho(u, G(v)) = 0 \quad \text{and} \quad \eta(u, v) = \varrho(v, F(u)) = 0.$$

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(ii) $\Rightarrow$ (iii): Let μ and η are FG-orbitally 0-lsc at (u,v) with respect to (z_0,y_0). From (a), there exist two sequence $\{z_n\}$ in $O_Z(z_0,y_0)$ and $\{y_n\}$ in $O_Y(z_0,y_0)$ such that $z_n \to u, y_n \to v$. We have also

$$\lim_{n\to\infty} \mu(z_n,y_n) = \lim_{n\to\infty} \rho(z_n,G(y_n)) \le \lim_{n\to\infty} \rho(z_n,z_{n+1}) = 0$$

and

$$\lim_{n\to\infty} \eta(z_n,y_n) = \lim_{n\to\infty} \varrho(y_n,F(z_n)) \le \lim_{n\to\infty} \varrho(y_n,y_{n+1}) = 0.$$

Since μ and η are FG-orbitally 0-lsc at (u,v) with respect to (z_0,y_0),

$$\rho(u,G(v)) = \inf\{\rho(u,w) : w \in G(v)\} = \mu(u,v) = 0 \quad \text{and}$$
$$\varrho(v,F(u)) = \inf\{\varrho(v,x) : x \in F(u)\} = \eta(u,v) = 0.$$

(iii) $\Rightarrow$ (i):Now let $\rho(u,G(v)) = \inf\{\rho(u,w) : w \in G(v)\} = 0$. Then we have $u \in \overline{G(v)}$. Since $G(v)$ is closed subset of Z, $\overline{G(v)} = G(v)$ and so $u \in G(v)$. Similarly if $\varrho(v,F(u)) = \inf\{\varrho(v,x) : x \in F(u)\} = 0$, then $v \in F(u)$.

We now assume that $u \in G(v)$ and $v \in F(u)$. Then $G(v) \subset (GF)(u)$ since $(GF)(u) = \bigcup_{a\in F(u)} G(a)$. Therefore $u \in (GF)(u)$. Similarly, $F(u) \subset (FG)(v)$ since $(FG)(v) = \bigcup_{b\in G(v)} F(b)$ and so $v \in (FG)(v)$, which completes the proof. □

Note that since inequality

$$\max\{\rho(z,z'),\varrho(y,y')\} \le \rho(z,z') + \varrho(y,y')$$

holds for all $z,z' \in Z$ and $y,y' \in Y$, we obtain the following result.

Corollary 1. *Let (Z,ρ) and (Y,ϱ) be two metric spaces and let F be a mapping of Z into $P(Y)$ and G be a mapping of Y into $P(Z)$. Suppose there exist $z' \in G(y)$ and $y' \in F(z)$ such that*

$$\rho(z,z') + \varrho(y,y') \le \gamma(z,y) - \gamma(z',y') \tag{5}$$

for all $z \in Z$ and $y \in Y$, where $\gamma : Z \times Y \to [0,\infty)$. If (Z,ρ) and (Y,ϱ) are related FG-orbitally complete for some $(z_0,y_0) \in Z \times Y$, then

(a) There exist two sequence $\{z_n\}$ in $O_Z(z_0,y_0)$ and $\{y_n\}$ in $O_Y(z_0,y_0)$ such that

$$\lim_{n\to\infty} z_n = u \in X \textit{ and } \lim_{n\to\infty} y_n = v \in Y,$$

(b) $\rho(z_n,u) + \varrho(y_n,v) \le \gamma(z_0,y_0)$,

(c) If F is a mapping of Z into $CL(Y)$ and G is a mapping of Y into $CL(Z)$, then the following statements are equivalent;

- *(i) $u \in G(v)$ and $v \in F(u)$.*
- *(ii) $\mu : Z \times Y \longrightarrow [0,\infty)$, $\mu(z,y) = \rho(z,G(y))$ and $\eta : Z \times Y \longrightarrow [0,\infty)$, $\eta(z,y) = \varrho(y,F(z))$ are FG-orbitally 0-lsc at (u,v) with respect to (z_0,y_0), where $\rho(z,G(y)) = \inf\{\rho(z,w) : w \in G(y)\}$ and $\varrho(y,F(z)) = \inf\{\varrho(y,x) : x \in F(z)\}$.*
- *(iii) $\rho(u,G(v)) = 0$ and $\varrho(v,F(u)) = 0$.*

Further, if $u \in G(v)$ and $v \in F(u)$, then $u \in (GF)(u)$ and $v \in (FG)(v)$.

Proof. We have

$$\begin{aligned}\max\{\rho(z,z'),\varrho(y,y')\} &\le \rho(z,z') + \varrho(y,y') \\ &\le \gamma(z,y) - \gamma(z',y').\end{aligned}$$

Then the results (a) and (c) follow immediately from Theorem 1.

To prove (b), Let $m > n$. Similarly as in proof of (b) in Theorem 1, using inequality (5), we get

$$\rho(z_n, z_m) + \varrho(y_n, y_m) \leq \sum_{i=n}^{m-1} [\rho(z_i, z_{i+1}) + \varrho(y_i, y_{i+1})]$$
$$\leq \sum_{i=0}^{m-1} [\gamma(z_i, y_i) - \gamma(z_{i+1}, y_{i+1})] \leq \gamma(z_0, y_0).$$

Letting m tends to infinity it follows that

$$\rho(z_n, u) + \varrho(y_n, v) \leq \gamma(z_0, y_0).$$

□

If we let
$$(Z, \rho) = (Y, \varrho),\ T = F = G \quad \text{and} \quad \gamma(z, y) = \vartheta(z),$$

where $\vartheta : Z \to [0, \infty)$, then from Corollary 1 we have the following multi-valued version of Bollenbacher and Hicks's result [16], which is a version of famous Caristi's fixed point theorem [19].

Corollary 2. *Let (Z, ρ) be a metric space and let T be a mapping of Z into $P(Z)$. Suppose there exists $z' \in T(z)$ such that*

$$\rho(z, z') \leq \vartheta(z) - \vartheta(z') \tag{6}$$

for each $z \in Z$, where $\vartheta : Z \to [0, \infty)$. If (Z, ρ) is T-orbitally complete for some $z_0 \in Z$, then

(a) *There exists a sequence $\{z_n\}$ in $O_Z(z_0)$ such that $\lim_{n\to\infty} z_n = u \in Z$,*
(b) $\rho(z_n, u) \leq \vartheta(z_0)$,
(c) *If T is a mapping of Z into $CL(Z)$, then the following statements are equivalent;*
  (i) $u \in T(u)$.
  (ii) $\mu : Z \longrightarrow [0, \infty)$, $\mu(z) = \rho(z, T(z))$ *is T-orbitally 0-lsc at u with respect to z_0, where* $\rho(z, T(z)) = \inf\{\rho(z, x) : x \in T(z)\}$.
  (iii) $d(u, T(u)) = 0$.

Corollary 3. *Let (Z, ρ) be a metric space and let T be a mapping of Z into $CB(Z)$. Suppose there exists $z' \in T(z)$ such that*

$$\rho(z', z'') \leq k\rho(z, z') \tag{7}$$

for each $z \in Z$ and for all $z'' \in T(z')$, where $0 \leq k < 1$. If (Z, ρ) is T-orbitally complete for some $z_0 \in Z$, then

(a) *There exists a sequence $\{z_n\}$ in $O_Z(z_0)$ such that $\lim_{n\to\infty} z_n = u \in Z$,*
(b) $\rho(z_n, u) \leq \sup_{z \in T(z_0)} \rho(z_0, z)$,
(c) *The following statements are equivalent;*
  (i) $u \in T(u)$.
  (ii) $\mu : Z \longrightarrow [0, \infty)$, $\mu(z) = \rho(z, T(z))$ *is T-orbitally 0-lsc at u with respect to z_0, where* $\rho(z, T(z)) = \inf\{\rho(z, x) : x \in T(z)\}$.
  (iii) $d(u, T(u)) = 0$.

Proof. Define the function ϑ on Z by $\vartheta(z) = \frac{1}{1-k}\big(\sup_{x \in T(z)} \rho(z, x)\big)$. Since $T(z)$ is bounded, ϑ is a mapping of Z into $[0, \infty)$. Then from inequality (7) we get

$$\sup_{z'' \in T(z')} \rho(z', z'') \leq k\Big(\sup_{z' \in T(z)} \rho(z, z')\Big). \tag{8}$$

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Thus, from inequality (8), we have

$$\sup_{z'\in T(z)} \rho(z,z') - k\Big(\sup_{z'\in T(z)} \rho(z,z')\Big) \\ \le \sup_{z'\in T(z)} \rho(z,z') - \sup_{z''\in T(z')} \rho(z',z'')$$

and

$$\sup_{z'\in T(z)} \rho(z,z') \le \frac{1}{1-k}\Big(\sup_{z'\in T(z)} \rho(z,z')\Big) - \frac{1}{1-k}\Big(\sup_{z''\in T(z')} \rho(z',z'')\Big) \\ = \vartheta(z) - \vartheta(z')$$

and so

$$\rho(z,z') \le \vartheta(z) - \vartheta(z').$$

Hence, the results follows since all the conditions of Corollary 2 are satisfied. □

We need the following definition for the next corollary.

Definition 6. *If we let F be a single valued mapping f of Z into Y and G be a single valued mapping g of Y into Z, then from (1) we get*

$$O_Z(z_0,y_0) = \{z_n : z_n = g(y_{n-1}), n = 1,2,\ldots\},$$
$$O_Y(z_0,y_0) = \{y_n : y_n = f(z_{n-1}), n = 1,2,\ldots\},$$

where $z_0 \in Z$ and $y_0 \in Y$.

Then the metric spaces (Z,ρ) and (Y,ϱ) are called related fg-orbitally complete for $(z_0,y_0) \in Z\times Y$ iff every Cauchy sequence in $O_Z(z_0,y_0)$ and $O_Y(z_0,y_0)$ converges to a point in Z and converges to a point in Y, respectively.

We finally give the following corollary for single valued mappings.

Corollary 4. *Let (Z,ρ) and (Y,ϱ) be two metric spaces and let f be a mapping of Z into Y and g be a mapping of Y into Z satisfying the inequalities*

$$\max\{\rho(z,g(y)),\varrho(y,f(z))\} \le \gamma(z,y) - \gamma(g(y),f(z)) \tag{9}$$

for all $z \in Z$ and $y \in Y$, where $\gamma : Z\times Y \to [0,\infty)$. If (Z,ρ) and (Y,ϱ) are related fg-orbitally complete fore some $(z_0,y_0) \in Z\times Y$, then

(a) $\lim_{n\to\infty} z_n = g(y_{n-1}) = u \in Z$ *and* $\lim_{n\to\infty} y_n = f(z_{n-1}) = v \in Y$, *for $z_0 \in Z$ and $y_0 \in Y$, exist.*

(b) $\max\{\rho(z_n,u),\varrho(y_n,v)\} \le \gamma(z_0,y_0)$,

(c) *$g(v) = u$ and $f(u) = v$ if and only if $\mu : Z\times Y \longrightarrow [0,\infty)$, $\mu(z,y) = \rho(z,g(y))$ and $\eta : Z\times Y \longrightarrow [0,\infty)$, $\eta(z,y) = \varrho(y,f(z))$ are fg-orbitally 0-lsc at (u,v) with respect to (z_0,y_0).*

Further if $u = g(v)$ and $v = f(u)$, then $u = (gf)(u)$ and $v = (fg)(v)$.

Proof. Define two mappings f of Z into $P(Y)$ and g of Y into $P(Z)$ by putting $F(z) = f(z)$ for all $z \in Z$ and $G(y) = g(y)$ for all $y \in Y$, respectively. It follows that F and G satisfy inequality (2). Then the results (a) and (b) follows since all the conditions of Theorem 1 are satisfied.

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Now we prove (c). Suppose that $f(u) = v$, $g(v) = u$ and $\{z_n\}$ and $\{y_n\}$ are sequences in $O_Z(z_0, y_0)$ and $O_Y(z_0, y_0)$ respectively with $z_n \to u$, $y_n \to v$. Then we get,

$$\lim_{n\to\infty} \mu(z_n, y_n) = \lim_{n\to\infty} \rho(z_n, g(y_n)) = \lim_{n\to\infty} \rho(z_n, z_{n+1}) = 0$$

and also $\mu(u, v) = \rho(u, g(v)) = \rho(u, u) = 0$.

Similarly we have $\lim_{n\to\infty} \eta(z_n, y_n) = 0$ and $\eta(u, v) = 0$. Thus μ and η are 0-lsc at (u, v).

Now μ, η are 0-lsc at (u, v) and let $z_n = g(y_{n-1}), y_n = f(z_{n-1})$. It follows from (a) that $\lim_{n\to\infty} \rho(z_n, z_{n+1}) = 0$ and $\lim_{n\to\infty} \varrho(y_n, y_{n+1}) = 0$. Then

$$\lim_{n\to\infty} \mu(z_n, y_n) = \lim_{n\to\infty} \rho(z_n, g(y_n)) = \lim_{n\to\infty} \rho(z_n, z_{n+1}) = 0 \quad \text{and}$$
$$\lim_{n\to\infty} \eta(z_n, y_n) = \lim_{n\to\infty} \varrho(y_n, f(z_n)) = \lim_{n\to\infty} \varrho(y_n, y_{n+1}) = 0.$$

Since μ, η are 0-lsc at (u, v), we have $d(u, g(v)) = \mu(u, v) = 0$ and $\varrho(v, f(u)) = \eta(u, v) = 0$ and so $f(u) = v$ and $g(v) = u$. □

4. Examples

We finally give two examples which support our main result.

Example 2. *Let $Z = [0, 1)$ and $Y = [0, 1/2)$ with the Euclidean metrics ρ and ϱ, respectively. Define the mappings $F : Z \to CL(Y)$ and $G : Y \to CL(Z)$ by*

$$F(z) = [0, z/4], \quad G(y) = [0, y/2]$$

for all $z \in Z$ and for all $y \in Y$. Then for $z_0 = 0 \in Z$ and $y_0 = 0 \in Y$, we have

$$O_Z(0, 0) = \{0, 0, 0, \dots\}, \quad O_Y(0, 0) = \{0, 0, 0, \dots\}.$$

Therefore, (Z, ρ) and (Y, ϱ) are related FG-orbitally complete.

If $z' = \frac{y}{2} \in G(y)$ and $y' = \frac{z}{4} \in F(z)$ are taken for each $z \in Z$ and $y \in Y$, then we get

$$\max\{\rho(z, z'), \varrho(y, y')\} = \max\left\{\left|z - \frac{y}{2}\right|, \left|y - \frac{z}{4}\right|\right\}$$
$$\leq \frac{3z}{2} + y = \gamma(z, y) - \gamma(z', y')$$

where $\gamma : Z \times Y \to [0, \infty), \gamma(z, y) = 2(z + y)$. Thus the inequality (2) is satisfied.

The sequences

$$\{z_n\} = \{0, 0, 0, \dots\}, \quad \{y_n\} = \{0, 0, 0, \dots\}$$

in $O_Z(0, 0)$ and $O_Y(0, 0)$ converge to 0. Also $0 \in F(0)$ and $0 \in G(0)$ and so $0 \in GF(0)$ and $0 \in FG(0)$.

Example 3. *Let $Z = [-1, 1]$ and $Y = [-1, 2]$ with the Euclidean metrics ρ and ϱ, respectively. Define the mappings $F : Z \to CL(Y)$ and $G : Y \to CL(Z)$ by*

$$F(z) = \{z/2, z\}, \quad G(y) = [-1, y/2]$$

for all $z \in Z$ and for all $y \in Y$.

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If $z' = \frac{y}{2} \in G(y)$ and $y' = \frac{z}{2} \in F(z)$ are taken for each $z \in Z$ and $y \in Y$, then we get

$$\max\{\rho(z,z'), \varrho(y,y')\} = \max\left\{|z - \frac{y}{2}|, |y - \frac{z}{2}|\right\}$$
$$\leq z + y = \gamma(z,y) - \gamma(z',y')$$

where $\gamma : Z \times Y \to [0,\infty), \gamma(z,y) = 2(z+y)$. Thus the inequality (2) is satisfied.

Take the points $z_0 = 1/2 \in Z$ and $y_0 = 0 \in Y$. If we choose $z_n \in G(y_{n-1})$ as $\frac{y_{n-1}}{2}$ and $y_n \in F(z_{n-1})$ as $\frac{z_{n-1}}{2}$, then we obtain the following sequences in $O_Z(1/2,0)$ and $O_Y(1/2,0)$, respectively.

$$\{z_n\} = \left\{0, \frac{1}{2^3}, \frac{1}{2^5}, \frac{1}{2^7}, \dots\right\}, \quad \{y_n\} = \{\frac{1}{2^2}, \frac{1}{2^4}, \frac{1}{2^6}, \dots\}$$

and so the sequences converge to 0. Also $0 \in F(0)$ and $0 \in G(0)$ and so $0 \in GF(0)$ and $0 \in FG(0)$.

Note that the closedness of the set $G(z)$ and $F(y)$, for all $z \in Z$ and for all $y \in Y$, is necessary condition in (c) of Theoremn 1. For example, if we take $G(y) = [-1, y/2)$ in example above, then $0 \notin G(0)$.

5. Conclusions

In this research article, by introducing the concept related orbitally completeness of two metric spaces for multi-valued mappings a theorem concerning the fixed points of the compositions of two multivalued mappings defined on two orbitally complete metric spaces is presented, and the relationship between the fixed points of these mappings is investigated. Some important results are also obtained as a corollary of the present main theorem. Finally, two concrete examples that illustrate the significance of our main theorem are provided. The findings presented in this paper are expected to serve as a foundation for further research in the field.

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