Solve the 3x+1 problem by the multiplication and division of binary numbers

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Abstract

The $3x+1$ problem is the following: Suppose we start with a positive integer, and if it is odd then multiply it by 3 and add 1, and if it is even, divide it by 2. Then repeat this process as long as you can. Will you eventually reach the integer 1, no matter what you started with? Collatz conjecture (or $3n+1$ problem) has been explored for about 85 years. In this paper, we convert an integer number from decimal to binary and convert the Collatz function to a binary function, which involves the multiplication and division of two binary numbers. Finally, by iterating the Collatz function, we eventually reach the integer number 1, thus completely solving the 3x+1 problem.

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1 Introduction

The $3x+1$ problem, also known as the Collatz conjecture, $3x+1$ mapping, Ulam conjecture, Kakutani's problem, Thwaites conjecture, Hasse's algorithm, or Syracuse problem [1], is one of the unsolved problems in mathematics. Paul Erdos (1913-1996) commented on the intractability of the $3x+1$ problem [2], stating that "Mathematics is not ready for those problems yet".

The $2x+1$ problem states that, for any positive integer x, if x is even, divide it by 2; if x is odd, multiply it by 3 and add 1. Repeating this process continuously leads to the conjecture that no matter which number is initially chosen, the result will always reach 1 eventually.

2 Terminology and notations

We use the notations as in [4,7], and describe a Collatz function as follows:

$$T(n) = \begin{cases} 3n+1, & \text{if } n \text{ is odd number,} \\ \frac{n}{2} & \text{if } n \text{ is even number.} \end{cases} \tag{1}$$

Let N denote the set of positive integers. For $n \in N$, and $k = 0, 1, 2, 3, \cdots$, $T^0(n)$ and $T^{k+1}(n)$ denote n and $T(T^k(n))$, respectively. Concerning the behavior of the iteration of the Collatz function, for any integer n, there must exist an integer r so that

$$T^r(n) = 1.$$

2.1 The modified Sarkovskii ordering and integer lattice

We convert the last row of numbers into the first column to get a modified Sarkovskii ordered integer lattice[6] as the following,

$$\begin{array}{lllllllllll}
1, & 3, & 5, & 7, & 9, & 11, & 13, & 15, & 17, & 19, & \cdots \\
2, & 2\cdot 3, & 2\cdot 5, & 2\cdot 7, & 2\cdot 9, & 2\cdot 11, & 2\cdot 13, & 2\cdot 15, & 2\cdot 17, & 2\cdot 19, & \cdots \\
2^2, & 2^2\cdot 3, & 2^2\cdot 5, & 2^2\cdot 7, & 2^2\cdot 9, & 2^2\cdot 11, & 2^2\cdot 13, & 2^2\cdot 15, & 2^2\cdot 17, & 2^2\cdot 19, & \cdots \\
2^3, & 2^3\cdot 3, & 2^3\cdot 5, & 2^3\cdot 7, & 2^3\cdot 9, & 2^3\cdot 11, & 2^3\cdot 13, & 2^3\cdot 15, & 2^3\cdot 17, & 2^3\cdot 19, & \cdots \\
2^4, & 2^4\cdot 3, & 2^4\cdot 5, & 2^4\cdot 7, & 2^4\cdot 9, & 2^4\cdot 11, & 2^4\cdot 13, & 2^4\cdot 15, & 2^4\cdot 17, & 2^4\cdot 19, & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

In the first row, they are odd numbers from left to right, that are $1, 3, 5, 7, 9, 11, 13, \cdots$. From the second row, each number is two times the number in its previous row, and so on.

2.2 The algebraic formula and Collatz graph

If we draw a line segment with an arrow between two digits in the lattice of integers in the modified Sarkovskii ordering, one being the original value x and the other being its value of the Collatz function $T(x)$, and then connect $T(x)$ to $T^2(x)$, and so on $T^2(x)$ to $T^3(x), \cdots$, we get a graph which can be called a *Collatz graph*.

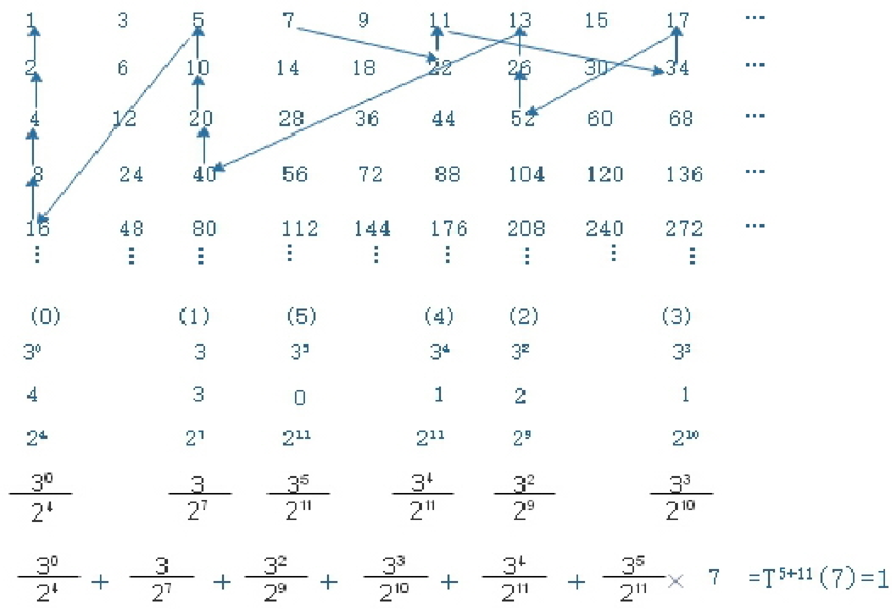

Fig. 1. The Collatz graph of $T^{16}(7) = T(5, 11, 7) = 1$ in the lattice of integers in the modified Sarkovskii ordering and the algebraic formula.

For different integers k and l, if there is a common vertex in their Collatz graphs, their graphs will overlap from that point onwards until they reach the minimum value of 1. Using the Collatz function $T(x)$, we can obtain an algebraic formula of $\frac{1}{2^4}, \frac{3}{2^7}, \frac{3^2}{2^9}, \cdots, \frac{3^m}{2^r} \cdot x$, where r is the number of vertical segments and m is the number of oblique segments in the Collatz graph,

$$T^{m+r}(n) = T(m, r, n) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \cdots + \frac{3^m}{2^r} \cdot x = 1.$$

For example, $n = 7$, $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1 \to \cdots$, the algebraic formula is

$$T^{16}(7) = T(5, 11, 7) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{11}} \cdot 7 = 1,$$

and the Collatz graph is Fig. 1.

And $n = 36$, $36 \to 18 \to 9 \to 28 \to 14 \to 7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1 \to \cdots$, the algebraic formula is

$$T^{21}(36) = T(6, 15, 36) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{13}} + \frac{3^6}{2^{15}} \cdot 36 = 1$$

and the Collatz graph is Fig. 2.

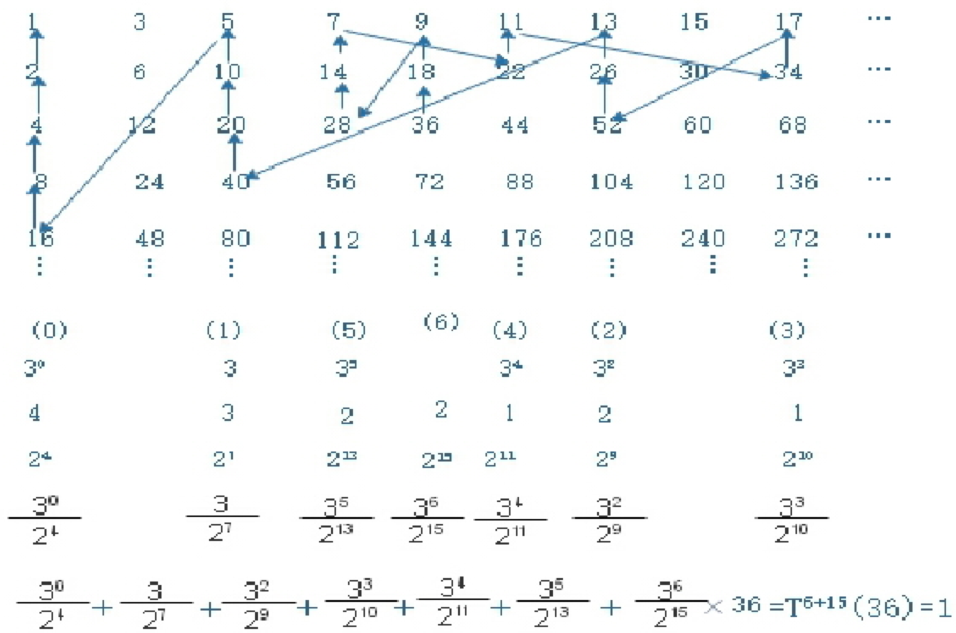

Fig. 2. The Collatz graph of $T^{21}(36) = T(6, 15, 36) = 1$ in the lattice of integers in the modified Sarkovskii ordering and the algebraic formula.

We observe that there is a property present,

Proposition 1 *For positive integers i, j, k, l, and $l_k, l_{k-1}, \cdots, l_1$, if $i > j$, then there is a recurrence relation*

$$T^i(n) = \frac{3^k}{2^l} T^j(n) + \frac{3^{k-1}}{2^{l_k}} + \cdots + \frac{3^2}{2^{l_3}} + \frac{3}{2^{l_2}} + \frac{1}{2^{l_1}}$$

where $k + l = i - j$, and $l \geq l_k \geq l_{k-1} \geq \cdots \geq l_1$.

For example, there are

$$T^3(97) = \frac{3}{2^2} \cdot 97 + \frac{1}{2^2} = 73$$

$$T^{18}(97) = \frac{3^7}{2^{11}} \cdot 97 + \frac{3^6}{2^{11}} + \frac{3^5}{2^9} + \frac{3^4}{2^7} + \frac{3^3}{2^6} + \frac{3^2}{2^5} + \frac{3}{2^2} + \frac{1}{2} = 107$$

We can get the recurrence formula about the Collatz function,

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$$
\begin{aligned}
T^{26}(97) &= \frac{3^3}{2^5} \cdot T^{18}(97) + \frac{3^2}{2^5} + \frac{3}{2^4} + \frac{1}{2^2} \\
&= \frac{3^{10}}{2^{16}} \cdot 97 + \frac{3^9}{2^{16}} + \frac{3^8}{2^{14}} + \frac{3^7}{2^{12}} + \frac{3^6}{2^{11}} + \frac{3^5}{2^{10}} + \frac{3^4}{2^7} + \frac{3^3}{2^6} + \frac{3^2}{2^5} + \frac{3}{2^4} + \frac{1}{2^2} \\
&= 91,
\end{aligned}
$$

3 Numerical example

Using the above Collatz graphs of the integer lattice of the modified Sarkovskii ordering, we give the following algebraic formulas,

$$T^{19}(9) = T(6,13,9) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{13}} + \frac{3^6}{2^{13}} \cdot 9 = 1,$$

$$
\begin{aligned}
T^{15}(23) = T(4,11,23) &= \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{11}} \cdot 23 = 1, \\
T^{17}(15) = T(5,12,15) &= \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{12}} + \frac{3^5}{2^{12}} \cdot 15 = 1, \\
T^{12}(17) = T(3,9,17) &= \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^9} \cdot 17 = 1, \\
T^{19}(61) = T(5,14,61) &= \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{14}} + \frac{3^5}{2^{14}} \cdot 61 = 1. \\
T^{16}(397) = T(5,11,397) &= \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{17}} + \frac{3^6}{2^{20}} + \frac{3^7}{2^{20}} \cdot 397 = 1,
\end{aligned}
$$

Example 2 *For the formula*

$$T(6,14,18) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{13}} + \frac{3^6}{2^{14}} \cdot 18 = 1,$$

we rewrite it as an integer equation,

$$3^6 \cdot 18 + 3^5 \cdot 2 + 3^4 \cdot 2^3 + 3^3 \cdot 2^4 + 3^2 \cdot 2^5 + 3 \cdot 2^7 + 2^{10} = 2^{14}.$$

PROOF. To calculate the power of 3 and the value of 18 using powers of 2,

$$
\begin{aligned}
3 &= 2 + 1 \\
3^2 &= 2^3 + 1 \\
3^3 &= 2^4 + 2^3 + 2 + 1 \\
3^4 &= 2^6 + 2^4 + 1 \\
3^5 &= 2^7 + 2^6 + 2^5 + 2^4 + 2 + 1 \\
3^6 &= 2^9 + 2^7 + 2^6 + 2^4 + 2^3 + 1 \\
18 &= 2^4 + 2
\end{aligned}
$$

substituting these expressions into the left-hand side of the above equation, one obtains,

$$
\begin{aligned}
&3^6 \cdot 18 + 3^5 \cdot 2 + 3^4 \cdot 2^3 + 3^3 \cdot 2^4 + 3^2 \cdot 2^5 + 3 \cdot 2^7 + 2^{10} \\
=&(2^9 + 2^7 + 2^6 + 2^4 + 2^3 + 1) \cdot (2^4 + 2) + (2^7 + 2^6 + 2^5 + 2^4 + 2 + 1) \cdot 2 \\
&+(2^6 + 2^4 + 1) \cdot 2^3 + (2^4 + 2^3 + 2 + 1) \cdot 2^4 + (2^3 + 1) \cdot 2^5 + (2 + 1) \cdot 2^7 + 2^{10},
\end{aligned}
$$

and get the value 2^{14} which is equal to the right value of the equation.

4 Convert the integer number from decimal to binary

Be inspired by the above, we use binary to rewrite the Collatz function (1) as the following formulas (2) and (3). We denote a binary number, which is a string of 0s and 1s, as $n = (1 \times \cdots \times)_2$, where $\times$ is either 1 or 0, e.g. $3 = (11)_2$,

$$
T(n) = T((1 \times \cdots \times)_2) = \begin{cases} (11)_2 \cdot (1 \times \cdots \times 1)_2 + 1, & \text{if } n \text{ is odd number,} \\ \frac{(1\times\cdots\times 10\cdots 00)_2}{(10)_2}, & \text{if } n \text{ is even number.} \end{cases} \tag{2}
$$

The result is

$$
T(n) = T((1 \times \cdots \times)_2) = \begin{cases} (1 \times \times \times 10 \cdots 0)_2, & \text{if } n \text{ is odd number,} \\ (1 \times \cdots \times 10 \cdots 0)_2, & \text{if } n \text{ is even number.} \end{cases} \tag{3}
$$

Namely, when n is an odd number, we multiply it with $(11)_2$ and add 1 to the end of the binary number. For example, $T(97) = T(1100001)$ in Fig. 3. When n is an even number, the division is equal to deleting the zero at the end in the binary number. We give the iteration of the Collatz function for $1, 5, 7, 9$, 97 in binary as the following five tables.

Example 3 *For positive integer 1, we manipulate the iteration of the Collatz function in both decimal and binary numbers,*

ith	*0*	*1*	*2*	*3*
decimal	*1*	*4*	*2*	*1*
binary	*1*	*100*	*10*	*1*

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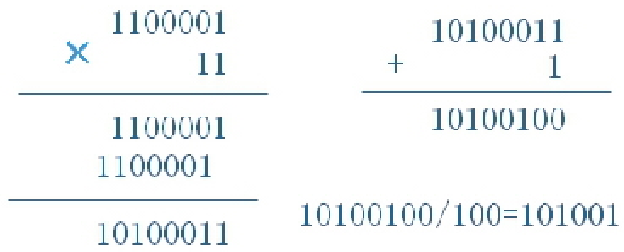

Fig. 3. For the Collatz function $T(97)$ in binary, the first step is the multiplication in left, the second step is division in the right bottom.

Example 4 *For positive integer $5 = (101)_2$, we manipulate the iteration of the Collatz function in both decimal and binary numbers,*

ith	*0*	*1*	*2*	*3*	*4*	*5*
decimal	*5*	*16*	*8*	*4*	*2*	*1*
binary	*101*	*10000*	*1000*	*100*	*10*	*1*

Example 5 *For $7 = (111)_2$, we manipulate the iteration of the Collatz function in both decimal and binary numbers.*

ith	*0*	*1*	*2*	*3*	*4*	*5*	*7*	*8*	*11*	*12*	*16*
decimal	*7*	*22*	*11*	*34*	*17*	*52*	*13*	*40*	*5*	*16*	*1*
binary	*111*	*10110*	*1011*	*100010*	*10001*	*110100*	*1101*	*101000*	*101*	*10000*	*1*

Example 6 *For $9 = (1001)_2$, we manipulate the iteration of the Collatz function in both decimal and binary numbers.*

ith	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*14*	*15*	*19*
decimal	*9*	*28*	*14*	*7*	*22*	*11*	*34*	*17*	*5*	*16*	*1*
binary	*1001*	*11100*	*1110*	*111*	*10110*	*1011*	*100010*	*10001*	*101*	*10000*	*1*

Example 7 *For $97 = (100100)_2$, we manipulate the iteration of the Collatz*

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function in binary as the following table,

0	97	**1100001**	30	206	11001110	60	425	**110101001**	90	866	11011
1	292	100100100	31	103	**1100111**	61	1276	10011111100	91	433	**1101**
2	146	10010010	32	310	100110110	62	638	1001111110	92	1300	10100
3	73	**1001001**	33	155	**10011011**	63	319	**100111111**	93	650	10100
4	220	11011100	34	466	111010010	64	958	1110111110	94	325	**1010**
5	110	1101110	35	233	**11101001**	65	479	**111011111**	95	976	11110
6	55	**110111**	36	700	1010111100	66	1438	10110011110	96	488	11110
7	166	11111010	37	350	101011110	67	719	**1011001111**	97	244	11110
8	83	**1111101**	38	175	**10101111**	68	2158	100001101110	98	122	11110
9	250	10100110	39	526	1000001110	69	1079	**10000110111**	99	61	**1111**
10	125	**1010011**	40	263	**100000111**	70	3238	110010100110	100	184	10111
11	376	11111010	41	790	11000010110	71	1619	**11001010011**	101	92	10111
12	188	**1111101**	42	395	**1100001011**	72	4858	1001011111010	102	46	10111
13	94	101111000	43	1186	10010100010	73	2429	**100101111101**	103	23	**1011**
14	47	10111100	44	593	**1001010001**	74	7288	1110001111000	104	70	10001
15	142	1011110	45	1780	11011110100	75	3644	111000111100	105	35	**1000**
16	71	**101111**	46	890	1101111010	76	1822	11100011110	106	106	11010
17	214	10001110	47	445	**110111101**	77	911	**1110001111**	107	53	**1101**
18	107	**1101011**	48	1336	10100111000	78	2734	101010101110	108	160	10100
19	322	101000010	49	668	1010011100	79	1367	**10101010111**	109	80	10100
20	161	**10100001**	50	334	101001110	80	4102	1000000000110	110	40	10100
21	484	111100100	51	167	**10100111**	81	2051	**100000000011**	111	20	10100
22	242	11110010	52	502	111110110	82	6154	1100000001010	112	10	1010
23	121	**1111001**	53	251	**11111011**	83	3077	**110000000101**	113	5	**101**
24	364	101101100	54	754	1011110010	84	9232	10010000010000	114	16	10000
25	182	10110110	55	377	**101111001**	85	4616	1001000001000	115	8	1000
26	91	**1011011**	56	1132	10001101100	86	2308	100100000100	116	4	100
27	274	100010010	57	566	1000110110	87	1154	10010000010	117	2	10
28	137	**10001001**	58	283	**100011011**	88	577	**1001000001**	118	1	1
29	412	110011100	59	850	1101010010	89	1732	11011000100			

Corollary 8 *The Collatz function makes an odd integer number in binary bigger by adding 1 or 2 bits to the left of the sequence of 1s and 0s, and an even integer number in binary smaller by deleting all zeros at the end of the sequence of 1s and 0s. Thus, although in some cases the value of The Collatz function T(x) may be bigger than x in decimal, in general, the iteration of the Collatz function will make an integer number smaller and smaller, eventually reaching the smallest positive integer number 1.*

We can rewrite the Collatz conjecture in binary, which makes it an easier problem to solve, thus allowing us to completely solve the Collatz conjecture.

Fact 9 *For any positive integer, under the Collatz function, the sequence of integer numbers in binary will eventually reach 1.*

PROOF. *For an odd binary integer, we multiply it by* $(11)_2$ *and add 1 in the last bit, the result number must be an even number in binary which at least one zero at the end. We delete all these zeros, namely it is the division. This is the above corollary 8. Thus, we repeat this process as long as we can, because the bits of the sequence in binary of a positive integer number is finite. Eventually, we must in finitely steps reach the smallest positive integer number 1.*

Remark 10 *We can say that the 3x+1 problem is a converse proposition of "period three implies chaos" [4], and it is also an example of any one positive integer number having a period of 3 in the Collatz function.*

5 Conclusions

We rewrite the Collatz function in binary, which makes the $3x + 1$ problem easier. The multiplications of $(11)_2$ and divisions of $(10)_2$ make the positive integer number smaller and smaller with the iterations of the Collatz function. In some cases, the value of the Collatz function $T(x)$ may be bigger than x, thus allowing us to completely solve the Collatz conjecture.

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